International Journal for Computational Civil and Structural Engineering, 21(3) 80–92 (2025)

DOI:10.22337/2587-9618-2025-21-3-80-92

DEFORMATION OF COMPOSITE SHELLS DILATING MATERIALS BEYOND ELASTIC LIMITS

Alexander A. Treshchev

Tula State University, Tula, RUSSIA

Annotation. A mathematical model is proposed for the deformation of a thin flat shell of positive Gaussian curvature with a rectangular contour loaded with a transverse load. The model takes into account the deformation of the shell both at the elastic stage of the material's operation and at the plastic stage. The model is developed for shells made of initially isotropic composite and polymer materials, for which physical linearity corresponding to the generalized Hooke's law is valid in the elastic stage of deformation. During the transition to the plastic region of deformation for these materials, the manifestation of dilatation and the dependence of the yield strength on the type of stress state are taken into account. The model describes all stages of deformation of the elastic shell, the stage of the initial appearance of plastic zones up to the formation of plastic hinges. The model is applicable to dilating materials whose plastic properties can be interpreted as ideal without hardening. In this regard, two plasticity conditions, specially developed for isotropic composites, are considered. The first of them was proposed in the works of E.V. Lomakin, and the second - in the publications of the author of the presented study. At the same time, the disadvantages of the first version of the condition are noted. In addition, the developed model can naturally function with the classical von Mises plasticity condition. Therefore, for quantitative and qualitative comparison of the results obtained, calculations were carried out using three variants of plasticity conditions. To construct the model, technical hypotheses traditional for thin shells were used within the framework of geometrically nonlinear representations of the Karman deformation components. Nonlinear resolving differential equations for the bending of shallow shells were obtained, which were processed by V.V. Petrov's two-step method of successive perturbations of parameters with subsequent numerical implementation. For specific calculations, a shell with a square plan, hinged along the entire contour and loaded with a distributed transverse load of constant intensity, was adopted. As a result of applying the developed mathematical model, the dependences of the calculated maximum deflections of a shell made of polymethyl methacrylate on the load intensity are presented. The fields of propagation of plasticity over the surfaces and through the thickness of the shell are constructed at individual load values. In addition, the values of the loads at which plasticity first develops on the surfaces of the shell and the ultimate loads corresponding to the formation of plastic hinges are given in tabular form. It is shown that when dilating materials go beyond the elastic limits, traditional theories of plasticity lead to significant errors in determining the states and ultimate loads for shallow shells. Considering the obvious advantages for the calculation of such structures, it is advisable to apply the plasticity conditions proposed by the author.

Keywords: plasticity conditions, shallow shell, dilatation, ideal plasticity, dependence of yield stress on the type of stress state, limit states, geometric nonlinearity

ДЕФОРМИРОВАНИЕ ОБОЛОЧЕК ИЗ КОМПОЗИТНЫХ ДИЛАТИРУЮЩИХ МАТЕРИАЛОВ ЗА ПРЕДЕЛАМИ УПРУГОСТИ

А.А. Трещев

Тульский государственный университет, Тула, РОССИЯ

Аннотация. Предложена математическая модель деформирования тонкой пологой оболочки положительной Гауссовой кривизны с прямоугольным контуром загруженной поперечной нагрузкой. В модели учтено деформирование оболочки как упругой стадии работы материала, так и в пластической. Модель разработана для оболочек, выполненных из начально изотропных композитных и полимерных материалов, у которых в упругой стадии деформирования справедлива физическая линейность, соответствующая обобщенному закону Гука. При переходе в пластическую область деформирования для

этих материалов учтено проявление дилатации и зависимость пределов текучести от вида напряженного состояния. Модель описывает все стадии деформирования оболочки упругую, стадию начального появления пластических зон вплоть до образования пластических шарниров. Модель применима для дилатирующих материалов, пластические свойства, которых можно интерпретировать как идеальные без упрочнения. В связи с этим рассмотрены два условия пластичности, специально разработанные применительно к изотропным композитам. Первое из них предложено в работах Е.В.Ломакина, а второе – в публикациях автора представленного исследования. При этом отмечены недостатки первого варианта условия. Кроме того, разработанная модель естественным образом может функционировать с классическим условием пластичности Мизеса. Поэтому для количественного и качественного сравнения получаемых результатов проведены расчеты с использованием трех вариантов условий пластичности. Для построения модели использованы традиционные для тонких оболочек технические гипотезы в рамках геометрически нелинейных представлений компонентов деформаций по Карману. Получены нелинейные разрешающие дифференциальные уравнения изгиба пологих оболочек, которые были обработаны двухшаговым методом последовательных возмущений параметров В.В.Петрова с последующей численной реализацией. Для конкретных расчетов принята квадратная в плане оболочка, шарнирно закрепленная по всему контуру и загруженная распределенной поперечной нагрузкой постоянной интенсивности. Как результат применения разработанной математической модели приведены зависимости рассчитанных максимальных прогибов оболочки, выполненной из полиметилметакрилата, от величины интенсивности нагрузки. Построены поля распространения пластичности по поверхностям и по толщине оболочки при отдельных величинах нагрузок. Кроме того в табличной форме приведены величины нагрузок, при которых впервые развивается пластичность на поверхностях оболочки и предельных нагрузок, соответствующих образованию пластических шарниров. Показано, что при выходе за пределы упругости дилатирующих материалов традиционные теории пластичности приводят к существенным погрешностям определения состояний и предельных нагрузок для пологих оболочек. Учитывая явные преимущества для расчета подобных конструкций целесообразно применять предложенные автором условия пластичности.

