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SOLUTION TO FILTRATION PROBLEM WITH LINEAR POROSITY FUNCTION

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Abstract: Injection techniques are an effective tool for road construction, repair and maintenance. They allow solving a wide range of problems related to soil reinforcement, pavement rehabilitation and drainage improvement, ensuring durability and reliability of road infrastructure. Injection of various compositions (cement, polymer, silicatized) into weak soils (peat, clay, silt) can increase their bearing capacity, reduce deformations and prevent roadway subsidence. In addition, injection compositions fill voids and cracks formed as a result of erosion, soil erosion or vibration from traffic, which prevents further destruction of the road base. The study of depth filtration in injectable soil reinforcement techniques is a key aspect for optimizing and improving the effectiveness of these technologies. It helps to understand how injection solutions penetrate the soil, how they interact with the soil, and how this process can be controlled to achieve the best results. In this paper, the problem of suspension filtration in a heterogeneous porous medium with a linear porosity function varying as a function of spatial coordinate is solved. The exact solution to the problem is found. Asymptotic solutions of the concentrations of suspended and retained particles for the case of insignificant porosity measurement are obtained.

Keywords: deep bed filtration, suspended and retained particles, suspension, porous medium, concentration function, asymptotic solution, porosity function

РЕШЕНИЕ ЗАДАЧИ ФИЛЬТРАЦИИ С ЛИНЕЙНОЙ ФУНКЦИЕЙ ПОРИСТОСТИ

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Аннотация: Инъекционные методы являются эффективным инструментом для строительства, ремонта и обслуживания автомобильных дорог. Они позволяют решать широкий спектр задач, связанных с укреплением грунта, восстановлением покрытия и улучшением дренажных свойств, обеспечивая долговечность и надежность дорожной инфраструктуры. Инъекции различных составов (цементных, полимерных, силикатизированных) в слабые грунты (торф, глина, ил) позволяют повысить их несущую способность, снизить деформации и предотвратить просадки дорожного полотна. Помимо этого, инъекционные составы заполняют пустоты и трещины, образовавшиеся в результате эрозии, размывания грунта или вибрации от транспорта, что предотвращает дальнейшее разрушение основания дороги. Изучение глубинной фильтрации при инъекционных методах укрепления грунтов является ключевым аспектом для оптимизации и повышения эффективности этих технологий. Оно помогает понять, как инъекционные растворы проникают в грунт, как происходит их взаимодействие с грунтом, и как можно контролировать этот процесс для достижения наилучших результатов. В работе решается задача фильтрации суспензии в неоднородной пористой среде с линейной функцией пористости, изменяющейся в зависимости от пространственной координаты. Найдено точное решение задания. Получены асимптотические решения концентраций взвешенных и осажденных частиц для случая незначительного измерения пористости.

Ключевые слова: глубинная фильтрация, взвешенные и осажденные частицы, суспензия, неоднородная пористая среда, асимптотическое решение, функция пористости

INTRODUCTION

Injection reinforcement of roadbeds is a method of repairing and reinforcing an existing road surface by injecting special liquid compositions (injection materials) into the space under the roadbed or into its structure. It is an effective way to restore the bearing capacity of the road base, fill voids, stabilize cracks and prevent further deterioration [1-3].

The injected material is injected under pressure through small holes (injectors) drilled into the road surface (fig. 1). It fills voids, cracks and pores in the road base, strengthening its structure and increasing its bearing capacity. After curing, the injection material creates a strong bond between the road surface layers and the subgrade, preventing further deterioration.



Figure 1. Injection of injectable material

Currently, there are several types of injection materials such as cement mortars, polymeric materials, microcements, and composite materials [4-7]. Each of the above types has its own disadvantages and advantages. Thus, cement mortars have high strength and relatively low cost, but have limited penetration ability into small cracks and are prone to shrinkage and cracking [8]. Polymeric materials (polyurethanes, epoxy resins) are characterized by high penetration ability, elasticity, resistance to water and chemicals, the disadvantages include their high cost and complexity of application [9-11].

The study of solid suspension filtration in porous media is a complex problem. Road pavement and road base are porous media (asphalt, crushed stone, sand, soil). The injection material must penetrate these pores and cracks to strengthen the structure. During the filtration process, larger suspension particles can become trapped in the porous medium, while smaller particles can continue to move through. This causes the composition of the injected material to change as it spreads deep into the roadway. The change in composition can affect the strength, elasticity and other characteristics of the cured material, which ultimately affects the durability of the reinforced section [12]. Also fine particles of injection material and fragments that have separated from the pavement can clog microcracks and pores, which reduces the water permeability of the road surface. This can lead to accumulation of water inside the road structure, which in winter will cause deterioration due to freezing and thawing.

