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ОБЩАЯ ИНФОРМАЦИЯ О ЖУРНАЛЕ

International Journal for Computational Civil and Structural Engineering

(Международный журнал по расчету гражданских и строительных конструкций)

Международный научный журнал “International Journal for Computational Civil and Structural Engineering (Международный журнал по расчету гражданских и строительных конструкций)” (IJCCSE) является ведущим научным периодическим изданием по направлению «Инженерные и технические науки», издаваемым, начиная с 1999 года (ISSN 2588-0195 (Online); ISSN 2587-9618 (Print) Continues ISSN 1524-5845). В журнале на высоком научно-техническом уровне рассматриваются проблемы численного и компьютерного моделирования в строительстве, актуальные вопросы разработки, исследования, развития, верификации, апробации и приложений численных, численно-аналитических методов, программно-алгоритмического обеспечения и выполнения автоматизированного проектирования, мониторинга и комплексного наукоемкого расчетно-теоретического и экспериментального обоснования напряженно-деформированного (и иного) состояния, прочности, устойчивости, надежности и безопасности ответственных объектов гражданского и промышленного строительства, энергетики, машиностроения, транспорта, биотехнологий и других высокотехнологичных отраслей.

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Журнал становится технологичнее. Издательство АСВ с сентября 2016 года является членом Международной ассоциации издателей научной литературы (Publishers International Linking Association (PILA)), осуществляющей свою деятельность на платформе CrossRef. Оригинальным статьям, публикуемым в журнале, будут присваиваться уникальные номера (индексы DOI – Digital Object Identifier), что значительно облегчит поиск метаданных и местонахождение полнотекстового произведения. DOI – это система определения научного контента в сети Интернет.

С октября 2016 года стал возможен прием статей на рассмотрение и рецензирование через онлайн систему приема статей Open Journal Systems на сайте журнала (электронная редакция): http://ijccse.iasv.ru/index.php/IJCCSE.

Автор имеет возможность следить за продвижением статьи в редакции журнала в личном кабинете Open Journal Systems и получать соответствующие уведомления по электронной почте.

В феврале 2018 года журнал был зарегистрирован в Directory of open access journals (DOAJ) (это один из самых известных поисковых сервисов в мире, который предоставляет открытый доступ к материалам и индексирует не только заголовки журналов, но и научные статьи), в сентябре 2018 года включен в продукты EBSCO Publishing.

В ноябре 2020 года журнал начал индексироваться в международной базе Scopus.

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ANALYSIS OF COMBINED DISKS WITH PIECEWISE THICKNESS

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Abstract: The combined constructions subjected to an action of expanding loads and consisting of separate sections are examined. Each of the mentioned sections has its own rigidity. These parts may be made from the same or from the various materials. The materials can be anisotropic or isotropic, homogeneous or inhomogeneous. The constructions under study have the round scheme and they are considered as circular disks with piecewise thickness. In the places of the separate parts conjugation the disks' thickness can be discontinuous or continuous. The analytical approach is used. The solutions are obtained in closed form and expressed in terms of Legendre functions, Legendre, Gegenbauer and Laguerre polynomials.

Keywords: combined disks, piecewise thickness, special functions.

РАСЧЕТ КОМБИНИРОВАННЫХ ДИСКОВ КУСОЧНО-ПЕРЕМЕННОЙ ТОЛЩИНЫ

Е.Б. Коренева

Московское высшее общевойсковое командное орденов Жукова, Ленина и Октябрьской Революции Краснознаменное училище, г. Москва, РОССИЯ

Аннотация: Изучаются комбинированные конструкции, работающие преимущественно на растяжение и состоящие из отдельных участков, каждый из которых обладает своим законом изменения жесткости. Эти участки могут быть сделаны из одного и того же или из различных материалов. Эти материалы могут быть анизотропными и изотропными, однородными и неоднородными. В рассматриваемых конструкциях в местах соединения отдельных частей толщина может быть непрерывной или иметь разрывы непрерывности. Изучаемые конструкции имеют в плане круговую форму и рассматриваются как круглые диски кусочно-переменной толщины. В данной работе для расчета подобных конструкций впервые используется аналитическая методика. Решения получены в замкнутом виде и выражены в функциях Лежандра, в полиномах Лежандра, Гегенбауэра, Лагерра.

Ключевые слова: комбинированные диски, кусочно-переменная толщина, специальные функции.

1. INTRODUCTION

In literature the considerable number of works are devoted for computation of plates and shells of various forms. For example the monographies [1] and [2] are to be mentioned. In the monography [3] the orthotropic and isotropic plates of variable thickness, subjected to the action of complicated loads are examined; the analytical methods were applied.

The modern software allows to investigate the similar constructions in detail. The numerical methods, in particular, the finite elements method, are widely used. The work [4] concerns the problem of buckling of orthotropic plates with free and rotationally restrained edges. 3D vibration of cross-ply laminated plates is studied in [5]. The oscillation problems of isotropic and orthotropic rectangular plates of linear thickness are considered in [6]. The work [7] is devoted to the numerical analysis of experimental research on buckling of closed shallow conical shells under external pressure. Free vibration analysis of a rotating varying-thickness-twisted blade with arbitrary boundary conditions is examined in [8]. The article [9] considers the optimization of three-dimensional up to yield bending behaviour using the full layer-wise theory for FGM rectangular plate subjected to thermo-mechanical loads. Comparative assessment of finite element modelling techniques for wind turbine rotors blades is represented in [10]. The work [11] concerns nonlinear primary resonance analysis of nanoshells. Geometrical influence on the vibration of layered plates is discussed in [12].

Some problems of statics, vibration and stability of thin-walled constructions are solved in [13] by the use of the equation decomposition method.

The elements with piecewise variable thickness occur in modern structures and buildings. First the analytical approach for the solution of similar problems was proposed in the works [14], [15]. In the mentioned works the circular plates resting on an elastic basis are examined. The inner part of these plates has the variable thickness and the outer part has the constant thickness. The conditions of the parts conjugation were fulfilled. The solutions were obtained in terms of Bessel functions. The problems of symmetric flexure of orthotropic and isotropic combined plates with piecewise thickness were considered in [16], [17]. The separate parts of these plates have various laws of cylindrical rigidity variation. In [16], [17] the solutions were obtained in terms of Gegenbauer and Laguerre polynomials.

In the present work the analytical method for the first time is applied for the analysis of the combined circular disks with piecewise variable thickness subjected to an action of expanding loads. The solutions of the problems under study are obtained in closed forms and expressed in terms of Legendre functions; Legendre, Gegenbauer and Laguerre polynomials.

2. THE BASIC SOLUTIONS EXPRESSED IN TERMS OF LEGENDRE FUNCTIONS

As it was mentioned above, the circular disks with piecewise variable thickness subjected to an action of expanding loads are analyzed.

We will write the differential equation, describing the symmetric deformation of the circular isotropic disks with the radially variable thickness and loaded by the surface stretching radial forces with the intensity q:

$$\frac{d^2N_r}{dx^2}+\left(\frac{1+\frac{2}{\alpha_0}}{x}-\frac{1}{D}\frac{dD}{dx}\right)\frac{dN_r}{dx}-$$

$$-\frac{1-\sigma}{\alpha_0 x}\frac{1}{D}\frac{dD}{dx}N_r+\frac{r_0 x^{\frac{1}{\alpha_0}-2}q_r}{\alpha_0^2}\times \qquad (1)$$

$$\times\left(2+\sigma-\alpha_0\frac{x}{D}\frac{dD}{dx}\right)=0,$$

where $D=\frac{Eh(x)}{1-\sigma^2}$ is the cylindrical rigidity for the tension; N_r - the normal stress; σ - Poisson's ratio; $x=\left(\frac{r}{r_0}\right)^{\alpha_0}$; r_0, α_0 - the parameters.

Further we will determine the laws of cylindrical rigidity variation which allow to receive the solutions in terms of Legendre functions. For this aim we compare the coefficients of the homogeneous differential equation, corresponding to (1), with the coefficients of the Legendre differential equation [3], [18], [19]:

$$\frac{d^2u}{dx^2}-\frac{2x}{1-x^2}\frac{du}{dx}\left[\frac{\nu(\nu-1)}{1-x^2}-\frac{\mu^2}{\left(1-x^2\right)^2}\right]u=0, \qquad (2)$$

where μ and ν are the parameters of the Legendre functions. Producing the comparision, we have

$$\frac{1+\frac{2}{\alpha_0}}{x}-\frac{1}{D}\frac{dD}{dx}=-\frac{2x}{1-x^2},$$

$$-\frac{1-\sigma}{\alpha_0 x}\frac{1}{D}\frac{dD}{dx}=\frac{\nu(\nu+1)}{1-x^2}-\frac{\mu}{\left(1-x^2\right)^2}.$$

We get that for the values of the parameters

$$\alpha_0 = -2,\ \mu = 0,\ \nu_{1,2} = \pm\sqrt{\frac{1}{4} + (1-\sigma)} \quad (3)$$

and for the following law for the rigidity for the tension

$$D = D_0\left(1 - x^2\right)^{-1}, \quad (4)$$

which corresponds to the thickness

$$\begin{aligned} h &= h_0\left(1 - x^2\right)^{-1}, 0 \le x < 1; \\ h &= h_0\left(x^2 - 1\right)^{-1}, 1 \le x < \infty; \end{aligned} \quad (5)$$

the solution of the homogeneous differential equation, corresponding to (1), has the form:

$$N_r = AP_\nu(x) + BQ_\nu(x). \quad (6)$$

Then we compare the coefficients of the corresponding to (1) differential equation with the coefficients of another Legendre equation:

$$\frac{d^2 v}{dx^2} - \frac{2(\mu+1)x}{1-x^2}\frac{dv}{dx} + \frac{(\nu-\mu)|\nu+\mu+1|}{1-x^2} v = 0, \quad (7)$$

which comes out of (2) by means of the following substitution:

$$u = \left(z^2 - 1\right)^{\frac{\mu}{2}} v.$$

As a result we get the following expression for the parameters:

$$\alpha_0 = -2,\ \nu_{1,2} = \frac{1}{2} \pm \sqrt{\frac{1}{4} + (1-\sigma-\mu)(\mu+1)}, \quad (8)$$

where μ is the arbitrary value.
We get the rigidity for the tension

$$D = D_0\left(1 - x^2\right)^{-(\mu+1)}, \quad (9)$$

as a result the thickness of disks is expressed in the form:

$$\begin{aligned} h &= h_0\left(1 - x^2\right)^{-(\mu+1)}, 0 \le x < 1; \\ h &= h_0\left(x^2 - 1\right)^{-(\mu+1)}, 1 \le x < \infty. \end{aligned} \quad (10)$$

In this case the solutions are represented in terms of adjoined Legendre functions:

$$N_r = \left[AP_\nu^\mu(x) + BQ_\nu^\mu(x)\right]\left(x^2 - 1\right)^{-\frac{\mu}{2}}. \quad (11)$$

We mark that the use of another Legendre equations for the consideration of disks of variable thickness symmetric deformation will not give any results.
Some disks' profiles for the cases when the solutions are given in terms of the adjoined Legendre functions are shown on the fig.1.

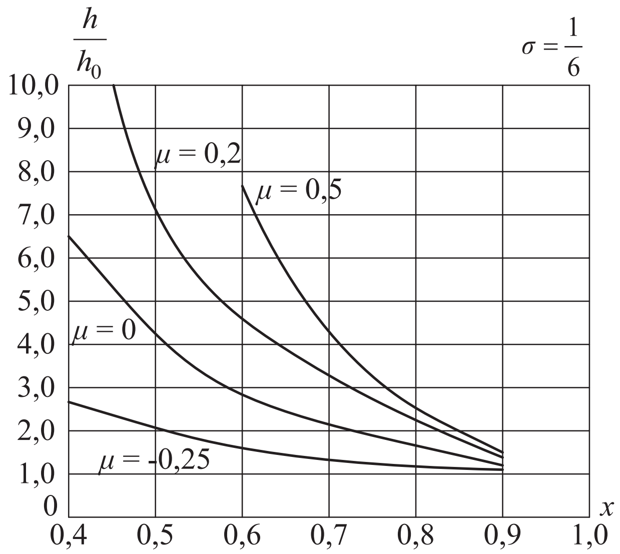

Figure 1. Some disks' profiles for the cases when the solutions are obtained in terms of the adjoined Legendre functions

We note the special case when the solutions are given in terms of the cone functions. We set $\nu = -\frac{1}{2} + ip$. Then the solution can be written in the following form:

$$N_r = AP^{\mu}_{-\frac{1}{2}+ip} + BQ^{\mu}_{-\frac{1}{2}+ip}\, x\,. \qquad (12)$$

The case of specific interest is when one of the adjoined Legendre functions parameters $\mu = \pm\left(\frac{1}{2}+n\right)$, when n is the positive integer; then the solutions of the equation (1) for the rigidities (4) and (9) are expressed in terms of Legendre polynomials. For example, the adjoined Legendre functions $P_{\nu}^{\frac{1}{2}}(x)$, $P_{\nu}^{-\frac{1}{2}}(x)$, $Q_{\nu}^{\frac{1}{2}}(x)$, $Q_{\nu}^{-\frac{1}{2}}(x)$ can be represented by the following formulae:

$$P_{\nu}^{\frac{1}{2}}(x) = \frac{1}{\sqrt{2\pi}}\left(x^2-1\right)^{-\frac{1}{4}} \times$$
$$\times\left[\left(x+\sqrt{x^2-1}\right)^{\nu+\frac{1}{2}} + \left(x+\sqrt{x^2-1}\right)^{-\nu-\frac{1}{2}}\right];$$
$$Q_{\nu}^{\frac{1}{2}}(x) = i\sqrt{\frac{\pi}{2}}\left(x^2-1\right)^{-\frac{1}{4}}\left(x+\sqrt{x^2-1}\right)^{-\nu-\frac{1}{2}};$$
$$P_{\nu}^{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi}}\frac{\left(x^2-1\right)^{-\frac{1}{4}}}{2\nu+1} \times \qquad (13)$$
$$\times\left[\left(x+\sqrt{x^2-1}\right)^{\nu+\frac{1}{2}} - \left(x+\sqrt{x^2-1}\right)^{-\nu-\frac{1}{2}}\right];$$
$$Q_{\nu}^{-\frac{1}{2}}(x) = i\sqrt{2\pi}\,\frac{\left(x^2-1\right)^{-\frac{1}{4}}}{2\nu+1}\left(x+\sqrt{x^2-1}\right)^{-\nu-\frac{1}{2}},$$

where $x \geq 1$.

Using the recurrence relations

$$(\nu-\mu+1)P_{\nu+1}^{\mu}(x) - (\nu+\mu+1)xP_{\nu}^{\mu}(x) =$$
$$= \sqrt{x^2-1}P_{\nu}^{\mu}(x);$$
$$P_{\nu-1}^{\mu}(x) - xP_{\nu}^{\mu}(x) = -(\nu-\mu+1)\sqrt{x^2-1}P_{\nu}^{\mu-1}(x);$$
$$xP_{\nu}^{\mu}(x) - P_{\nu+1}^{\mu}(x) = -(\nu+\mu)\sqrt{x^2-1}P_{\nu}^{\mu-1}(x),$$

from these formulae the expressions for the functions $P_{\nu}^{\frac{3}{2}}(x)$, $P_{\nu}^{\frac{5}{2}}(x)$ etc. can be consequently received. For instance, we can write:

$$Q_{\nu}^{1,5}(x) = \frac{i\sqrt{\pi}}{\sqrt{2}}\left(x^2-1\right)^{-\frac{3}{4}}\left(x+\sqrt{x^2-1}\right)^{-\nu-\frac{1}{2}} \times$$
$$\times\left[x + \left(\nu+\frac{1}{2}\right)\sqrt{x^2-1}\right];$$
$$P_{\nu}^{1,5}(x) = \frac{1}{\sqrt{2}}\left(x^2-1\right)^{-\frac{3}{4}}\left(x+\sqrt{x^2-1}\right)^{-\nu-\frac{1}{2}} \times$$
$$\times\left[\left(\nu-\frac{1}{2}\right)x\left(x+\sqrt{x^2-1}\right) - \left(\nu+\frac{1}{2}\right) - x -\right.$$
$$\left. - \left(\nu+\frac{1}{2}\right)\sqrt{x^2-1}\right].$$

Next the particular solution of inhomogeneous equation (1) is to be considered. For this purpose the Cauchy functions for the solution received above are to be obtained.

The Wronskian for the solutions of the Legendre equation is used for this aim:

$$W(x) = \frac{K(\mu,\nu)}{1-x^2};$$
$$K(\mu,\nu) = \qquad (14)$$
$$= \frac{e^{i\mu\pi}2^{2\mu}\Gamma\left(1+\frac{\mu}{2}+\frac{\nu}{2}\right)\Gamma\left(\frac{1}{2}+\frac{\mu}{2}+\frac{\nu}{2}\right)}{\Gamma\left(1+\frac{\nu}{2}-\frac{\mu}{2}\right)\Gamma\left(\frac{1}{2}+\frac{\nu}{2}-\frac{\mu}{2}\right)},$$

where $\Gamma(x)$ - gamma-function.

The Cauchy functions for the solution (6) and the rigidity (4) are:

$$Y_1(x_1;x) = \nu\{-[x_1Q_{\nu}(x_1) - Q_{\nu-1}(x_1)]P_{\nu}(x) +$$
$$+ [x_1P_{\nu}(x_1) - P_{\nu-1}(x_1)]Q_{\nu}(x)\};$$
$$Y_2(x_1;x) = \left(x_1^2-1\right)[Q_{\nu}(x_1)P_{\nu}(x) - \qquad (15)$$
$$- P_{\nu}(x_1)Q_{\nu}(x)],$$

in this case $\mu = 0$ and, hence, $K(\mu,\nu) = 1$.

The Cauchy functions for the solution (11) and the disk's rigidity (9) are:

$$Y_1(x_1;x)=\frac{\left(x_1^2-1\right)^{-\frac{\mu}{2}}\left(x^2-1\right)^{-\frac{\mu}{2}}}{K(\mu,\nu)}\times$$
$$\times\left\{-\left[(\nu-\mu)x_1Q_\nu^\mu(x_1)-(\nu+\mu)Q_{\nu-1}^\mu(x_1)\right]P_\nu^\mu(x)+\right.$$
$$\left.+\left[(\nu-\mu)x_1P_\nu^\mu(x_1)-(\nu+\mu)P_{\nu-1}^\mu(x_1)\right]Q_\nu^\mu(x)\right\}; \quad (16)$$
$$Y_2(x_1;x)=\frac{\left(x_1^2-1\right)^{-\frac{\mu}{2}+1}\left(x^2-1\right)^{-\frac{\mu}{2}}}{K(\mu,\nu)}\times$$
$$\times\left\{Q_\nu^\mu(x_1)P_\nu^\mu(x)-P_\nu^\mu(x_1)Q_\nu^\mu(x)\right\}.$$

For the real x, μ, ν the Wronskian $W(x)$ is complex-valued, if $\mu\neq\frac{n+1}{2}$, where n is integer. In this case $W(x)$ contains the factor $e^{i\mu\pi}$. However the Cauchy functions will be real, because the adjoined Legendre function contains the similar factor.

Thus, the particular solution of the inhomogeneous equation (1) is:

$$N_C=\int_{x_1}^{x}f(z)Y_2(z;x)dz\,, \quad (17)$$

where $f(z)$ is the right part of (1).

It is recommended to determine numerically the values of (17) for the actual parameters.

Next we go to the cases when the solutions are expressed in terms of orthogonal polynomials.

3. THE BASIC SOLUTIONS IN TERMS OF GEGENBAUER POLYNOMIALS

For receiving of the solutions we compare the coefficients of the homogeneous equation, corresponding to (1), with the coefficients of Jacobi equation [18], [20]. The analysis shows that the solution is possible, when the parameters $\alpha=\beta$; it corresponds to the case of the ultraspherical Gegenbauer polynomials $C_m^\lambda(x)$.

The differential equation for the Gegenbauer polynomials is

$$y''-\frac{2\lambda+1}{1-x^2}xy'+\frac{m(m+2\lambda)}{1-x^2}y=0\,. \quad (18)$$

Fulfilling the above-mentioned comparision, we get the following parameters

$$\alpha_0=-2,$$
$$m_{1,2}=\lambda\pm\sqrt{\lambda^2+\frac{(1-\sigma)}{2}(2\lambda+1)} \quad (19)$$

and the rigidity for the tension is

$$D=D_0\left(1-x^2\right)^{-\left(\lambda+\frac{1}{2}\right)},0\le x<1;$$
$$D=D_0\left(x^2-1\right)^{-\left(\lambda+\frac{1}{2}\right)},0\le x<\infty. \quad (20)$$

The solution of the homogeneous equation has the following form:

$$N_1=AC_m^\lambda(x)+B\left(\frac{1-x}{2}\right)^{\frac{1}{2}-\lambda}\times$$
$$\times F\left(-n-\lambda+\frac{1}{2},n+\lambda+\frac{1}{2};\frac{3}{2}-\lambda;\frac{1-x}{2}\right), \quad (21)$$

where $F(\ )$ is the hypergeometric function.

For the disks with the rigidity (20) and the solution (21) the Cauchy functions are to be determined. The expression for Wronskian is

$$W(x)=\frac{\Gamma(m+2\lambda)}{m!\,\Gamma(2\lambda)}\left(\frac{1}{2}-\lambda\right)2^{2\lambda+1}\times$$
$$\times\left(1-x^2\right)^{-\lambda-\frac{1}{2}}. \quad (22)$$

Further we obtain the following formulae for the Cauchy functions:

$$Y_1(x_1;x)=\frac{m!\,\Gamma(2\lambda)\left(1-x_1^2\right)^{\lambda+\frac{1}{2}}}{2^{2\lambda+1}\left(\frac{1}{2}-\lambda\right)\Gamma(m+2\lambda)}\left\{\left[\left(\frac{\lambda}{2}-\frac{1}{4}\right)\left(\frac{1-x_1}{2}\right)^{-\lambda-\frac{1}{2}}F\left(-m-\lambda+\frac{1}{2},m+\lambda+\frac{1}{2};\frac{3}{2}-\lambda;\frac{1-x_1}{2}\right)-\right.\right.$$

$$\left.-\left(\frac{1-x_1}{2}\right)^{-\frac{1}{2}-\lambda}\frac{\frac{1}{4}-(m+\lambda)^2}{\frac{3}{2}-\lambda}F\left(-m-\lambda+\frac{3}{2},m+\lambda+\frac{3}{2};\frac{5}{2}-\lambda;\frac{1-x_1}{2}\right)\right]C_m^\lambda(x)+$$

$$\left.+\left[\frac{mx_1}{1-x_1^2}C_m^\lambda(x_1)-\frac{m+2\lambda-1}{1-x_1^2}C_{m-1}^\lambda(x_1)\right]\left(\frac{1-x}{2}\right)^{\frac{1}{2}-\lambda}F\left(-m-\lambda+\frac{1}{2},m+\lambda+\frac{1}{2};\frac{3}{2}-\lambda;\frac{1-x}{2}\right)\right\}; \quad (23)$$

$$Y_2(x_1;x)=\frac{n!\,\Gamma(2\lambda)\left(1-x_1^2\right)^{\lambda+\frac{1}{2}}}{2^{2\lambda+1}\left(\frac{1}{2}-\lambda\right)\Gamma(m+2\lambda)}\left\{\left(\frac{1-x_1}{2}\right)^{\frac{1}{2}-\lambda}F\left(-m-\lambda+\frac{1}{2},m+\lambda+\frac{1}{2};\frac{3}{2}-\lambda;\frac{1-x_1}{2}\right)C_m^\lambda(x)+\right.$$

$$\left.+C_m^\lambda(x_1)\left(\frac{1-x}{2}\right)^{\frac{1}{2}-\lambda}F\left(-m-\lambda+\frac{1}{2},m+\lambda+\frac{1}{2};\frac{3}{2}-\lambda;\frac{1-x}{2}\right)\right\}.$$

The particular solution of the inhomogeneous equation is determined by means of the expression (17).

4. THE BASIC SOLUTIONS IN TERMS OF LAGUERRE POLYNOMIALS

Let us determine the possibility to obtain the solutions in terms of Laguerre polynomials $L_m^{(\alpha)}(x)$ [18], [20]. The differential equation for these polynomials is

$$y''+\frac{\alpha+1-x}{x}y'+\frac{m}{x}y=0. \quad (24)$$

After the described above transformations we get the following parameters:

$$\alpha_0=-\frac{2}{\alpha},\ \alpha=\frac{2m}{1-\sigma}. \quad (25)$$

The rigidity for the tension is

$$D=D_0\mathrm{e}^x. \quad (26)$$

The general solution of the homogeneous equation for the case under study is

$$N_r=AL_m^\alpha(x)+Bx^{-\alpha}{}_1F_1(-n-\alpha;1-\alpha;x), \quad (27)$$

where ${}_1F_1(x)$ is the confluent hypergeometric function.

The Wronskian for the solution (27) is determined by the following formula:

$$W(x)=\binom{m+\alpha}{m}\alpha x^{-\alpha-1}e^x,$$

where

$$\binom{m+\alpha}{m}=\frac{(m+\alpha-m+1)m}{m!}=\frac{(\alpha+1)m}{m!};$$
$$(\alpha+1)_m=(\alpha+1)(\alpha+2)...(\alpha+m). \quad (28)$$

The Cauchy functions for the solutions (27) are

$$\begin{aligned} Y_1(x_1;x) &= \binom{m+\alpha}{m}^{-1} \alpha^{-1} e^{-x_1} x^{\alpha+1} \times \\ &\times \Big\{ \Big[\alpha x_1^{-\alpha-1} {}_1F_1(-m-\alpha;1-\alpha;x_1) + \\ &+ x_1^{-\alpha} \frac{m+\alpha}{1-\alpha} {}_1F_1(-m-\alpha+1;2-\alpha;x_1) \Big] \times \\ &\times L_m^{\alpha}(x) - \left[\frac{m}{x_1} L_m^{\alpha}(x_1) - \frac{m+\alpha}{x_1} L_{m-1}^{\alpha}(x_1) \right] \times \\ &\times x^{-\alpha} F(-m-\alpha;1-\alpha;x) \Big\}; \\ Y_2(x_1;x) &= \binom{m+\alpha}{m}^{-1} \alpha^{-1} e^{-x_1} x^{\alpha+1} \times \\ &\times \Big\{ -L_m^{(\alpha)}(x_1) x^{\alpha} {}_1F_1(-m-\alpha;1-\alpha;x) + \\ &+ x_1^{\alpha} {}_1F_1(-m-\alpha;1-\alpha;x_1) L_{-m}^{\alpha}(x) \Big\}. \end{aligned} \quad (29)$$

Here, the particular solution is also determined with the use of the formula (17), where the function $Y_2(x_1;x)$ is determined by means the expression (29).

5. THE STATEMENT OF THE COMPUTATION PROBLEM OF THE CIRCULAR DISK WITH PIECEWISE THICKNESS

The disks, subjected to an action of the stretching loads and consisting of two sections with different laws of thickness variation, are under study. The disks' profile has a gap in the place of these parts conjugation (fig.2).

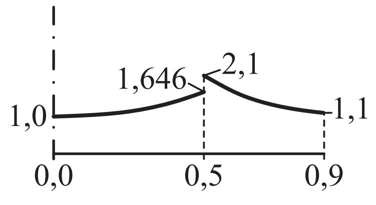

Figure 2. The profile of the disk with piecewise thickness

The rigidity for the tension in the first section is approximated by the formula (26). The normal stress when $0{,}0 \le x \le 0{,}5$ is determined in the following way:

$$\begin{aligned} N_r^{(1)} &= A_1 L_m^{\alpha}(x) + B_1 x^{-\alpha} {}_1F_1(-n-\alpha;1-\alpha;x) + \\ &+ N_C^{(1)}(x). \end{aligned} \quad (30)$$

The rigidity for the tension in the second section is approximated by the formula (9). We have when $0{,}5 \le x \le 0{,}9$:

$$\begin{aligned} N_r^{(2)} &= N_r^{(1)} + \left[A_2 P_\nu^{\mu}(x) + B_2 Q_\nu^{\mu}(x) \right] \times \\ &\times \left(x^2-1\right)^{-\frac{\mu}{2}} + N_C^{(2)}(x), \end{aligned} \quad (31)$$

where $N_C^{(1)}$, $N_C^{(2)}$ are the particular solutions determined by means the expression (17).

6. THE CONCLUSION

In the present work the exact analytical solutions of computational problems of circular disks with piecewise variable thickness, subjected to an action of expanding loads, are obtained. The constructions under study consist of two or several sections. Each of the mentioned parts has its own law of thickness variation. These sections can be made from the same or from the different materials, which can be homogeneous or inhomogeneous, isotropic and anisotropic. In the places of conjugation the disk's thickness can be continuous or discontinuous. The received solutions are obtained in terms of Legendre functions and Legendre, Gegenbauer and Laguerre polynomials.

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INTELLIGENT AND DIGITAL TECHNOLOGIES IN THE CONSTRUCTION OBJECTS TECHNICAL DIAGNOSTICS

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Abstract. The intelligent and digital technologies implementation into the civil engineer expert activities is able to provide alternative solutions to different qualifications specialists. This will increase the speed of data processing and the reliability of the expert opinion on the technical condition of the operated construction objects as well as will allow assessing their real residual resource for making a decision on the possibility of further exploitation. The ontology and the original technology for the main system components determination are necessary to achieve the goal (the confinement model) used for the structural and functional analysis of the knowledge system. The entire technologies set for the architectural and construction objects (digital documentation, scan results, thermal imager data, non-destructive survey methods data), data about structure's defects and damage, appropriate software as well as intelligent technologies (fuzzy logic, neural networks) are used for more thoroughly diagnose certain construction parts and transmit digital information to determine the technical condition category. Also, this data set can be used for following situations: a control of the dynamics of changes in the technical state of a construction object, an improvement the accuracy of determining the scope of repair work, an enhancement the quality of project documentation, methods for assessing the quality of restoration work and measures for the conservation of architectural monuments, etc.

Keywords: intelligent and digital technologies, ontology, confinement-model, architectural and construction objects, category of technical condition.

ИНТЕЛЛЕКТУАЛЬНЫЕ И ЦИФРОВЫЕ ТЕХНОЛОГИИ В ТЕХНИЧЕСКОЙ ДИАГНОСТИКЕ ОБЪЕКТОВ СТРОИТЕЛЬСТВА

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Аннотация. Внедрение интеллектуальных и цифровых технологий в экспертную деятельность инженеров-строителей способно предоставлять альтернативные решения специалистам разной квалификации. Это повышает скорость обработки данных и достоверность экспертного заключения о техническом состоянии эксплуатируемых объектов строительства для принятия решения о возможности их дальнейшей эксплуатации, а также позволяет оценить их реальный остаточный ресурс. Для структурного и функционального анализа системы знаний использованы: онтологический анализ и оригинальная технология (конфайнмент-моделирование) определения основных компонентов системы, необходимых для достижения цели - определения категории технического состояния строительного объекта. Для более тщательной диагностики отдельных деталей конструкций архитектурных и строительных объектов и передачи цифровой информации используется полный набор технологий (цифровая документация, результаты сканирования, данные тепловизора, данные методов неразрушающего контроля), данные о дефектах и повреждениях конструкции, соответствующее программное обеспечение, а также интеллектуальные технологии (нечеткая логика, нейронные сети). Также, этот набор данных может быть использован для: контроля динамики изменения технического состояния объекта строительства, повышения точности определения объема ремонтных работ, повышения качества проектной документации, способов оценки качества реставрационных работ и мероприятий по консервации памятников архитектуры и др.

Ключевые слова: интеллектуальные и цифровые технологии, онтология, конфайнмент-модель, архитектурно-строительные объекты, категория технического состояния

1. INTRODUCTION

Exploitation of any construction structure in accordance with the specific regulations demands technical diagnostics conduction. Often, conclusions of the technical examination are approved in the short time under uncertainty. Decisions are usually based on the opinions of the specialists considering their professional experience. Lack of sufficient knowledge leads to the erroneous conclusions and might be the cause of the premature failure of the construction object. Accordingly, there is growing attention to the development of intelligent expert systems in the complex field, such as diagnostics, inspection and monitoring of buildings structures. Such systems serve as a tool for information support for the development and decision-making processes. Deep knowledge in the field of construction and experience of an expert are required to address the problems of constructions survey, such as the complex nature of the building structures along with lack of regulatory documents connections, multiple, incomplete, inaccurate and contradictory results of an engineering object diagnostics, insufficient formulated assessment criteria, the problem of the "length of the technical conditions scale" (GOST (State Standard) scale, including 4 categories) and the blurring of the categories boundaries [1].

The intelligent and digital technologies introduction into the civil engineer expert activities provides the alternative solutions to the different qualifications specialists. The main task of the intelligent technologies is the knowledge processing related to solving complex issues in which logical (or semantic) information prevails over computational information [2, 3]. Taking the implementation into the process of technical condition diagnostic an additional technological level is advisable for this purpose. It should include the computer processing of the accumulated knowledge and practical experience. At the same time the knowledge embedded in a computer-oriented knowledge-base (KB) must be organized and formalized so that the user can quickly receive the necessary information. The structure of a comprehensive intelligent system for diagnostics of mass construction objects elaborated considering the current level of the computer technology development along with

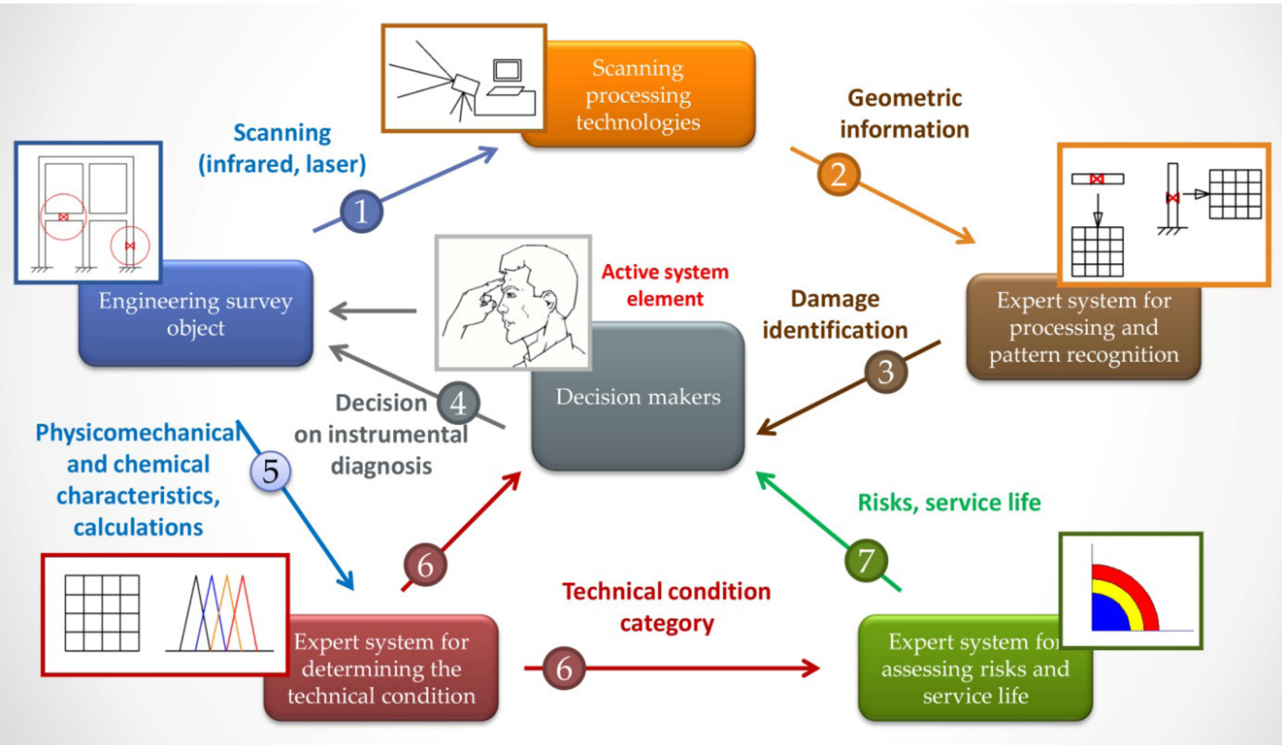

Figure 1. The structure of an integrated intelligent system for diagnostics of construction objects.

evolution of the telemetry methods, digital and intelligent technologies. The developed structure consists of several expert systems (ES) which can operate separately or as a whole system (Fig. 1).
Integration of the buildings technical condition diagnostics results into a holistic system mixed with the telemetry possibilities, laser scanning, intelligent and digital technologies, mathematical modeling enable to accelerate data processing and increase the reliability of the expert opinion about the technical condition of the exploited building constructions. Such system based on the identification of defects in load-bearing and enclosing structures as well as foundations and other constructions, also will allow evaluating the real residual constructions resource for decisions about the reconstruction, overhaul and destruction. An integrated approach to solving all these problems will allow obtaining a multiplier effect, while each included expert system expands due to the other system elements capabilities. In the near prospect the expert systems and neural network combination with the traditional programming technology will provide a revolutionary breakthrough in the applications integration of ready-made intelligently interacting modules.
The current state of the expert systems and neural networks development in the construction sector can be characterized as a stage of the increasing interest as evidenced by the many publications of the authors: D.A. Pospelov, T.A.Gavrilova, V.F. