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PARTICLE TRANSPORT WITH FINITE FILTRATION TIME

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Abstract: Particle transport by a fluid flow occurs in many applied construction problems, including pumping mortar into porous soil, creating watertight diaphragm walls, and constructing dams and underwater structures. A model of deep bed filtration of suspensions and colloids in a homogeneous porous medium with a finite number of vacancies for retained particles is considered. A suspension of constant concentration is injected into the inlet of a porous medium containing clean water. If the sediment growth rate remains positive as the sediment concentration approaches the upper limiting value, the filtration process continues for a finite time. In this case, the filtration function that specifies the sediment growth rate in the mathematical model is not blocking. At each point of the porous medium, sedimentation begins from the moment the concentration front passes and ends after a finite period of time depending on the distance to the porous medium inlet. A global exact solution to the problem is constructed in the filtration domain, which consists of two zones. In the zone bordering the concentration front, the solution has a standard form, and in the zone adjacent to the upper limiting values of the concentrations of suspended and retained particles, it has the form of a traveling wave.

Keywords: deep bed filtration, porous medium, filtration function, retained particles concentration, exact solution

ПЕРЕНОС ЧАСТИЦ С КОНЕЧНЫМ ВРЕМЕНЕМ ФИЛЬТРАЦИИ

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Аннотация: Перенос частиц потоком жидкости встречается во многих прикладных строительных задачах, в том числе при закачке раствора в пористый грунт, при создании водонепроницаемых стен в грунте и при конструировании плотин и подводных сооружений. Рассматривается модель глубинной фильтрации суспензий и коллоидов в однородной пористой среде, которая имеет конечное число вакантных мест для задержанных частиц. На вход пористой среды, содержащей чистую воду, впрыскивается суспензия постоянной концентрации. Если скорость роста осадка остается положительной при приближении концентрации осадка к предельной величине, то процесс фильтрации продолжается конечное время. В этом случае функция фильтрации, задающая скорость роста осадка в математической модели, не является блокирующей. В каждой точке пористой среды осаждение осадка начинается с момента прохождения фронта концентраций и заканчивается через конечный промежуток времени, зависящий от расстояний до входа пористой среды. Построено глобальное точное решение задачи в области фильтрации, которая состоит из двух зон. В зоне, граничащей с фронтом концентраций, решение имеет стандартный вид, а в зоне, примыкающей к предельным значениям концентраций взвешенных и осажденных частиц - вид бегущей волны.

Ключевые слова: глубинная фильтрация, пористая среда, функция фильтрации, концентрация осажденных частиц, точное решение

1. INTRODUCTION

Filtration of small particles in porous media occurs in natural phenomena and in technological processes and is one of the main problems of underground hydromechanics. The transport and sedimentation of particles in porous soil must be taken into account when designing foundations, tunnels and underground structures [1-6].

When filtering a suspension in a porous medium, sediment formation can be caused by various reasons. Electrical, gravitational and hydrodynamic forces act on particles in a fluid flow. If the sizes of pores and particles are close, then the size - exclusion capture mechanism plays an important role - particles freely pass through large pores and get stuck in small pores [7-10].

When the number of particles in the carrier liquid is small, the suspended particles do not interact with each other and the sediment growth rate is proportional to the suspended particles concentration. The proportionality coefficient between the sediment growth rate and the suspended concentration is called the filtration function. As a rule, the sediment growth rate decreases with increasing sediment concentration and becomes zero at a certain sediment limit. In this case, particle sedimentation ceases and suspended particles pass through the porous medium without hindrance, since all vacant places for retained particles are filled. The corresponding filtration functions are called blocking. For a monodisperse suspension, the filtration function is linear and is called the Langmuir coefficient. If the suspension contains different particles, the filtration function is nonlinear [11-14].

As a rule, mathematical models of such processes assume an infinite time to reach the maximum sediment. However, in some cases, the sediment growth rate remains significant throughout the filtration time and the maximum sediment is formed in a finite time. This model is considered in the article. The filtration function that ensures a finite filtration time can be blocking or non-blocking [15-18].

Mathematical difficulties in constructing a solution to a model with a non-blocking filtration function are associated with the ambiguity of the solution when the maximum sediment is reached. They were overcome using the condition of continuity of sediment concentration in a porous medium. In addition to the exact solution found, an example with an unlimited filtration function is considered.

2. MODEL OF PARTICLE TRANSPORT

The model of particle transport in a porous medium includes a mass balance equation and a sediment growth equation [19]. The initial and boundary conditions describe the injection of a constant concentration suspension into a porous medium that does not contain suspended or settled particles.

In the domain $\Omega = \{x \geq 0,\, t \geq 0\}$ consider the system

$$\frac{\partial C}{\partial t} + \frac{\partial C}{\partial x} + \frac{\partial S}{\partial t} = 0, \qquad (1)$$

$$\frac{\partial S}{\partial t} = \Lambda(S)C, \qquad (2)$$

where filtration function $\Lambda(S)$ is continuous and positive for $0 \leq S \leq S_m$, $S_m > 0$, and $\Lambda(S) = 0$ for $S > S_m$. At the point S_m the function $\Lambda(S)$ is non-zero: $\Lambda(S_M) = \Lambda_M > 0$.