Ключевые слова: условия пластичности, пологая оболочка, дилатация, идеальная пластичность, зависимость пределов текучести от вида напряженного состояния, предельные состояния, геометрическая нелинейность

1. INTRODUCTION

In construction and other branches of technology, polymer, composite and similar materials are increasingly being used as structural materials. The physical and mechanical behavior of these materials in most cases contradicts traditional theories of elastoplastic deformation. This is especially evident after the stresses reach the yield strength. As a result of experimental studies, the dependence of the yield strength of composites on the type of stress state, accompanied by plastic compressibility or loosening, called dilatation (dilatancy) was established [1 – 7]. Even widely known and long-used materials in technology, such as concrete and cast iron, have similar properties [8 – 11]. To determine the plastic states of dilating materials, a number of phenomenological conditions were proposed [11 – 17], and directly for modern composites, the relations of limit states proposed by Corresponding Member of the Russian Academy of Sciences E.V. Lomakin have found wide application. [3]:

$$F(\sigma_{ij}) = f(\xi^*)\sigma_i = k\,, \qquad (1)$$

where $f(\xi^*)$ – a function determined depending on the type of stress state; $k = \sqrt{3}\tau_s$; τ_s – yield strength established at pure shear; $\xi^* = \sigma / \sigma_i$ – parameter of the type of stress state; $\sigma = \delta_{ij}\sigma_{ij} / 3$ – medium stress; $\sigma_i = \sqrt{3S_{ij}S_{ij} / 2}$ – stress intensity; $S_{ij} = \sigma_{ij} - \delta_{ij}\sigma$ – stress tensor deviator; δ_{ij} – Kronecker symbol.

In this case, the type of function $f(\xi^*)$ was established as a result of processing experimental data on the yield strength of a particular material under a wide range of types of stress states. For some specific types of this function, condition (1) can be reduced to the previously considered criteria of strength and ductility, that is, it partially has a generalizing character in relation to previously known ones [6, 11, 14]. The parameter $\xi^* = \sigma / \sigma_i$ or its deformation analogue, apparently, was first used in the works of A.V. Berezin, Yu.N. Rabotnov, E.V. Lomakin. and V.M. Panferov [2, 18, 19]. However, recently, separate publications have appeared [20], in which these parameters are characterized as those proposed by their authors. The plasticity condition (1) was repeatedly used to solve many applied problems, the results of which were often quite acceptable.

2. RECOMMENDED CONDITION OF PLASTICITY

Obviously, the parameter $\xi^* = \sigma / \sigma_i$ has a variation interval $\pm\infty$, and this can lead to mathematical uncertainties. This parameter has been repeatedly criticized even at the level of deformation theories [21 – 24]. When using this parameter, the approximation of experimental data on determining the yield strengths corresponding to various types of stress states, the function $f(\xi^*)$ often exists in a limited interval ξ^*, that is, the plasticity condition (1) does not work in many cases. Therefore, the author of the presented work at one time proposed a different plasticity condition [23, 24], free from such shortcomings, to determine the limiting states of dilating composites:

$$F(\sigma_{ij}) = f(\xi)\tau = k_\tau, \qquad (2)$$

where $\xi = \sigma / S_0$ – is the normalized normal stress on the octahedral site (parameter change interval ± 1); $\tau = \sqrt{S_i S_i / 3}$ – tangential octahedral stress ($\tau = \sigma_i \sqrt{2} / 3$); $S_0 = \sqrt{\sigma^2 + \tau^2}$ – norm of the vector space associated with the octahedral area; $k_\tau = \sqrt{2/3}\tau_S$.

Condition (2) has undergone deep experimental testing and theoretical testing. Optimal options for the function $f(\xi)$, that approximates experimental data for a wide range of composites are presented in [23, 24]. In particular, materials such as polymethyl methacrylate, phenol plastic AG-4V, graphite composites VPP and MPG-6, cast iron MSCh38-60 are considered, for which options for linear, exponential, piecewise linear and piecewise exponential approximation of the function $f(\xi)$ are adopted, respectively.

Based on the associated law technique, plastic flow equations are formulated. Therefore, in the presented work, this condition (2) was used to construct a model of transverse bending of flat shells of positive Gaussian curvature beyond the elasticity limit.

3. PROBLEM STATEMENT AND SHELL CALCULATION

Thin shells of a square outline with a constant surface curvature are considered, such that a mathematical model of their mechanics was built using the Kirchhoff–Love hypotheses and the Karman formalism [25]. In this case, the connections between the significant components of the shell deformation at any point of its body with the displacements of the middle surface were determined in the traditional way:

$$e_{11} = \varepsilon_{11} - z\frac{\partial^2 w}{\partial x_1^2}; \quad e_{22} = \varepsilon_{22} - z\frac{\partial^2 w}{\partial x_2^2};$$

$$e_{12} = \varepsilon_{12} - z\frac{\partial^2 w}{\partial x_1 \partial x_2}, \qquad (3)$$

where w – is the deflection of the middle surface; x_1, x_2 – Cartesian coordinates of the plan of the middle surface; ε_{ij} – straines in the middle surface:

$$\varepsilon_{11} = \frac{\partial u}{\partial x_1} - k_1 w + \frac{1}{2}\left(\frac{\partial w}{\partial x_1}\right)^2;$$

$$\varepsilon_{22} = \frac{\partial \upsilon}{\partial x_2} - k_2 w + \frac{1}{2}\left(\frac{\partial w}{\partial x_2}\right)^2;$$

$$2\varepsilon_{12} = \frac{\partial \upsilon}{\partial x_1} + \frac{\partial u}{\partial x_2} + \frac{\partial w}{\partial x_1}\frac{\partial w}{\partial x_2};$$

u, υ – displacement of the middle surface in the direction of the corresponding axes x_1 and x_2; k_1, k_2 – curvature of the middle surface of the shell in the corresponding directions.

It should be noted here that, in contrast to the elastic scheme of the shell, the resulting straines ε_{ij}, п appear not only due to the influence of membrane forces, but are also further aggravated due to the displacement of its neutral surfaces relative to the middle, caused by the difference in stress values reaching the yield strength in opposite zones of the section.

Considering that the materials considered [23, 24], when deformed within the elastic range, do not exhibit a dependence of mechanical properties on the kind of stress state, the relationship equations between stresses and strains were established according to the generalized Hooke's law [25]:

$$\sigma_{11} = \frac{E}{1-\nu^2}(e_{11} + \nu \cdot e_{22});$$

$$\sigma_{22} = \frac{E}{1-\nu^2}(e_{22} + \nu \cdot e_{11});$$

$$\tau_{12} = \frac{E}{1+\nu} e_{12}, \qquad (4)$$

where E – is the elastic modulus of the material; ν – Poisson's ratio.