Underestimating the influence of seepage can lead to ineffective strengthening, shortening the service life of the pavement and even its destruction. It is therefore necessary to pay attention to all aspects of the interaction between the injection material and the porous structure of the pavement when designing and carrying out injection reinforcement works.

The concept of porosity is one of the most important in the study of deep filtration of suspension in porous media [13-15]. This characteristic determines many of its properties of porous media, including permeability, specific surface area, thermal conductivity, strength and sorption capacity. It characterizes the proportion of the volume occupied by pores (voids) in relation to the total volume of the material.

Homogeneous porosity is characterized by the uniform distribution of pores throughout the volume of the material, as well as the closeness of their size and shape. This means that the porosity (pore volume fraction) is almost the same at any point in the material, and that there are no significant differences in the size and shape of individual pores. Many articles are devoted to depth filtration of suspension in a homogeneous porous medium [16-20].

The study of filtration of porous medium with inhomogeneous porosity is of great interest [21-24]. Variable porosity is characterized by non-uniform distribution of pores, for example, there may exist regions with high and low porosity. In addition, the pore diameter can vary over a wide range. Both micropores and macropores may be present. There can also be differences in pore shape (e.g. prismatic, cylindrical).

In this paper the problem of depth filtration in a heterogeneous porous medium for the case of variable porosity function, linearly varying depending on the spatial coordinate x is solved. Exact solutions for the concentrations of suspended and retained particles, as well as asymptotic solutions for the case of small changes in porosity are obtained.

MATHEMATICAL MODEL

Let consider a dimensionless problem of depth filtration in a medium with variable porosity with unknown concentration functions of suspended particles $C(x,t)$ and retained particles $S(x,t)$:

$$\varphi(x)\frac{\partial C}{\partial t}+\frac{\partial C}{\partial x}+\frac{\partial S}{\partial t}=0, \qquad (1)$$

$$\frac{\partial S}{\partial t}=\Lambda(S)C. \qquad (2)$$

Here $\varphi(x)$ is a variable porosity function, $\Lambda(S)$ is a filtration function. The problem is solved in the domain $\begin{cases}0\le x\le 1,\\ t\ge 0.\end{cases}$

At the inlet of the filter at $x=0$ the concentration of suspended particles is equal to one, i.e.

$$C(0,t)=1, \qquad (3)$$

at the initial moment of time $t=0$ the concentrations of suspended and retained particles are equal to zero:

$$C(x,0)=0;\quad S(x,0)=0. \qquad (4)$$

Let the filtration function be of the form $\Lambda(S)=S_m-S$, where S_m is the maximum value of sediment, and the porosity function be of the form $\varphi(x)=a+bx$. Then equations (1) and (2) take the form

$$(a+bx)\frac{\partial C}{\partial t}+\frac{\partial C}{\partial x}+\frac{\partial S}{\partial t}=0, \qquad (5)$$

$$\frac{\partial S}{\partial t}=(S_m-S)C. \qquad (6)$$

The curve $t=ax+b\frac{x^2}{2}$, called the concentration front, divides the domain $\begin{cases}0\le x\le 1,\\ t\ge 0\end{cases}$ into two subdomains. Before the front at $t<ax+b\frac{x^2}{2}$, the concentrations of both types of particles are zero. Behind the front, the concentrations of suspended and retained particles are positive values. At the front the concentrations are $C\left(x,ax+b\frac{x^2}{2}\right)>0$ and $S\left(x,ax+b\frac{x^2}{2}\right)=0$.