Unique solution of the system (1), (2) is determined by boundary-initial conditions

$$C\big|_{x=0} = 1, \qquad (3)$$

$$C\big|_{t=0} = 0, \quad S\big|_{t=0} = 0. \qquad (4)$$

Concentration front $t = x$ divides the domain Ω into two subdomains $\Omega_0 = \{x \geq 0,\, 0 \leq t < x\}$ and $\Omega_1 = \{x \geq 0,\, t > x\}$. In Ω_0 the system (1)-(4) has zero solution: $C = 0,\, S = 0$. In Ω_1 the solution is positive. Solution C has a break on the concentration front; solution S is continuous in Ω and

$$S\big|_{t=x} = 0. \qquad (5)$$

In the domain Ω_1 the solution to the problem (1) - (4) coincides with the solution to problem (1) - (3), (5).

According to condition (3) at the porous medium inlet $x = 0$ Eq. (2) takes the form

$$\frac{\partial S}{\partial t} = \Lambda(S). \quad (6)$$

For $0 \le S < S_m$ filtration function $\Lambda(S)$ is continuous. Solution to Eq. (6) is given by implicit formula

$$\int_0^{S_0} \frac{dS}{\Lambda(S)} = t, \quad (7)$$

where

$$S_0(t) = S(x,t)\big|_{x=0}. \quad (8)$$

Denote

$$t_0 = \int_0^{S_m} \frac{dS}{\Lambda(S)}. \quad (9)$$

For $0 \le t \le t_0$ the solution (8) is given by the formula (7).
For $t > t_0$ Eq. (6) takes the form

$$\frac{\partial S}{\partial t} = 0. \quad (10)$$

Solution to Eq. (15) with initial condition $S_0(t_0) = S_m$ is constant:

$$S_0(\tau) = S_m. \quad (11)$$

At the inlet $x = 0$ the solution $S_0(t)$ increases from zero at $t = 0$ to S_m at $t = t_0$ and is constant at $t > t_0$.
To obtain the solution at the concentration front substitute Eq. (2) into Eq. (1)

$$\frac{\partial C}{\partial t} + \frac{\partial C}{\partial x} + \Lambda(S)C = 0. \quad (12)$$

Solution to Eq. (12) with boundary condition (3) is

$$C^-(x) = e^{-\Lambda(0)x}. \quad (13)$$

The line $t = x + t_0$ divides the domain Ω_1 into two zones

$$V_1 = \{x \ge 0,\ x \le t \le x + t_0\},$$
$$V_2 = \{x \ge 0,\ t > x + t_0\}.$$

V_1 is a standard filtration zone and V_2 is a traveling wave zone [20, 21].

3. EXACT SOLUTION

In zone V_1 the solution to system (1)-(3), (5) is obtained by the standard methods [22-24]. Express C from Eq. (2):

$$C = \frac{\partial S / \partial t}{\Lambda(S)}, \quad (14)$$

and substitute into Eq. (12):

$$\frac{\partial}{\partial t}\left(\frac{\partial S / \partial t}{\Lambda(S)}\right) + \frac{\partial}{\partial x}\left(\frac{\partial S / \partial t}{\Lambda(S)}\right) + \frac{\partial S}{\partial t} = 0. \quad (15)$$

Using the formula

$$\frac{\partial}{\partial x}\left(\frac{\partial S / \partial t}{\Lambda(S)}\right) = \frac{\partial^2 S / \partial t \partial x}{\Lambda(S)} - \frac{\partial S}{\partial t}\frac{\partial S}{\partial x}\frac{\Lambda'(S)}{\Lambda^2(S)} = \frac{\partial}{\partial t}\left(\frac{\partial S / \partial x}{\Lambda(S)}\right),$$

change the order of differentiation in (15):

$$\frac{\partial}{\partial t}\left(\frac{\partial S / \partial t}{\Lambda(S)}\right) + \frac{\partial}{\partial t}\left(\frac{\partial S / \partial x}{\Lambda(S)}\right) + \frac{\partial S}{\partial t} = 0. \quad (16)$$

Integrate (16) in *t*:

$$\frac{\partial S / \partial t}{\Lambda(S)} + \frac{\partial S / \partial x}{\Lambda(S)} + S = K(x). \quad (17)$$

The integration constant $K(x)$ is determined from condition (5):

$$S\big|_{t=x} = 0 \;\Rightarrow\; \left(\frac{\partial S}{\partial x} + \frac{\partial S}{\partial t}\right)\Big|_{t=0} = 0 \;\Rightarrow\; K(x) = 0\,.$$

Eq. (17) takes the form

$$\frac{\partial S / \partial t}{\Lambda(S)} + \frac{\partial S / \partial x}{\Lambda(S)} + S = 0\,. \qquad (18)$$

Denote characteristic variables

$$\tau = t - x, \quad x = x\,.$$

In the characteristic variables Eq. 18) takes the form

$$\frac{\partial S / \partial x}{\Lambda(S)} + S = 0\,. \qquad (19)$$

Solution to Eq. (19) with boundary condition (8) has the form

$$\int_{S}^{S_0(\tau)} \frac{dS}{S\Lambda(S)} = x\,. \qquad (20)$$

In Cartesian variables the solution (20) takes the form

$$\int_{S(x,t)}^{S_0(t-x)} \frac{dS}{S\Lambda(S)} = x\,. \qquad (21)$$

Differentiate Eq. (21) in t:

$$\frac{\partial S_0 / \partial t}{S_0\Lambda(S_0)} - \frac{\partial S / \partial t}{S\Lambda(S)} = 0\,. \qquad (22)$$