During the transition of the shell deformation process from the elastic stage to the plastic fluidity of the material, the material can develop in one zone of sections, or in two (stretched and compressed). Such shell conditions are aggravated due to the dependence of the yield strength on the type of stress state, as shown in Fig. 1.

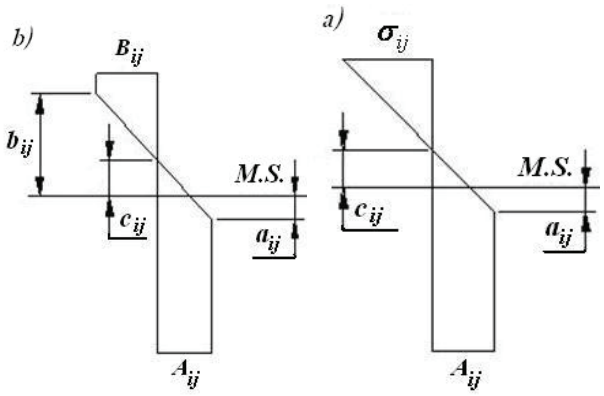


Figure 1. Diagrams of stress distribution across shell sections beyond elasticity: а – one-sided plasticity; б – bilateral plasticity

In Fig. 1 A_{ij} indicates the stresses that cause plasticity in the lower zone, and through B_{ij} – in the upper zone. The functions A_{ij} and B_{ij} are calculated in accordance with the plasticity condition (2). In the process of plastic deformation of the shell material, it can be distinguished into two stages: the state of one-sided plasticity and the state of two-sided plasticity (Fig. 1). The plasticity condition is first triggered in the outer fibers of one surface of the shell, and with increasing load, bilateral plasticity develops.

The problem of transverse bending of the shell is formulated within the framework of the theory of small elastoplastic deformations. In this case, as shown above, the traditional fundamentals of the technical theory of thin shells are used, and the stress-strain diagrams of materials have a physical yield strength (there is a yield plateau), which allows the use of a

model of an ideal elastoplastic body. In addition, the loading of the shell is assumed to be simple.

Considering that the bending of the shell is considered at all stages of elastoplastic deformation, it is convenient to present the dependence of stresses on strains in a single general form:

$$\sigma_{ij} = F_{ij}(r_{ij} - z\Delta_{ij}); \quad i,j = 1, 2, \qquad (4^*)$$

where $F_{11} = F_{22} = \frac{E}{1-\nu^2}$; $F_{12} = \frac{E}{1+\nu}$;

$r_{11} = \varepsilon_{11} + \nu\varepsilon_{22}$; $r_{22} = \varepsilon_{22} + \nu\varepsilon_{11}$; $r_{12} = \varepsilon_{12}$;

$\Delta_{11} = w,_{11} + \nu w,_{22}$; $\Delta_{22} = w,_{22} + \nu w,_{11}$;

$\Delta_{12} = w,_{12}$; $\chi_{11} = w,_{11}$; $\chi_{22} = w,_{22}$; $\chi_{12} = w,_{12}$.

The coordinates of the neutral surface of the shell $x_3 = c_{ij}$ are determined based on the stress being equal to zero $\sigma_{ij} = 0$. Then we have $c_{ij} = r_{ij} / \Delta_{ij}$. The positions of the boundaries of the plastic zones in terms of stresses a_{ij} and b_{ij} are determined from the obvious conditions, that $\sigma_{ij} = A_{ij}$ at $x_3 = a_{ij}$ and $\sigma_{ij} = B_{ij}$ at $x_3 = b_{ij}$ (see Fig. 1), from which we obtain

$$a_{ij} = \frac{F_{ij} \cdot r_{ij} - A_{ij}}{F_{ij} \cdot \Delta_{ij}}; \; b_{ij} = \frac{F_{ij} \cdot r_{ij} - B_{ij}}{F_{ij} \cdot \Delta_{ij}}. \qquad (5)$$

In the general case, in the process of elastoplastic deformation of the shell, its mechanical behavior, taking into account the accepted assumptions, is reduced to solving a system of three nonlinear differential equilibrium equations [25]:

$$N_{11},_1 + N_{12},_2 = 0; \quad N_{12},_1 + N_{22},_2 = 0;$$
$$M_{11},_{11} + 2M_{12},_{12} + M_{22},_{22} +$$
$$N_{11}(w,_{11} + k_1) + 2N_{12}w,_{12} + N_{22}(w,_{22} + k_2) + q = 0, \qquad (6)$$

where q – is the intensity of the transverse load; N_{ij}, M_{ij} – membrane forces, bending and torque moments.

Forces and moments are determined by integrating stresses over the thickness of the shell according to known rules:

$$N_{ij} = \int_{-h/2}^{h/2} \sigma_{ij} dx_3; \; M_{ij} = \int_{-h/2}^{h/2} \sigma_{ij} x_3 dx_3. \qquad (7)$$

In the elastic stage of shell deformation, system (6), taking into account traditional concepts (3), (4) and (7), is reduced to three nonlinear differential equations for displacements u, υ and w, the solution of which is known from the literature [25].

Taking into account the transformation of the equations of state to a general form that is valid at all stages of shell deformation (4), the parameters r_{ij} will be determined by the equal areas of stress diagrams distributed over the thickness of the shell, corresponding to forces (7). Thus, with the development of one-sided plasticity, we have

$$N_{ij} = \int_{-h/2}^{a_{ij}} \sigma_{ij} dx_3 + \int_{a_{ij}}^{h/2} A_{ij} dx_3, \qquad (8)$$

and with bilateral plasticity –

$$N_{ij} = \int_{-h/2}^{b_{ij}} B_{ij} dx_3 + \int_{b_{ij}}^{a_{ij}} \sigma_{ij} dx_3 + \int_{a_{ij}}^{h/2} A_{ij} dx_3. \qquad (9)$$