EXACT SOLUTION TO THE PROBLEM

Taking into account that at the filter inlet $C(0,t)=1$, we obtain from equation (6)

$$\int_0^{S(0,t)}\frac{dS}{S_m-S}=\int_0^t dt,$$

from which we obtain

$$S(0,t)=S_m\left(1-e^{-t}\right).$$

Then from equation (6) we have $C = \frac{\partial S / \partial t}{S_m - S}$, let us substitute the obtained expression into equation (5):

$$(a+bx)\frac{\partial}{\partial t}\left(\frac{\partial S/\partial t}{S_m - S}\right)+\frac{\partial}{\partial x}\left(\frac{\partial S/\partial t}{S_m - S}\right)+\frac{\partial S}{\partial t}=0.$$

Taking into account that the obtained equality is equal to the equation

$$\frac{\partial}{\partial t}\left((a+bx)\frac{\partial S/\partial t}{S_m - S}\right)+\frac{\partial}{\partial t}\left(\frac{\partial S/\partial x}{S_m - S}\right)+\frac{\partial S}{\partial t}=0,$$

we obtain

$$(a+bx)\frac{\partial S/\partial t}{S_m - S}+\frac{\partial S/\partial x}{S_m - S}+\frac{\partial S}{\partial t}=0.$$

Moving on to the new variables $\begin{cases}\tau = t - ax - b\dfrac{x^2}{2},\\ x = x,\end{cases}$ we have

$$\frac{\partial S}{\partial x}+\left(S_m - S\right)S=0.$$

Solving the obtained equation with separating variables and taking into account the fact that $S_0(0,t)=S_0(0,\tau)$, we obtain the exact solution behind the concentration front

$$S(x,\tau)=\frac{S_m\left(e^{\tau}-1\right)}{e^{S_m x}+e^{\tau}-1}.$$

Then

$$S(x,t)=\frac{S_m\left(e^{t-ax-b\frac{x^2}{2}}-1\right)}{e^{S_m x}+e^{t-ax-b\frac{x^2}{2}}-1} \qquad (7)$$

and

$$C(x,t)=\frac{e^{t-ax-b\frac{x^2}{2}}}{e^{S_m x}+e^{t-ax-b\frac{x^2}{2}}-1}. \qquad (8)$$

ASYMPTOTIC SOLUTION FOR SMALL CHANGES IN POROSITY

Let us consider the case of small porosity variation, i.e. let $\varphi(x)=a+\varepsilon x$, where ε takes a small value. Let us decompose the concentrations of suspended and precipitated particles, as well as the filtration function by a small parameter

$$S(x,t)=S_0(x,t)+\varepsilon S_1(x,t)+\frac{\varepsilon^2}{2}S_2(x,t)+.., \quad (9)$$

$$C(x,t)=C_0(x,t)+\varepsilon C_1(x,t)+\frac{\varepsilon^2}{2}C_2(x,t)+..., (10)$$

$$\Lambda(S)=S_m - S_0(x,t)-\varepsilon S_1(x,t)-\frac{\varepsilon^2}{2}S_2(x,t)+...(11)$$

Substituting expressions (9) and (11) into (5) and grouping the summands by powers of ε, we obtain:

at ε^0: $$a\frac{\partial S_0(x,t)}{\partial t}+\frac{\partial S_0(x,t)}{\partial x}+\left(S_m - S_0(x,t)\right)S_0(x,t)=0; \qquad (12)$$

at ε^1: $$x\frac{\partial S_0(x,t)}{\partial t}+a\frac{\partial S_1(x,t)}{\partial t}+\frac{\partial S_1(x,t)}{\partial x}+\left(S_m - 2S_0(x,t)\right)S_1(x,t)=0; \qquad (13)$$

at ε^2: $$x\frac{\partial S_1(x,t)}{\partial t}+\frac{a}{2}\frac{\partial S_2(x,t)}{\partial t}+\frac{1}{2}\cdot\frac{\partial S_2(x,t)}{\partial x}+\frac{1}{2}\left(S_m - 2S_0(x,t)\right)S_2(x,t)-S_1^2(x,t)=0. \qquad (14)$$

To find $S_1(x,t)$, $S_2(x,t)$ and $S_3(x,t)$ in equations (12)-(14) we will make a transition to the variables $\begin{cases}\tau = t - ax, \\ x = x.\end{cases}$ Equation (12) is transformed to the form

$$\frac{\partial S_0(x,\tau)}{\partial x} + \left(S_m - S_0(x,\tau)\right) S_0(x,\tau) = 0,$$

Then

$$S_0(x,\tau) = \frac{S_m S_0(0,\tau)}{S_m e^{S_m x} - S_0(0,\tau) e^{S_m x} + S_0(0,\tau)}.$$