Substitute Eqs. (2) and (6) in the formula (22):

$$\frac{1}{S_0} - \frac{C}{S} = 0\,. \qquad (23)$$

Express C from Eq. (23):

$$C(x,t) = \frac{S(x,t)}{S_0(t-x)}\,. \qquad (24)$$

Formula (24) defines the Riemann invariant of the system (1)-(3), (5).
Denote the solution at the border $t = x + t_0$ of zone V_1:

$$S_1(x) = S(x,t)\big|_{t=x+t_0}\,, \qquad (25)$$

At the starting point of the border $x = 0,\ t = t_0$ and the solution is $S_0(t_0) = S_m$. Formula (21) takes the form

$$\int_{S_1(x)}^{S_m} \frac{dS}{S\Lambda(S)} = x\,. \qquad (26)$$

To obtain the solution in the travelling wave zone V_2 multiply Eq. (19) by $\Lambda(S)$:

$$\frac{\partial S}{\partial x} + S\Lambda(S) = 0\,. \qquad (27)$$

In zone V_2 the solution to Eq. (27) with condition

$$S(x,\tau)\big|_{x=0} = S_m\,, \qquad (28)$$

is not unique, because filtration function $\Lambda(S)$ has a break at the point $S = S_m$. For a fixed $\tau > t_0$ the solution is constant $S = S_m$ at $0 \le x \le x_m(\tau)$ and decrease at $x \ge x_m(\tau)$, where $x = x_m(\tau)$ is a continuous increasing function and

$$x_m(t_0) = 0\,. \qquad (29)$$

Denote the line Γ separating subzones U_1 and U_2 of constant and non-constant solution

$$\Gamma = \{(x_m(\tau),\ \tau),\ \tau \ge t_0\}\,.$$

Solution in the domain U_1 is

$$S(x,\tau)=S_m,\quad C(x,\tau)=1. \qquad (30)$$

Note that Riemann invariant (24) holds in subzone U_1.

On line Γ

$$S(x,\tau)\big|_{x=x_m(\tau)}=S_m. \qquad (31)$$

The function $x=x_m(\tau)$ is determined from the continuity of the solution on Γ. Consider the tangent vector $l=\left(x_m'(\tau),-1\right)$ to the line Γ at the point $\left(x_m(\tau),\tau\right)$. From Eq. (31)

$$\left.\frac{\partial S}{\partial l}\right|_{(x_m(\tau),\tau)}=0. \qquad (32)$$

Calculate the derivative (32) using Eq. (2) and (27)

$$\left.\frac{\partial S}{\partial l}\right|_{(x_m(\tau),\tau)}=\frac{\partial S}{\partial x}x_m'(\tau)+\frac{\partial S}{\partial \tau}= \\ =-S\Lambda(S)x_m'(\tau)+\Lambda(S)C. \qquad (33)$$

Equate right parts of Eqs. (32) and (33) and reduce the multiplier $\Lambda(S)$:

$$-Sx_m'(\tau)+C=0 \qquad (34)$$

Substitution of solution (30) in Eq. (34) yields

$$x_m'(\tau)=\frac{1}{S_m}. \qquad (35)$$

Solution to Eq. (35) with initial condition (29) is

$$x_m(\tau)=\frac{\tau-t_0}{S_m}. \qquad (36)$$

In subzone U_2 the solution to Eq. (19) with condition (31) is

$$\int_S^{S_m}\frac{dS}{S\Lambda(S)}=x-x_m(\tau). \qquad (37)$$

It follows from Eq. (37) that the solution in subzone U_2 has the form of the traveling wave

$$S=\tilde{S}\left(x-x_m(\tau)\right). \qquad (38)$$

According to formula (29) at $\tau=\tau_0$ the formula (37) coincides with (26). Formula (38) takes the form

$$S=S_1\left(x-x_m(\tau)\right). \qquad (39)$$

Differentiate Eq. (37) in τ:

$$-\frac{\partial S/\partial\tau}{S\Lambda(S)}=-x_m'(\tau). \qquad (40)$$

Substitution of Eq. (2) and (39) into Eq. (40) yields Riemann invariant (24) in subzone U_2.

$$C=\tilde{C}\left(x-x_m(\tau)\right)=\frac{S_1\left(x-x_m(\tau)\right)}{S_m}. \qquad (41)$$

In Cartesian variables the formula (36) takes the form

$$x=\frac{t-x-t_0}{S_m}. \qquad (42)$$

Express *x* from (42):

$$x_\Gamma(t)=\frac{t-t_0}{S_m+1}. \qquad (43)$$

The boundary Γ between subzones U_1 and U_2 is a straight line (43) starting from *OX* axis at $t=t_0$.

The solution (39) takes the form

$$S=S_1\left(x-x_\Gamma(t)\right). \qquad (44)$$

In Cartesian variables, the formula (41) takes the form

$$C\left(x - x_{\Gamma}(t)\right) = \frac{S_1\left(x - x_{\Gamma}(t)\right)}{S_m}. \qquad (45)$$

The various zones into which the filtration domain is divided and the type of solutions in these zones are shown in Figure 1.