Obviously, the moments will be determined in accordance with rules (7) as follows:

with the development of one-sided plasticit –

$$M_{ij} = \int_{-h/2}^{a_{ij}} \sigma_{ij} x_3 dx_3 + \int_{a_{ij}}^{h/2} A_{ij} x_3 dx_3 ; \quad (10)$$

when bilateral plasticity occurs –

$$M_{ij} = \int_{-h/2}^{b_{ij}} B_{ij} x_3 dx_3 + \int_{b_{ij}}^{a_{ij}} \sigma_{ij} x_3 dx_3 + \int_{a_{ij}}^{\frac{h}{2}} A_{ij} x_3 dx_3 . \quad (11)$$

The set of equations (4*), (10), (11) with the simplest transformations gives expressions for moments in the form:

$$M_{ij} = R_{ij} + S_{ij} \cdot \Delta_{ij}, \quad (12)$$

where

$$R_{ij} = \frac{A_{ij}}{2}\left(\frac{h^2}{4} - a_{ij}^{\ 2}\right) + \frac{F_{ij} \cdot r_{ij}}{2}\left(a_{ij}^{\ 2} - \frac{h^2}{4}\right),$$

$$S_{ij} = -\frac{F_{ij}}{3}\left(a_{ij}^{\ 3} + \frac{1}{8} b_{ij}^{\ 3}\right)$$ – with one-sided plasticit;

$$R_{ij} = \frac{A_{ij}}{2}\left(\frac{h^2}{4} - a_{ij}^2\right) + \frac{B_{ij}}{2}\left(b_{ij}^2 - \frac{h^2}{4}\right) +$$

$$+ \frac{F_{ij}}{2} r_{ij}(a_{ij}^2 - b_{ij}^2), \; S_{ij} = -F_{ij}(a_{ij}^3 - b_{ij}^3)/3$$

– with bilateral plasticity.

In this case, the set of dependencies (4*), (8), (9) after transformations allows us to obtain expressions for membrane forces:

$$N_{ij} = T_{ij} + C_{ij} \cdot \Delta_{ij}, \quad (13)$$

where

$$T_{ij} = A_{ij}\left(\frac{h}{2} - a_{ij}\right) + F_{ij} \cdot r_{ij}\left(a_{ij} + \frac{h}{2}\right),$$

$$C_{ij} = -\frac{F_{ij}}{2}\left(a_{ij}^{\ 2} - \frac{h^2}{4}\right)$$ – with one-sided plasticit;

$$T_{ij} = A_{ij}\left(\frac{h}{2} - a\right) + B_{ij}\left(b - \frac{h}{2}\right) + ;$$

$$+ F_{ij} r_{ij}(a_{ij} - b_{ij}); \; C_{ij} = -F_{ij} \cdot (a_{ij}^2 - b_{ij}^2)$$

– with bilateral plasticity.

Considering dependencies (3), (4), (4*), (5), (12), (13) together, and introducing them into the equilibrium equations (6), we obtain a mathematical model that describes the deformation of the shell from dilating materials in all three stages. This model is represented by a system of three nonlinear differential equations:

$$\frac{\partial(T_{11} + C_{11} \cdot \Delta_{11})}{\partial x_1} + \frac{\partial(T_{12} + C_{12} \cdot \Delta_{12})}{\partial x_2} = 0;$$

$$\frac{\partial(T_{12} + C_{12} \cdot \Delta_{12})}{\partial x_1} + \frac{\partial(T_{22} + C_{22} \cdot \Delta_{22})}{\partial x_2} = 0;$$

$$\frac{\partial^2(R_{11} + S_{11} \cdot \Delta_{11})}{\partial x_1^2} + \frac{\partial^2(R_{22} + S_{22} \cdot \Delta_{22})}{\partial x_2^2} +$$

$$+ \frac{\partial^2(R_{12} + S_{12} \cdot \Delta_{12})}{\partial x_1 \partial x_2} =$$

$$= -[q + (T_{11} + K_{11} \cdot \Delta_{11}) \cdot \frac{\partial^2 w}{\partial x_1^2} +$$

$$+ (T_{22} + K_{22} \cdot \Delta_{22}) \cdot \frac{\partial^2 w}{\partial x_2^2} +$$

$$+ 2(T_{12} + K_{12} \cdot \Delta_{12}) + K_1(T_{11} + K_{11} \cdot \Delta_{11}) +$$

$$+ K_2(T_{22} + K_{22} \cdot \Delta_{22})]. \quad (14)$$

It should be noted that the dimensions of the depths of development of plastic deformations along the thickness of the shell a_{ij} and b_{ij} do not depend on the direction ($a_{11} = a_{22} = a_{12}$, $b_{11} = b_{22} = b_{12}$), since these parameters are

determined by the totality of stresses σ_{ij} arising in accordance with the plasticity condition (2), and not by a separate stress component. Therefore, you can easily omit the indices in these parameters, using the general notations a and b for them. In order for system (14) to be closed, it is necessary to add boundary conditions, which in the presented work are assumed to be the same along the entire support contour as for hinged fastening (for example, with: $x_1 = l$: $w = 0$, $\upsilon = 0$, $u = 0$, $M_{11} = 0$ (l – the length of the support contour along the corresponding axis)).

The solution of the system of nonlinear differential equations was carried out within the framework of the two-step method of successive perturbations of parameters proposed by V.V. Petrov [26]. Before applying the two-step method procedure, it is necessary to linearize the resolving differential equations (14) and boundary conditions using the method of successive loadings [26]. The numerical implementation of the considered model was carried out using the finite-difference method. When assigning mesh sizes, the convergence of the numerical procedure was assessed.

For specific calculations, a flat shell, square in plan, 1000×1000 mm in size, 10 mm thick, made of polymethyl methacrylate, was adopted. The lifting boom of the middle surface of the shell in the center of the plan was taken to be 100 mm. For polymethyl methacrylate, the following mechanical characteristics were accepted: elastic modulus E=3300 MPa; Poisson's ratio ν=0,3; plasticity constant k_τ=58,9 MPa. A function that depends on the kind of stress state $f(\xi)$, and is included in the plasticity condition (2) was adopted in accordance with the data on processing experimental studies of limit states [23, 24] in exponential form:

$$f(\xi) = \exp(A_1 \xi), \qquad (15)$$

where A_1=0,424.