Passing to the old variables and taking into account that $S_0(0,\tau) = S_0(0,t) = S_m\left(1 - e^{-t}\right)$, we obtain

$$S_0(x,t) = \frac{S_m(e^{t-ax} - 1)}{e^{S_m x} + e^{t-ax} - 1}.$$

The transition to new variables in equation (13) leads it to the linear equation

$$\frac{\partial S_1(x,\tau)}{\partial x} + \left(S_m - 2S_0(x,\tau)\right) S_1(x,\tau) = $$
$$= -xS_m \frac{e^{\tau + S_m x}}{\left(e^{S_m x} + e^{\tau} - 1\right)^2}.$$

Given that $S_1(0,\tau) = 0$, we obtain the solution

$$S_1(x,\tau) = -\frac{S_m x^2 e^{S_m x + \tau}}{2\left(e^{S_m x} + e^{\tau} - 1\right)^2},$$

consequently,

$$S_1(x,t) = -\frac{S_m x^2 e^{S_m x + t - ax}}{2\left(e^{S_m x} + e^{t-ax} - 1\right)^2}.$$

After the transition to new variables equation (14) takes the form

$$\frac{\partial S_2(x,\tau)}{\partial x} + \left(S_m - 2S_0(x,\tau)\right) S_2(x,\tau) =$$
$$= 2S_1^2(x,\tau) - 2x\frac{\partial S_1(x,\tau)}{\partial \tau}.$$

Solving it as linear, substituting found $S_0(x,\tau)$, $S_1(x,\tau)$ and taking into account $S_2(0,\tau) = 0$, we obtain

$$S_2(x,\tau) = \frac{S_m e^{S_m x + \tau} x^4 \left(e^{S_m x} - e^{\tau} - 1\right)}{4\left(e^{S_m x} + e^{\tau} - 1\right)^3},$$

wherefrom

$$S_2(x,t) = \frac{S_m e^{S_m x + t - ax} x^4 \left(e^{S_m x} - e^{t-ax} - 1\right)}{4\left(e^{S_m x} + e^{t-ax} - 1\right)^3}.$$

Let us find $C_0(x,t)$, $C_1(x,t)$ and $C_2(x,t)$. For this purpose, we substitute expressions (9), (10) into equation (6) and equate the coefficients at the same degrees of ε:

$$\text{at } \varepsilon^0: \quad \frac{\partial S_0(x,t)}{\partial t} = \left(S_m - S_0(x,t)\right) C_0(x,t); \tag{15}$$

$$\text{at } \varepsilon^1: \quad \frac{\partial S_1(x,t)}{\partial t} = \left(S_m - S_0(x,t)\right) C_1(x,t) - S_1(x,t) C_0(x,t); \tag{16}$$

$$\text{at } \varepsilon^2: \quad \frac{1}{2}\frac{\partial S_2(x,t)}{\partial t} = \left(S_m - S_0(x,t)\right)\frac{C_2(x,t)}{2} - S_1(x,t) C_1(x,t) - \frac{1}{2} S_2(x,t) C_0(x,t). \tag{17}$$

Substituting already known $S_0(x,t)$, $S_1(x,t)$ and $S_2(x,t)$, we obtain

$$C_0(x,t)=\frac{e^{t-ax}}{e^{S_m x}+e^{t-ax}-1},$$

$$C_1(x,t)=-\frac{x^2e^{t-ax}\left(e^{S_m x}-1\right)}{2\left(e^{S_m x}+e^{t-ax}-1\right)^2},$$

$$C_2(x,t)=\frac{x^4e^{t-ax}\left(e^{S_m x}-1\right)\left(e^{S_m x}-e^{t-ax}-1\right)}{4\left(e^{S_m x}+e^{t-ax}-1\right)^3}.$$

Thus, the asymptotic solutions have the form

$$S(x,t)=\frac{S_m(e^{t-ax}-1)}{e^{S_m x}+e^{t-ax}-1}-\frac{\varepsilon}{2}\frac{S_m x^2e^{S_m x+t-ax}}{\left(e^{S_m x}+e^{t-ax}-1\right)^2}+\frac{\varepsilon^2}{8}\frac{S_m e^{S_m x+t-ax}x^4\left(e^{S_m x}-e^{t-ax}-1\right)}{\left(e^{S_m x}+e^{t-ax}-1\right)^3}+\ldots,$$

$$C(x,t)=\frac{e^{t-ax}}{e^{S_m x}+e^{t-ax}-1}-\frac{\varepsilon}{2}\frac{x^2e^{t-ax}\left(e^{S_m x}-1\right)}{\left(e^{S_m x}+e^{t-ax}-1\right)^2}+\frac{\varepsilon^2}{8}\frac{x^4e^{t-ax}\left(e^{S_m x}-1\right)\left(e^{S_m x}-e^{t-ax}-1\right)}{\left(e^{S_m x}+e^{t-ax}-1\right)^3}+\ldots$$

RESEARCH RESULTS

Fig. 2 and fig. 4 show the exact (black lines) and asymptotic (red lines) solutions of suspended and retained particle concentrations, respectively, as a function of the *x* coordinate for $a=1$, $S_m=1$ and $\varepsilon=0.2$ at *t*=5. In fig. 3 and fig. 5 relative errors of asymptotic solutions of suspended and retained particle concentrations at the same parameters are shown.