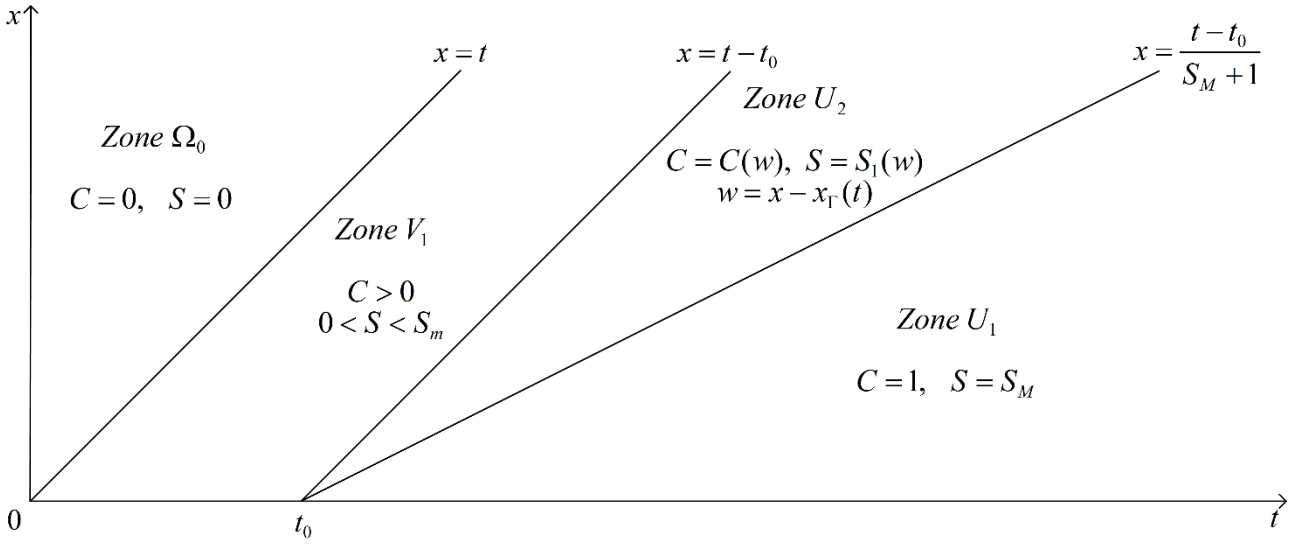


Figure 1. Solution diagram of the model with finite filtration time

4. EXAMPLE

As an example, consider an unlimited filtration function. Let

$$\Lambda(S) = \begin{cases} \dfrac{1}{\sqrt{1-S}}, & 0 \le S < 1; \\ 0, & S \ge 1. \end{cases}$$

Formula (7) takes the form

$$\int_0^{S_0} \sqrt{1-S}\, dS = t. \qquad (46)$$

Integration of (46) gives the solution in explicit form:

$$S_0 = 1 - \left(1 - \frac{3}{2}t\right)^{2/3}. \qquad (47)$$

From formula (47)

$$t_0 = \frac{2}{3}.$$

Finally, the solution at the inlet $x = 0$ is

$$S_0(t) = \begin{cases} 1 - \left(1 - \dfrac{3}{2}t\right)^{2/3}, & 0 \le t < \dfrac{2}{3}; \\ 1, & t \ge \dfrac{2}{3}. \end{cases} \qquad (48)$$

The deposit concentration increases from zero to 1 at finite time $t_0 = 2/3$ and becomes constant after reaching the upper limit $S_m = 1$. The graph of deposit concentration versus time at the inlet of the porous medium is shown in Fig. 2.

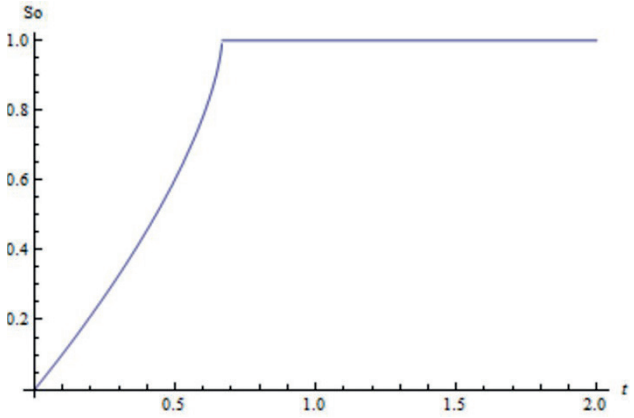


Figure 2. Retained particles concentration at the inlet of the porous medium

At the concentration front the suspended concentration is given by the formula

$$C^{-}(x) = e^{-x}.$$

Calculate the solution in zone $V_1 = \{x \geq 0,\ 0 \leq t - x \leq \frac{2}{3}\}$. The formula (26) takes the form

$$\int_{S(x,\tau)}^{S_0(\tau)} \frac{\sqrt{1-S}\,dS}{S} = x. \qquad (49)$$

Calculate the integral in Eq. (49):

$$2\left(\sqrt{1-S_0} - \sqrt{1-S}\right) - \\ -\ln\frac{\left(1+\sqrt{1-S_0}\right)}{\left(1-\sqrt{1-S_0}\right)}\frac{\left(1-\sqrt{1-S}\right)}{\left(1+\sqrt{1-S}\right)} = x. \qquad (50)$$

Substitute formula (47) into Eq. (50):

$$2\left(\sqrt[3]{1-1.5\tau} - \sqrt{1-S}\right) - \\ -\ln\frac{\left(1+\sqrt[3]{1-1.5\tau}\right)}{\left(1-\sqrt[3]{1-1.5\tau}\right)}\frac{\left(1-\sqrt{1-S}\right)}{\left(1+\sqrt{1-S}\right)} = x. \qquad (51)$$

In Cartesian variables the Eq. (51) takes the form

$$2\left(\sqrt[3]{1-1.5(t-x)} - \sqrt{1-S}\right) - \\ -\ln\frac{\left(1+\sqrt[3]{1-1.5(t-x)}\right)}{\left(1-\sqrt[3]{1-1.5(t-x)}\right)}\frac{\left(1-\sqrt{1-S}\right)}{\left(1+\sqrt{1-S}\right)} = x. \qquad (52)$$

Eq. (52) and the Riemann invariant (24) are the solution in zone V_1.