The calculation of the shell was carried out taking into account three variants of plasticity conditions: 1) use of condition (2); 2) according to condition (1) taking into account the corresponding function ξ^* and constant $k = \sqrt{3}\tau_s$ [3]; 3) application of the Mises condition, which does not take into account the dependence of the yield strength on the type of stress state. The load on the shell was assumed to be uniformly distributed over the entire surface. The load step interval was taken for all calculation options at a constant 5 kPa in the elastic stage of deformation, and when the yield strength was reached – 5 kPa, decreasing to 100 Pa as the intensity of deflections increased. Taking into account the symmetry of the load case and the geometry of the shell in plan, its fourth part was calculated, covered with a mesh of size 15×15, which ensured the necessary accuracy of the solution.

As the results of the calculations in Fig. 2 shows the processes of increase in the maximum relative deflections of the shell in its plan center w/h as the intensity of the transverse load q increases using three models of plasticity conditions. The solid line shows the dependence of the growth of deflections obtained using condition (2), the dashed line - in accordance with the E.V. Lomakin model (1), and the dash-dotted line - with the classical Mises model. In Fig. 2, the following designations for the characteristic states of the shell are introduced: I.1 – the occurrence of plastic deformations during calculations using the plasticity condition (2); I.2 – moment of formation of a plastic hinge when calculating according to condition (2); II.1 – the occurrence of plastic deformations during calculations using the plasticity condition (1); II.2 – moment of formation of a plastic hinge when calculating according to condition (1); III.1 – the occurrence of plastic deformations during calculations using the von Mises limit state; III.2 – formation of a plastic hinge when calculating using the von Mises limit state.

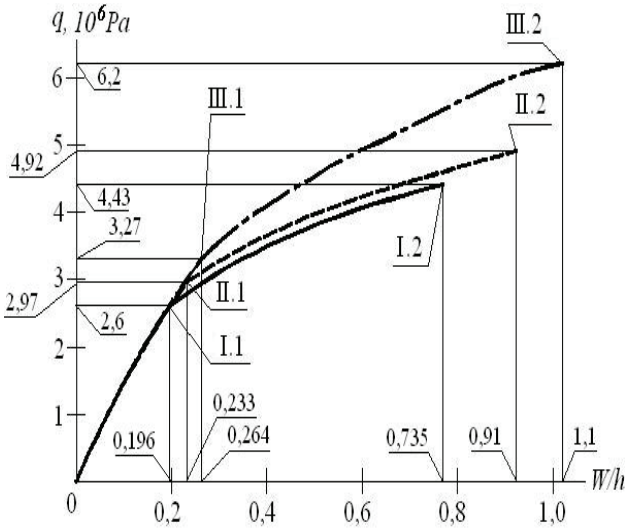
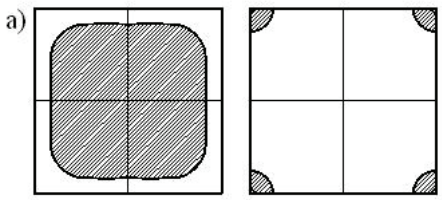


Figure 2. Increase in maximum deflections depending on load intensity

In Fig. 3 and 4 demonstrate the trends in the propagation of areas of plastic deformation along the surfaces of the shell (upper and lower). Pictures of plasticity zones are also calculated for three variants of plasticity conditions and are presented at two characteristic values of load intensity. The areas of the shell surfaces in which plastic deformations occurred are shaded.

Fig. 5 and 6 illustrate the deep distribution of zones of plastic deformation along the thickness of the shell along the axis of symmetry, which is the coordinate axis and along the diagonal. Pictures of the depth penetration of plasticity zones are also calculated for three variants of plasticity conditions and are presented at two characteristic values of load intensity. The areas of deep development of plastic deformations along the thickness of the shell are shaded as follows: solid lines characterize the boundaries of plasticity zones corresponding to condition (2); dashed lines – condition (1); dash-dotted lines – to the Mises condition.



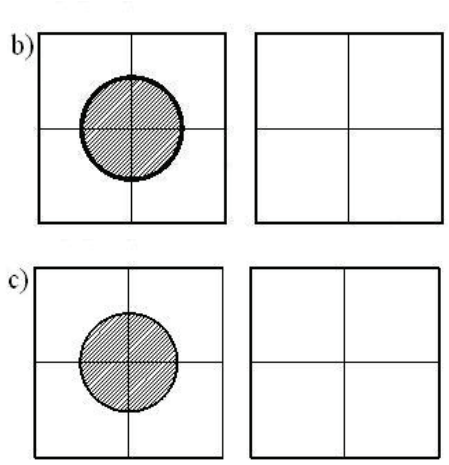


Figure 3. Development of plastic zones along the lower and upper surfaces of the shell at a load intensity of 3,85 MPa: a) – calculation according to condition (2); b) – calculation according to condition (1); c) – calculation according to the Mises condition

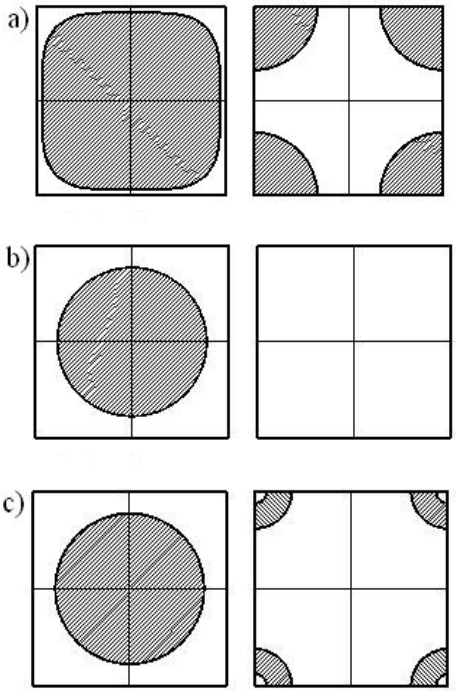


Figure 4. Development of plasticity zones along the lower and upper surfaces of the shell at a load intensity of 4,12 MPa: a) – calculation according to condition (2); b) – calculation according to condition (1); c) – calculation according to the Mises condition

Additionally, to illustrate the characteristic states in the process of elastoplastic deformation of a shell made of polymethyl methacrylate, Table 1 shows numerical parameters for the intensity of loads that first cause plasticity and the formation of plastic hinges, calculated for three variants of limit state conditions. The percentages of discrepancy between the values of limit states when calculating using conditions (1) and (2) in comparison with the Mises theory are given in parentheses.

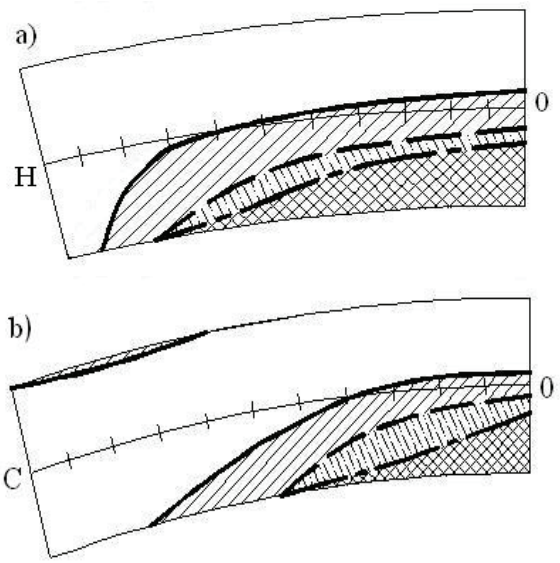
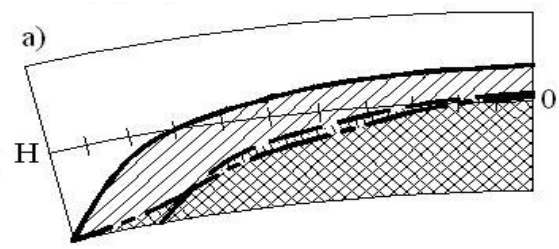


Figure 5. Development of plasticity zones along the thickness of the shell at a value load intensity 3,85 MPa: a) along the axis of symmetry along the coordinate x_1; b) along the diagonal of the shell (0 – center of the shell plan; H – boundary hingedly supported contour; C – corner point of the shell plan)



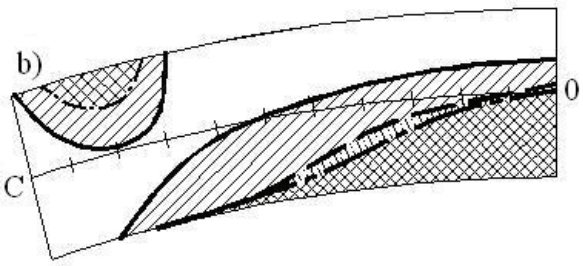


Figure 6. Development of plasticity zones along the thickness of the shell at a value load intensity 4,12 MPa: a) along the axis of symmetry along the coordinate x_1; b) along the diagonal of the shell (0 – center of the shell plan; H – boundary hingedly supported contour; C – corner point of the shell plan)

Table 1

Variant of the plasticity condition	Load intensity at appearance of plasticity, MPa		Load intensity upon the onset of plastic hinges, MPa
	On the top surface	On the bottom surfaces	
Treshchev (2)	3,85 (28,1%)	2,6 (25,8%)	4,43 (40,0%)
Lomakin (1)	4,82 (2,3%)	2,97 (10,0%)	4,92 (26,0%)
Mises	4,93	3,27	6,20

4. ANALYSIS OF THE RESULTS OBTAINED

Analysis of the results of calculating the elastic-plastic states of the shell shown in Fig. 2 shows that when plastic deformations appear at the initial stage, the values of maximum deflections obtained using conditions (2) and (1) differ insignificantly and do not exceed 9%. With increasing load, the discrepancy between these values tends to maintain the specified interval. At the same time, there are significant differences in the values of deflections calculated with conditions (2) and (1) in relation to the calculation results taking into account the Mises theory. These differences with developed plasticity zones reach 30%. To prevent violation of the rigor of the calculation, changes in the parameters of the stress-strain states of the shell

were analyzed in order to detect local unloadings, which were not established until the formation of plastic hinges. Additional data from Table 1 demonstrate the fact that the load intensity causing the appearance of plasticity in the lower fibers of the shell, calculated using equation (2) differs from option (1) by 14,2%. The difference in calculating maximum deflections according to model (2) in comparison with the classical Mises theory reaches 25,8%, and for model (1) – 10%. Similar differences in the magnitude of the loads causing the appearance of plastic deformations in the upper fibers of the shell are 25,2%, 28,1% and 2,3%, respectively (see Table 1).

The intensity of the load causing the formation of plastic hinges, calculated using condition (2), differs from the data of the Mises theory by 40%, and for the version using equation (1) – by 26%. Between themselves, the values of the indicated loads calculated on the basis of conditions (2) and (1) differ by 11,1%.

Analyzing the areas of development of plasticity on the surfaces and along the thickness of the shell at a load of 3,85 MPa, it can be noted that the patterns calculated using condition (1) and Mises are close to each other, differing in the depth of penetration (see Fig. 3 and 5). In this case, condition (2) gives qualitative differences from the previous two options, since plastic deformations already occur in the corner sections of the upper fibers, and on the lower surface the plasticity region is more developed. An increase in load does not lead to significant changes in the qualitative patterns of plasticity development on surfaces, but numerical discrepancies are found in the sizes of these zones between the calculation options under conditions (2) and (1). At the same time, the qualitative and quantitative pictures of the areas occupied by plasticity in the upper surfaces of the shell differ when calculated with condition (2) and according to the Mises theory. This can be detected in particular at 4,12 MPa, when, when calculating using condition (2), plasticity extends completely to the corner zones of the shell, while the Mises theory predicts that plastic deformations, developing from the corner regions to the central ones, retain elasticity directly in the corners.

5. CONCLUSION

The presented mathematical model of the deformation of thin shells made of dilating materials and the calculations carried out using the example of polymethyl methacrylate clearly show that when determining the states of such structures beyond elasticity, traditional theories of plasticity lead to significant errors. These errors manifest themselves when determining the limiting states of the formation of plastic hinges, when the Mises theory leads to an error in the values of the maximum loads on a hinged shell of up to 40% (polymethyl methacrylate material), and in the values of deflections – up to 49,7%.

It should also be noted that due to the advantages of the plasticity condition (2), its use is more appropriate for calculating shells beyond elasticity, which has been confirmed experimentally.

REFERENCES

1. **Ainbinder S.B.** The influence of hydrostatic pressure on the mechanical properties of polymer materials / S.B. Ainbinder, M.G. Laka, I.Yu. Majors // Polymer Mechanics. 1965. No. 1. P. 65 – 75.
2. Resistance to deformation and fracture of isotropic graphite materials under complex stress conditions / **A.V. Berezin** et al. // Problems of strength. 1979. No. 2. P. 60 – 65.
3. **Lomakin E.V.** Dependence of the limiting state of composite and polymer materials on the type of stress state / E.V. Lomakin // Mechanics of composite materials. – 1988. – No. 1. P. 3 – 9.
4. **Fridman A.M.** Study of the destruction of carbon-graphite materials under complex stress conditions / A.M. Fridman, Yu.P. Anufriev, V.N. Barabanov // Problems of strength. – 1973. – No. 1. – pp. 52–55.

5. **Zhukov A.M.** Strength properties of polymethyl methacrylate under biaxial tension / A.M. Zhukov // Ing. Sat. – 1960. – Vol. 1. – Issue. 2. – P. 200 – 204.
6. **Goldman A.Ya**. Strength of structural plastics / A.Ya. Goldman. – L.: Mechanical Engineering, 1979. – 320 p.
7. **Ainbinder S.B.** Properties of polymers at high pressures / S.B. Einbinder, K.I. Alksne, E.L. Tyupina, M.G. Laka. – M., 1973.
8. **Bazant Z.P.** Endochronic Theory of Inelasticity and Failure of Concrete / Z.P.Bazant, P.D.Bhat // Journal of the Engineering Mechanics Division, ASCE. – 1976. – Vol. 102. – No. EM4. – P. 701–722.
9. **Tasuji M.E.** Stress-Strain Response and Fracture of Concrete in Biaxial Loading / M.E.Tasuji, F.O.Slate, A.H.Nilson // ACI Journal. – 1979. – No. 7. – P. 806 – 812.
10. **Leonov M.Ya.** Dependences between strains and stresses for semi-brittle bodies / M.Ya. Leonov, V.A. Panyaev, K.N. Rusinko // Ing. magazine MSB. – 1967. – No. 6. – P. 26 – 32.
11. **Yagn Yu.I.** Strength and ductility of modified cast iron under various stress states / Yu.I.Yagn, V.V. Evstratov // Reports Academy of Sciences of the USSR. – 1957. – Vol. 113. – No. 3. – P. 573 – 575.
12. **Goldenblat I.I.** Strength criteria for structural materials / I.I.Goldenblat, V.A.Kopnov. – M.: Mashinostroenie, 1968. – 192 p.
13. **Pisarenko G.S.** Deformation and strength of materials under complex stress state / G.S. Pisarenko, A.A. Lebedev. – Kyiv: Naukova Dumka, 1976. – 416 p.
14. **Balandin P.P.** On the issue of strength hypotheses / P.P. Balandin // Bulletin of engineers and technicians. – 1937. – No. 1. – P. 37 – 41.
15. **Geniev G.A.** On the issue of generalizing the theory of concrete strength / G.A.Geniev, V.N.Kissyuk // Concrete and reinforced concrete. – 1965. – No. 2. – P. 16 – 19.
16. **Tolokonnikov L.A.** On the shape of the limiting surface of an isotropic body / L.A.Tolokonnikov // Applied mechanics. – 1969. – Issue. 10. – Vol. 5. – P. 123 – 126.
17. **Green R.J.** A plasticity theory for porous solid / R.J.Green // Int. J. Mech. Sci. – Vol. 14. –1972. – P. 215 – 227.
18. **Lomakin E.V.** Relations of the theory of elasticity for an isotropic body of different moduli / E.V. Lomakin, Yu.N. Rabotnov // News Academy of Sciences of the USSR. MSB. – 1978. – No. 6. – P. 29–34.
19. **Panferov V.M.** Theory of elasticity and deformation theory of plasticity for bodies with different properties for compression, tension and torsion / V.M. Panferov // Reports Academy of Sciences of the USSR. – 1968. – Vol. 180. – No. 1. – P. 41 – 44.
20. Relationships between stresses and strains in a nonlinearly deformable body. Part 1. Basic principles and relationships of the mechanics of a deformable solid body / **V.M. Kruglov**, S.V. Bakushev, A.I. Shein, V.T. Erofeev, S.D.S. Al Dulaimi, A.A. Tomilov // Expert: Theory and practice. – 2023. – No. 4(23). – P. 154 – 163.
21. **Treshchev A.A.** Theory of deformation and strength of materials with initial and induced sensitivity to the type of stress state. Defining relations / A.A. Treshchev. – M.; Tula: RAACS; Tula State University, 2016. – 326 p.
22. **Treshchev A.A.** Theory of deformation and strength of differently resistant materials / A.A.Treshchev. – Tula: Tula State University, 2020. – 359 p.
23. **Treshchev A.A.** Dependence of limit states of structural materials on the type of stress state / A.A.Treshchev // News of higher educational institutions. Construction. – 1999. – No. 10. – P. 9–18.
24. **Treshchev A.A.** On the theory of plasticity of dilating materials of different resistance / A.A. Treshchev // Problems of mechanical

engineering and automation. – International magazine. – 2003. – No. 2. – pp. 58–62.
25. **Timoshenko S.P.** Plates and shells / S.P. Timoshenko, S. Voinovsky–Krieger. – M.: Fizmatgiz, 1966. – 636 p.
26. **Petrov V.V.** Methods for calculating structures made of nonlinear deformable material / V.V. Petrov, I.V. Krivoshein. – M.: ASV, 2009. – 208 p.