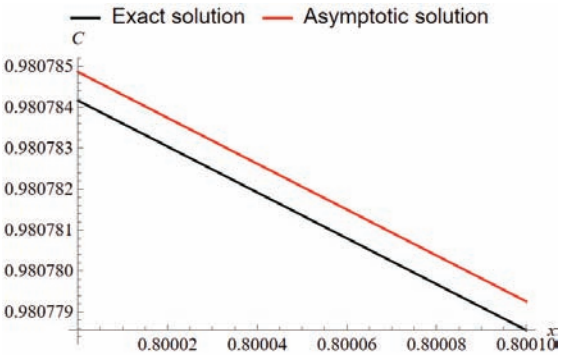


Figure 2. Exact and asymptotic solution of suspended particle concentration at t=5

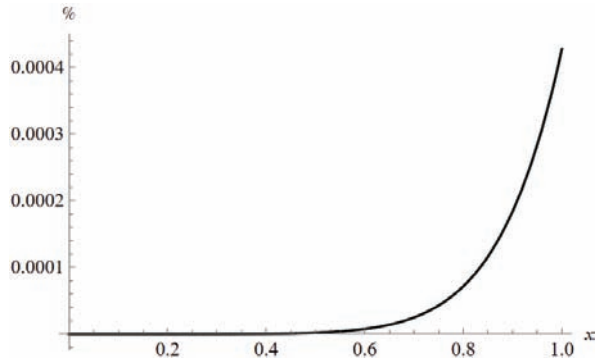


Figure 3. Relative error of asymptotic solution of suspended particle concentration at t=5

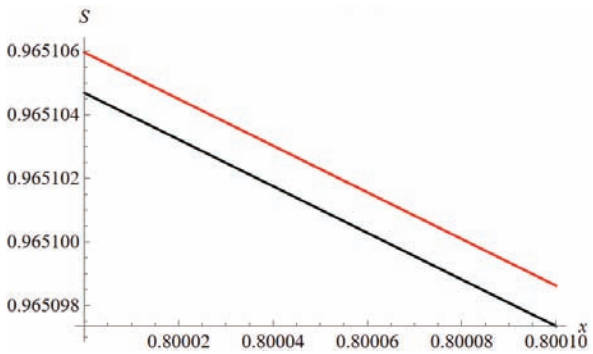


Figure 4. Exact and asymptotic solution of retained particle concentration at t=5

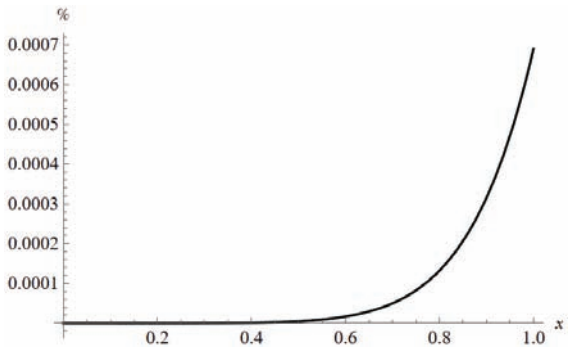


Figure 5. Relative error of the asymptotic solution of the concentration of retained particles at t=5

Figs. 6-9 are similar to Figs. 2-5 for $\varepsilon = 1$.

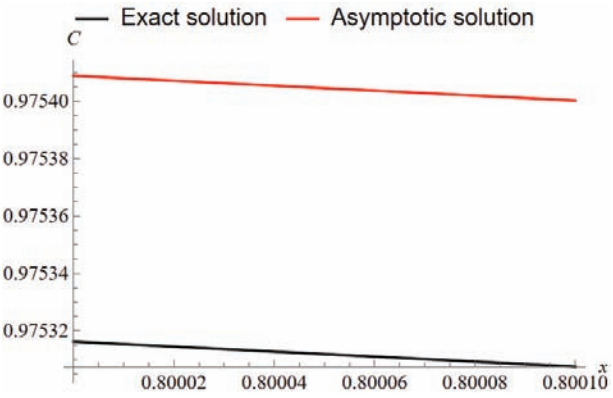


Figure 6. Exact and asymptotic solution of suspended particle concentration at t=5

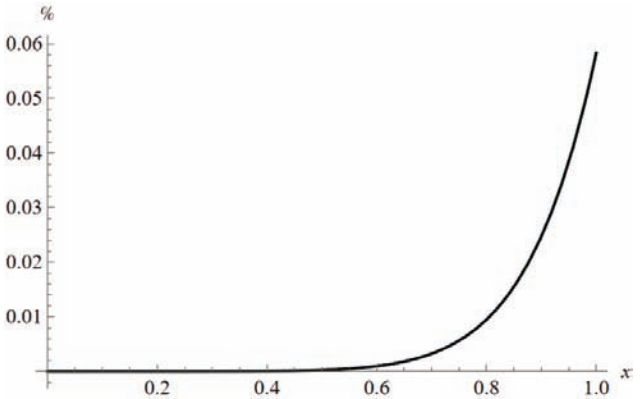


Figure 7. Relative error of asymptotic solution of suspended particle concentration t=5

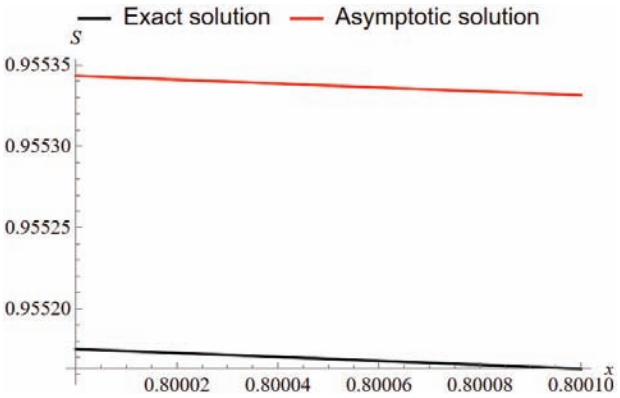


Figure 8. Exact and asymptotic solution of retained particle concentration t=5

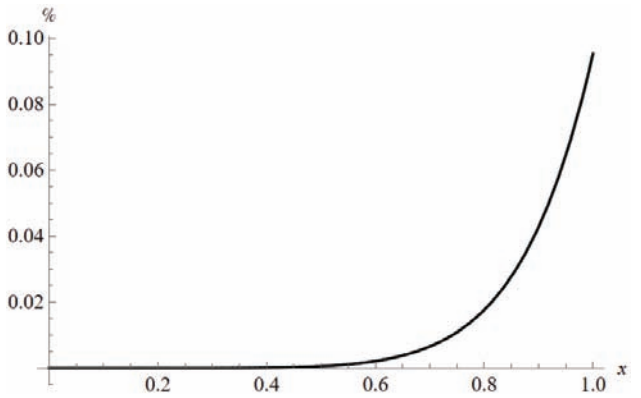


Figure 9. Relative error of the asymptotic solution of the concentration of retained particles at t=5

The obtained dependences show that with increasing parameter ε, the relative errors of asymptotic solutions of both suspended and retained particle concentrations increase significantly. The errors have a monotonically increasing character. It should be noted that the error of the asymptotic solution of the concentration of retained particles is higher than the error of the asymptotic solution of the concentration of suspended particles. Thus, at the filter output at x=1: for $\varepsilon = 0.2$ the relative error for $S(x,t)$ is 0.0007 %, for $C(x,t)$ is 0.0004 %, for the relative error for $S(x,t)$ is 0.0952 %, for $C(x,t)$ is 0.0584 %, which indicates the difference of these values in 1.6 times.

Fig. 10 and fig. 12 show the exact (black lines) and asymptotic (red lines) solutions of suspended and retained particle concentrations, respectively, as a function of time t for $a = 1$, $S_m = 1$ and $\varepsilon = 0.2$ at x=0.5.

Fig. 11 and fig. 13 show the relative errors of asymptotic solutions of suspended and retained particle concentrations at the same parameters.

Figs. 14-17 are similar to figs. 10-13 and are obtained for the value $\varepsilon = 1$.