In the domain $V_2 = \{x \geq 0,\ t > x + \frac{2}{3}\}$ the boundary of zones U_1 and U_2 is

$$x_\Gamma(t) = \frac{t}{2} - \frac{1}{3}.$$

In subzone $U_2 = \left\{t \geq \frac{2}{3},\ 0 \leq x \leq \frac{t}{2} - \frac{1}{3}\right\}$ implicit formula (37) for the solution takes the form

$$\int_S^1 \frac{\sqrt{1-S}\,dS}{S} = x - \tau + \frac{2}{3}. \qquad (53)$$

Calculate the integral in the left side of Eq. (53) similarly to formula (49):

$$\int_S^1 \frac{\sqrt{1-S}\,dS}{S} = 2\left(z - \frac{1}{2}\ln\frac{1+z}{1-z}\right)\Bigg|_{\sqrt{1-S}}^{0} = \\ = -2\sqrt{1-S} - \ln\frac{\left(1-\sqrt{1-S}\right)}{\left(1+\sqrt{1-S}\right)}.$$

Eq. (53) takes the form

$$-2\sqrt{1-S} - \ln\frac{\left(1-\sqrt{1-S}\right)}{\left(1+\sqrt{1-S}\right)} = x - \tau + \frac{2}{3}. \qquad (54)$$

In Cartesian variables, the implicit solution (54) takes the form of the travelling wave

$$-2\sqrt{1-S} - \ln\frac{\left(1-\sqrt{1-S}\right)}{\left(1+\sqrt{1-S}\right)} = 2x - t + \frac{2}{3}. \qquad (55)$$

The travelling wave $S(w)$, $w = 2x - t + 2/3$ is shown in Figure 3.

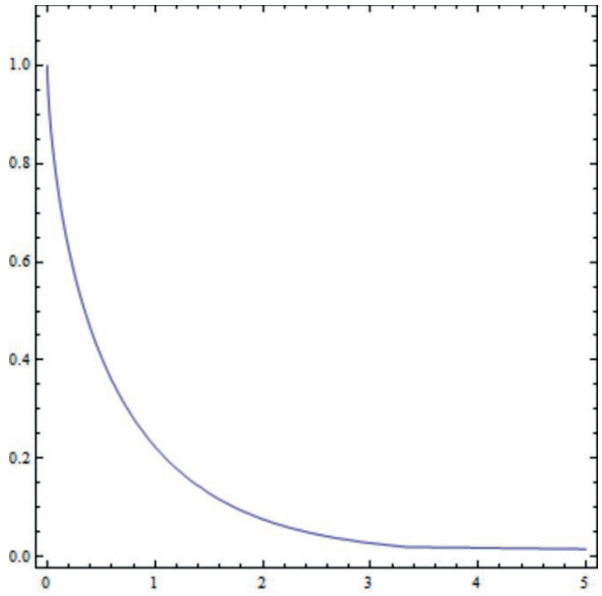


Figure 3. Solution in travelling wave zone.

In subzone $U_1 = \left\{ t \geq \frac{2}{3},\ \frac{t}{2} - \frac{1}{3} \leq x \leq t - \frac{2}{3} \right\}$ concentrations reach maximum values

$$S = 1, \quad C = 1. \tag{56}$$

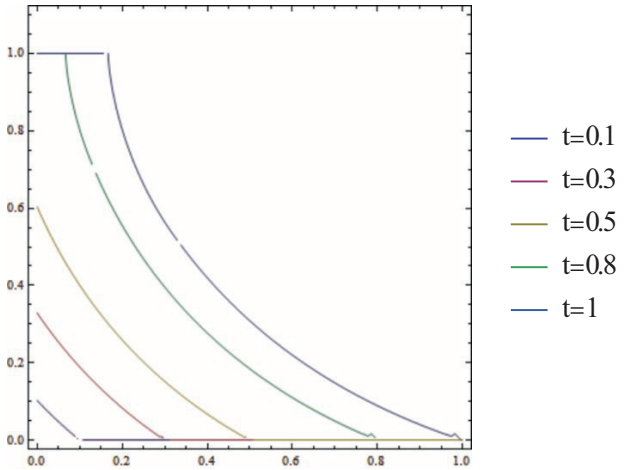


Figure 4. Global retained concentration profiles

A global exact solution to the model is obtained by combining local solutions (52), (55) and (56) in three zones. The filtration domain is the time interval during which the concentration of deposited particles decreases from 1 to zero. The global deposit concentration profiles - the dependence of retained particles concentrations on spatial coordinate x at different times are shown in Figure 4 [25-28].

5. DISCUSSION

In particle transport problems, suspended and retained particle concentrations grow with time to limiting values. Filtration time determines the time dependence of suspended and retained particle concentrations. If time filtration is infinite, then the retained concentration is given by a single equation, the solution is smooth everywhere. At a finite filtration time, the domain of non-zero concentrations (behind the front) is divided into three zones: the standard filtration zone, the filtration zone with solutions in the form of traveling waves, and the zone of upper limiting values of suspended and retained particle concentrations. In the first two zones, the retained concentrations are given by different equations, their boundary is a straight line, which is not a characteristic of the original system. On this line, the derivatives of the solution are discontinuous.