СПИСОК ЛИТЕРАТУРЫ

1. **Айнбиндер С.Б.** Влияние гидростатического давления на механические свойства полимерных материалов / С.Б.Айнбиндер, М.Г.Лака, И.Ю.Майорс // Механика полимеров. 1965. № 1. С. 65 – 75.
2. Сопротивление деформированию и разрушению изотропных графитовых материалов в условиях сложного напряженного состояния / **А.В.Березин** и др. // Проблемы прочности. 1979. № 2. С. 60 – 65.
3. **Ломакин Е.В.** Зависимость предельного состояния композитных и полимерных материалов от вида напряженного состояния / Е.В.Ломакин // Механика композитных материалов. – 1988. – № 1. С. 3 – 9.
4. **Фридман А.М.** Исследование разрушения углеграфитовых материалов в условиях сложного напряженного состояния / А.М.Фридман, Ю.П.Ануфриев, В.Н.Барабанов // Проблемы прочности. – 1973. – №1. – С. 52–55.
5. **Жуков А.М.** Прочностные свойства полиметилметакрилата при двухосном растяжении / А.М.Жуков // Инж. сб. – 1960. – Т. 1. – Вып. 2. – С. 200 – 204.
6. **Гольдман А.Я.** Прочность конструкционных пластмасс / А.Я.Гольдман. – Л.: Машиностроение, 1979. – 320 с.
7. **Айнбиндер С.Б**. Свойства полимеров при высоких давлениях / С.Б.Айнбиндер, К.И.Алксне, Э.Л.Тюпина, М.Г.Лака. – М., 1973.
8. **Bazant Z.P.** Endochronic Theory of Inelasticity and Failure of Concrete / Z.P.Bazant, P.D.Bhat // Journal of the Engineering Mechanics Division, ASCE. – 1976. – Vol. 102. – № EM4. – P. 701–722.
9. **Tasuji M.E.** Stress-Strain Response and Fracture of Concrete in Biaxial Loading / M.E.Tasuji, F.O.Slate, A.H.Nilson // ACI Journal. – 1979. – №7. – P. 806–812.
10. **Леонов М.Я.** Зависимости между деформациями и напряжениями для полухрупких тел / М.Я.Леонов, В.А.Паняев, К.Н.Русинко // Инж. журн. МТТ. – 1967. – № 6. – С. 26 – 32.
11. **Ягн Ю.И.** Прочность и пластичность модифицированного чугуна при различных напряженных состояниях / Ю.И.Ягн, В.В. Евстратов // Докл. АН СССР. – 1957. – Т. 113. - №3. – С. 573-575.
12. **Гольденблат И.И.** Критерии прочности конструкционных материалов / И.И.Гольденблат, В.А.Копнов. – М.: Машиностроение, 1968. – 192 с.
13. **Писаренко Г.С.** Деформирование и прочность материалов при сложном напряженном состоянии / Г.С.Писаренко, А.А.Лебедев. – Киев: Наукова думка, 1976. – 416 с.
14. **Баландин П.П.** К вопросу о гипотезах прочности / П.П.Баландин // Вестник инженеров и техников. – 1937. – № 1. – С. 37 – 41.
15. **Гениев Г.А.** К вопросу обобщения теории прочности бетона / Г.А.Гениев, В.Н.Киссюк // Бетон и железобетон. – 1965. – № 2. – С. 16 – 19.
16. **Толоконников Л.А.** О форме предельной поверхности изотропного тела / Л.А.Толоконников // Прикладная механика. – 1969. – Вып. 10. – Том 5. – С. 123 – 126.

17. **Green R.J.** A plasticity theory for porous solid / R.J.Green // Int. J. Mech. Sci. – Vol.14. –1972. – P. 215 – 227.
18. **Ломакин Е.В.** Соотношения теории упругости для изотропного разномодульного тела / Е.В.Ломакин, Ю.Н.Работнов // Изв. АН СССР. МТТ. – 1978. – №6. – С. 29–34.
19. **Панферов В.М.** Теория упругости и деформационная теория пластичности для тел с различными свойствами на сжатие, растяжение и кручение / В.М.Панферов // Докл. АН СССР. – 1968. – Т. 180. – №1. – С. 41–44.
20. Зависимости между напряжениями и деформациями в нелинейно деформируемом теле. Часть 1. Основные принципы и соотношения механики деформируемого твердого тела / **В.М.Круглов**, С.В.Бакушев, А.И.Шеин, В.Т.Ерофеев, С.Д.С. Аль Дулайми, А.А.Томилов // Эксперт: Теория и практика. – 2023. – №4(23). – С. 154 – 163.
21. **Трещев А.А.** Теория деформирования и прочности материалов с изначальной и наведенной чувствительностью к виду напряженного состояния. Определяющие соотношения / А.А.Трещев. – М.; Тула: РААСН; ТулГУ, 2016. – 326 с.
22. **Трещев А.А.** Теория деформирования и прочности разносопротивляющихся материалов / А.А.Трещев. – Тула: ТулГУ, 2020. – 359 с.
23. **Трещев А.А.** Зависимость предельных состояний конструкционных материалов от вида напряженного состояния / А.А.Трещев // Известия высших учебных заведений. Строительство. – 1999. – №10. – С. 9–18.
24. **Трещев А.А.** К теории пластичности дилатирующих разносопротивляющихся материалов / А.А.Трещев // Проблемы машиностроения и автоматизации. – Международный журнал. – 2003. – №2. – С. 58–62.
25. **Тимошенко С.П.** Пластинки и оболочки / С.П.Тимошенко, С.Войновский–Кригер. – М.: Физматгиз, 1966. – 636 с.
26. **Петров В.В.** Методы расчета конструкций из нелинейно-деформируемого материала / В.В.Петров, И.В.Кривошеин. – М.: АСВ, 2009. – 208 с.

Aleksander Anatolyevich Treshchev, Doctor of Technical Sciences, Professor, Head of the Department of Construction, Building Materials and Structures, Tula State University, Corresponding Member of RAACS; 300012, Russia, Tula, Lenin Avenue. 92. E-mail: taa58@yandex.ru.

Александр Анатольевич Трещев, доктор технический наук, профессор, заведующий кафедрой «Строительство, строительные материалы и конструкции» Тульского государственного университета, член-корреспондент РААСН; 300012, г. Тула, Проспект Ленина, 92. E-mail: taa58@yandex.ru