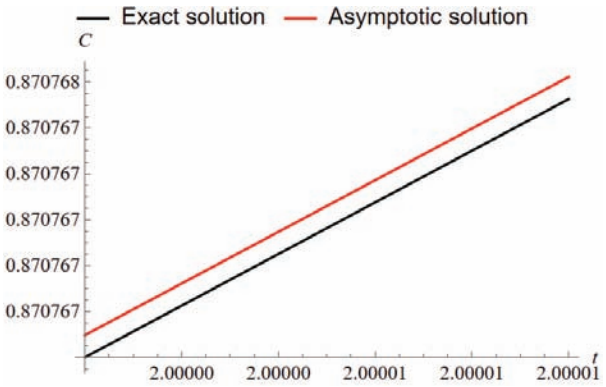


Figure 10. Exact and asymptotic solution of suspended particle concentration at x=0.5

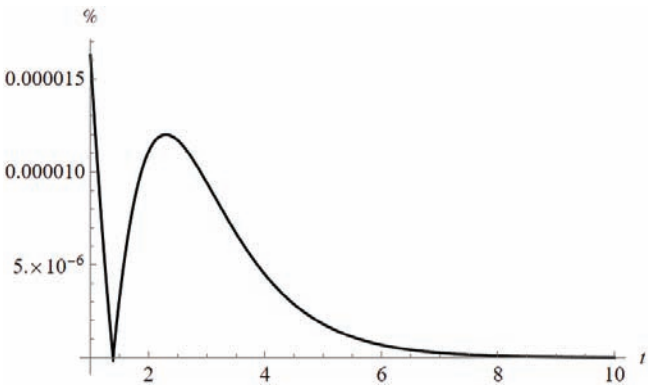


Figure 11. Relative error of asymptotic solution of suspended particle concentration at x=0.5

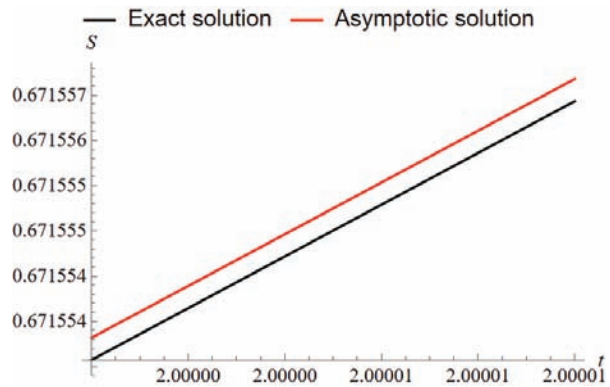


Figure 12. Exact and asymptotic solution of retained particle concentration at x=0.5

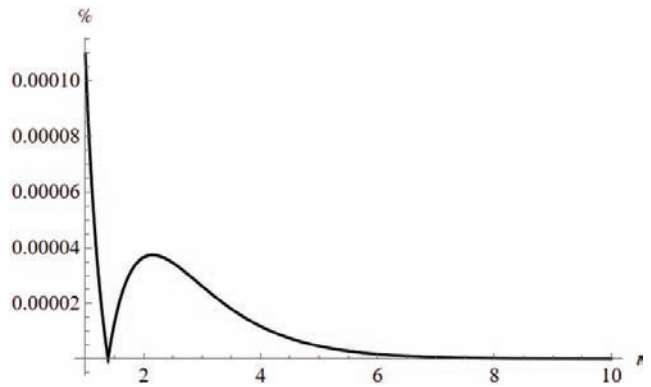


Figure 13. Relative error of asymptotic solution of retained particle concentration at x=0.5

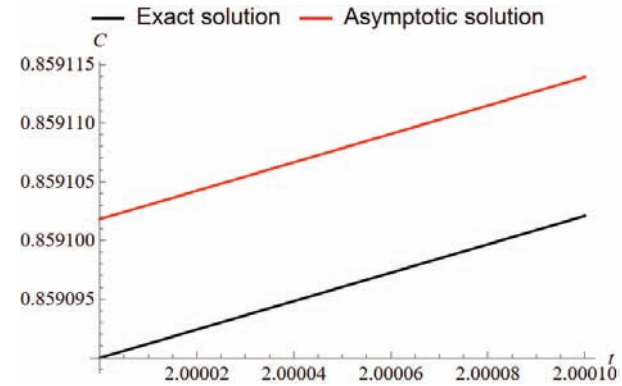


Figure 14. Exact and asymptotic solution of suspended particle concentration at x=0.5