One-dimensional models are rarely encountered in practical problems. However, the presence of exact solutions of one-dimensional models can significantly speed up numerical calculations of two-dimensional and three-dimensional problems. For this, the flow of a suspension in a porous medium must be represented as a bundle of one-dimensional jets - trajectories of particle transport in pores [29].

We studied the problem of particle transport in a homogeneous porous medium, in which at the initial moment there were no suspended and retained particles. The initial presence of particles in a porous medium complicates the problem, since the porous medium becomes inhomogeneous. Such a model will be considered separately.

Exact solutions to the direct filtration problem are used to solve the inverse problem - finding

the filtration function for known suspended concentrations at the inlet and outlet [30-32].

6. CONCLUSIONS

The study of the particle model transport with non - blocking filtration function leads to the following conclusions:

- The filtration time is finite, i.e. the retained particles concentration in the porous medium increases from zero to the limit value in a finite time.
- At each point in the porous medium, the filtration time depends linearly on the distance to the inlet.
- An exact solution to the model in explicit form is obtained. The solution has a different form in the standard filtration zone and in the traveling wave zone.
- The concentrations of suspended and retained particles are related by a simple algebraic relationship - the Riemann invariant.

REFERENCES

1. **Zhou Z., Zang H., Wang S., Du X., Ma D., Zhang J.** Filtration Behavior of Cement-Based Grout in Porous Media // Transport in Porous Media, 2018, vol. 125, pp. 435–463.
2. **Xie B., Cheng H., Wang X., Yao Z., Rong C., Zhou R., Zhang L., Guo L., Yu H., Xiong, W., Xiang X.** Theoretical Research on Diffusion Radius of Cement-Based Materials Considering the Pore Characteristics of Porous Media // Materials, 2022, vol. 15, 7763.
3. **Zhu G., Zhang Q., Liu R., Bai J., Li W., Feng X.** Experimental and Numerical Study on the Permeation Grouting Diffusion Mechanism considering Filtration Effects // Geofluids, 2021, 6613990.
4. **Wang X., Cheng H., Yao Z., Rong C., Huang X., Liu X.** Theoretical Research on Sand Penetration Grouting Based on Cylindrical Diffusion Model of Tortuous Tubes // Water, 2022, vol. 14, 1028.
5. **Faramarzi L., Rasti A., Abtahi S.M.** An Experimental Study of the Effect of Cement and Chemical Grouting on the Improvement of the Mechanical and Hydraulic Properties of Alluvial Formations // Journal of Construction & Building Materials, 2016, vol. 126, pp. 32–43.
6. **Tsuji M., Kobayashi S., Mikake S., Sato T., Matsui H.** Post-Grouting Experiences for Reducing Groundwater Inflow at 500 m Depth of the Mizunami Underground Research Laboratory, Japan // Procedia Engineering, 2017, vol. 191, pp. 543–550.
7. **Rabinovich A., Bedrikovetsky P., Tartakovsky D.** Analytical model for gravity segregation of horizontal multiphase flow in porous media // Physics of Fluids, 2020, vol. 32 (4), pp. 1–15.
8. **Kuzmina L., Osipov Y.** Deep bed filtration with multiple pore-blocking mechanisms // MATEC Web of Conferences, 2018, vol. 196, 04003, pp. 1–6.
9. **Kuzmina L.I., Osipov Y.V., Sosedka M.G.** Modelling of filtration with two capture mechanisms // Lecture Notes in Civil Engineering, 2023, vol. 282, pp. 21–31.
10. **Santos A., Bedrikovetsky P.** Size exclusion during particle suspension transport in porous media: stochastic and averaged equations // Computational and Applied Mathematics, 2004, vol. 23(2-3), pp. 259–284.
11. **Altoe J.E., Bedrikovetsky P., Siqueira A.G., de Souza A.L.S., Shecaira F.S.** Correction of basic equations for deep bed filtration with dispersion // Journal of Petroleum Science and Engineering, 2006, vol. 51, pp. 68–84.
12. **Iritani, E., Katagiri, N.** Developments of Blocking Filtration Model in Membrane Filtration // KONA Powder and Particle Journal, 2016, vol. 33, pp. 179–202.
13. **Tien, C., Ramarao, B., Yasarla, R.** A blocking model of membrane filtration //