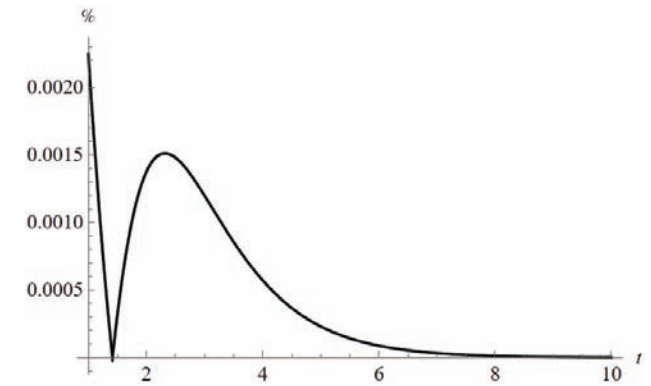


Figure 15. Relative error of asymptotic solution of suspended particle concentration at x=0.5

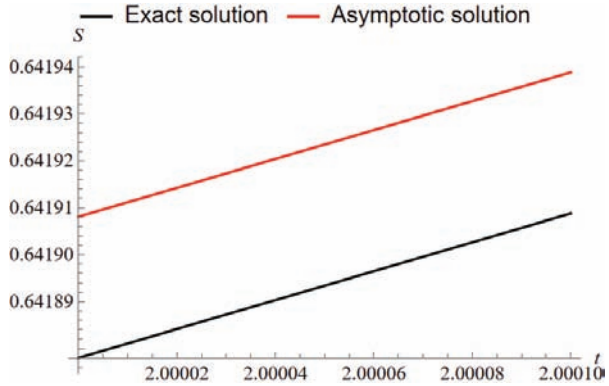


Figure 16. Exact and asymptotic solution of retained particle concentration at x=0.5

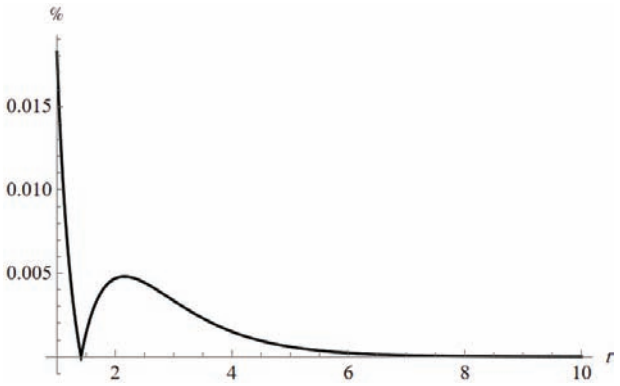


Figure 17. Relative error of asymptotic solution of retained particle concentration at x=0.5

The plots show that the relative errors of suspended and retained particles do not behave monotonically. At first, the errors decrease, reaching the zero value, then increase, reaching the maximum value, and then decrease again. At the same time, increasing the value of the parameter ε does not affect the maximum point: for $C(x,t)$ at $\varepsilon = 0.2$ $t_{\max} = 2.3044$, at $\varepsilon = 1$ $t_{\max} = 2.3169$; for $S(x,t)$ at $\varepsilon = 0.2$ $t_{\max} = 2.1528$, at $\varepsilon = 1$ $t_{\max} = 2.1572$. The larger the value of the parameter ε, the larger the relative errors of the asymptotic solutions of both types of particles.

CONCLUSION

The paper presents the solution of the problem of depth filtration with linear porosity function. Exact solutions of suspended $C(x,t)$ and retained $S(x,t)$ particle concentrations are obtained. With Taylor series expansion of functions $C(x,t)$ and $S(x,t)$ in powers of ε, the formulas of asymptotic solutions of suspended and retained particles concentrations with porosity function $\varphi(x) = a + \varepsilon x$, where ε takes a small value, are derived. As the value of the parameter ε increases, an increase in the relative errors of the asymptotic solutions for both types of particles is observed. As we move along the filter, the relative errors of the asymptotic solutions for the concentrations of suspended and retained particles monotonically increase, reaching the highest value at the filter outlet at $x=1$. The plots of relative errors as a function of time t at a fixed value of x have a non-monotonic character. Near the concentration front the errors reach their highest values, then there is their decrease to zero value, after which there is an increase in the error up to some value at $t_{\max}$, then again decrease to zero value. Thus, the use of asymptotic formulas for solutions of suspended and retained particle concentrations is reasonable for arbitrary time intervals.

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