Chemical Engineering Science, 2014, vol. 111, pp. 421–431.
14. **Kuzmina L.I., Osipov Yu.V.** Determining the Lengmur coefficient of the filtration problem // International Journal for Computational Civil and Structural Engineering, 2020, vol. 16(4), pp. 48–54.
15. **Bedrikovetsky, P.** Upscaling of stochastic micro model for suspension transport in porous media // Transport in Porous Media, 2008, vol. 75, pp. 335–369.
16. **Kuzmina L.I., Osipov Yu.V., Vetoshkin N.V.** Calculation of long-term filtration in a porous medium. // International Journal for Computational Civil and Structural Engineering, 2018, vol. 14(1), pp. 92-101.
17. **Kuzmina L., Osipov Y.** Long-term filtration of particles in a porous medium // BIO Web of Conferences, 2024, vol. 107, 03003.
18. **Kuzmina L.I., Nazaikinskii V.E., Osipov Y.V.** On a Deep Bed Filtration Problem with Finite Blocking Time // Russian Journal of Mathematical Physics, 2019, vol. 26(1), pp. 130–134.
19. **Herzig J.P., Leclerc D.M., Le Goff P.** Flow of Suspensions Through Porous Media – Application to Deep Filtration // Journal of Industrial & Engineering Chemistry, 1970, vol. 62(8), pp. 8–35.
20. **Logan J.D.** Transport Modeling in Hydrogeochemical Systems, Interdisciplinary Applied Mathematics, vol 15, Springer, New York, 2001.
21. **Kuzmina L., Osipov Y.** Traveling Wave Solution to Filtration Model in Porous Medium // Advances in Transdisciplinary Engineering, 2023, vol. 43, pp. 450–455.
22. **Courant R., Hilbert D.** Partial differential equations, Reprint of the 1962 Original, Edited, Wiley-InterScience, New York, 1989.
23. **Polyanin A.D., Manzhirov A.V.** Handbook of mathematics for engineers and scientists, Chapman & Hall/CRC, Boca Raton, FL, 2006.
24. **Vyazmina E.A., Bedrikovetskii P.G., Polyanin A.D.** New classes of exact solutions to nonlinear sets of equations in the theory of filtration and convective mass transfer // Theoretical Foundations of Chemical Engineering, 2007, vol. 41(5), pp. 556–564.
25. **Wang D., Ge L., He J., Zhang W., Jaisi D.P., Zhou D.** Hyperexponential and nonmonotonic retention of polyvinylpyrrolidone-coated silver nanoparticles in an Ultisol // Journal of Contaminant Hydrology, 2014; vol. 164, pp. 35–48.
26. **Leisi R., Bieri J., Roth N.J., Ros C.** Determination of parvovirus retention profiles in virus filter membranes using laser scanning microscopy // Journal of Membrane Science, 2020, vol. 603, 118012.
27. **Yuan H., Shapiro A.A.** A mathematical model for non-monotonic deposition profiles in deep bed filtration systems // Chemical Engineering Journal, 2011, vol. 166, pp. 105–115.
28. **Malgaresi G., Collins B., Alvaro P., Bedrikovetsky P.** Explaining non-monotonic retention profiles during flow of size-distributed colloids // Chemical Engineering Journal, 2019, vol. 375, 121984.
29. **Bedrikovetsky, P.** Mathematical theory of oil and gas recovery: with applications to ex-USSR oil and gas fields, Springer Science & Business Media, Des Moines, 2013.
30. **Alvarez A.C., Bedrikovetsky P.G., Hime G., Marchesin A.O., Marchesin D., Rodrigues J.R.** A fast inverse solver for the filtration function for flow of water with particles in porous media // Inverse Problems, 2006, vol. 22, pp. 69–88.
31. **Alvarez A.C., Hime G., Marchesin D., Bedrikovetsky P.G.** The inverse problem of determining the filtration function and permeability reduction in flow of water with

particles in porous media // Transport in Porous Media, 2007, vol. 70(1), pp. 43–62.
32. **Kuzmina L., Osipov Y.** Inverse Filtration Problem of Multiparticle Suspension // Advances in Transdisciplinary Engineering, 2024, vol. 62, , pp. 2-11.
33. **Du X., Wong G.K.** Predicting filtration coefficient and formation damage coefficient for particle flow in porous media using machine learning // Results in Engineering, 2025, vol. 25, 104545.

СПИСОК ЛИТЕРАТУРЫ

1. **Zhou Z., Zang H., Wang S., Du X., Ma D., Zhang J.** Filtration Behavior of Cement-Based Grout in Porous Media // Transport in Porous Media, 2018, vol. 125, pp. 435–463.
2. **Xie B., Cheng H., Wang X., Yao Z., Rong C., Zhou R., Zhang L., Guo L., Yu H., Xiong, W., Xiang X.** Theoretical Research on Diffusion Radius of Cement-Based Materials Considering the Pore Characteristics of Porous Media // Materials, 2022, vol. 15, 7763.
3. **Zhu G., Zhang Q., Liu R., Bai J., Li W., Feng X.** Experimental and Numerical Study on the Permeation Grouting Diffusion Mechanism considering Filtration Effects // Geofluids, 2021, 6613990.
4. **Wang X., Cheng H., Yao Z., Rong C., Huang X., Liu X.** Theoretical Research on Sand Penetration Grouting Based on Cylindrical Diffusion Model of Tortuous Tubes // Water, 2022, vol. 14, 1028.
5. **Faramarzi L., Rasti A., Abtahi S.M.** An Experimental Study of the Effect of Cement and Chemical Grouting on the Improvement of the Mechanical and Hydraulic Properties of Alluvial Formations // Journal of Construction & Building Materials, 2016, vol. 126, pp. 32–43.
6. **Tsuji M., Kobayashi S., Mikake S., Sato T., Matsui H.** Post-Grouting Experiences for Reducing Groundwater Inflow at 500 m Depth of the Mizunami Underground Research Laboratory, Japan // Procedia Engineering, 2017, vol. 191, pp. 543–550.
7. **Rabinovich A., Bedrikovetsky P., Tartakovsky D.** Analytical model for gravity segregation of horizontal multiphase flow in porous media // Physics of Fluids, 2020, vol. 32 (4), pp. 1–15.
8. **Kuzmina L., Osipov Y.** Deep bed filtration with multiple pore-blocking mechanisms // MATEC Web of Conferences, 2018, vol. 196, 04003, pp. 1–6.
9. **Kuzmina L.I., Osipov Y.V., Sosedka M.G.** Modelling of filtration with two capture mechanisms // Lecture Notes in Civil Engineering, 2023, vol. 282, pp. 21–31.
10. **Santos A., Bedrikovetsky P.** Size exclusion during particle suspension transport in porous media: stochastic and averaged equations // Computational and Applied Mathematics, 2004, vol. 23(2-3), pp. 259–284.
11. **Altoe J.E., Bedrikovetsky P., Siqueira A.G., de Souza A.L.S., Shecaira F.S.** Correction of basic equations for deep bed filtration with dispersion // Journal of Petroleum Science and Engineering, 2006, vol. 51, pp. 68–84.
12. **Iritani, E., Katagiri, N.** Developments of Blocking Filtration Model in Membrane Filtration // KONA Powder and Particle Journal, 2016, vol. 33, pp. 179–202.
13. **Tien, C., Ramarao, B., Yasarla, R.** A blocking model of membrane filtration // Chemical Engineering Science, 2014, vol. 111, pp. 421–431.
14. **Kuzmina L.I., Osipov Yu.V.** Determining the Lengmur coefficient of the filtration problem // International Journal for Computational Civil and Structural Engineering, 2020, vol. 16(4), pp. 48–54.
15. **Bedrikovetsky, P.** Upscaling of stochastic micro model for suspension transport in porous media // Transport in Porous Media, 2008, vol. 75, pp. 335–369.
16. **Kuzmina L.I., Osipov Yu.V., Vetoshkin N.V.** Calculation of long-term filtration in a porous medium. // International Journal for

Computational Civil and Structural Engineering, 2018, vol. 14(1), pp. 92-101.
17. **Kuzmina L., Osipov Y.** Long-term filtration of particles in a porous medium // BIO Web of Conferences, 2024, vol. 107, 03003.
18. **Kuzmina L.I., Nazaikinskii V.E., Osipov Y.V.** On a Deep Bed Filtration Problem with Finite Blocking Time // Russian Journal of Mathematical Physics, 2019, vol. 26(1), pp. 130–134.
19. **Herzig J.P., Leclerc D.M., Le Goff P.** Flow of Suspensions Through Porous Media – Application to Deep Filtration // Journal of Industrial & Engineering Chemistry, 1970, vol. 62(8), pp. 8–35.
20. **Logan J.D.** Transport Modeling in Hydrogeochemical Systems, Interdisciplinary Applied Mathematics, vol 15, Springer, New York, 2001.
21. **Kuzmina L., Osipov Y.** Traveling Wave Solution to Filtration Model in Porous Medium // Advances in Transdisciplinary Engineering, 2023, vol. 43, pp. 450–455.
22. **Courant R., Hilbert D.** Partial differential equations, Reprint of the 1962 Original, Edited, Wiley-InterScience, New York, 1989.
23. **Polyanin A.D., Manzhirov A.V.** Handbook of mathematics for engineers and scientists, Chapman & Hall/CRC, Boca Raton, FL, 2006.
24. **Vyazmina E.A., Bedrikovetskii P.G., Polyanin A.D.** New classes of exact solutions to nonlinear sets of equations in the theory of filtration and convective mass transfer // Theoretical Foundations of Chemical Engineering, 2007, vol. 41(5), pp. 556–564.
25. **Wang D., Ge L., He J., Zhang W., Jaisi D.P., Zhou D.** Hyperexponential and nonmonotonic retention of polyvinylpyrrolidone-coated silver nanoparticles in an Ultisol // Journal of Contaminant Hydrology, 2014; vol. 164, pp. 35–48.
26. **Leisi R., Bieri J., Roth N.J., Ros C.** Determination of parvovirus retention profiles in virus filter membranes using laser scanning microscopy // Journal of Membrane Science, 2020, vol. 603, 118012.
27. **Yuan H., Shapiro A.A.** A mathematical model for non-monotonic deposition profiles in deep bed filtration systems // Chemical Engineering Journal, 2011, vol. 166, pp. 105–115.
28. **Malgaresi G., Collins B., Alvaro P., Bedrikovetsky P.** Explaining non-monotonic retention profiles during flow of size-distributed colloids // Chemical Engineering Journal, 2019, vol. 375, 121984.
29. **Bedrikovetsky, P.** Mathematical theory of oil and gas recovery: with applications to ex-USSR oil and gas fields, Springer Science & Business Media, Des Moines, 2013.
30. **Alvarez A.C., Bedrikovetsky P.G., Hime G., Marchesin A.O., Marchesin D., Rodrigues J.R.** A fast inverse solver for the filtration function for flow of water with particles in porous media // Inverse Problems, 2006, vol. 22, pp. 69–88.
31. **Alvarez A.C., Hime G., Marchesin D., Bedrikovetsky P.G.** The inverse problem of determining the filtration function and permeability reduction in flow of water with particles in porous media // Transport in Porous Media, 2007, vol. 70(1), pp. 43–62.
32. **Kuzmina L., Osipov Y.** Inverse Filtration Problem of Multiparticle Suspension // Advances in Transdisciplinary Engineering, 2024, vol. 62, , pp. 2-11.
33. **Du X., Wong G.K.** Predicting filtration coefficient and formation damage coefficient for particle flow in porous media using machine learning // Results in Engineering, 2025, vol. 25, 104545.

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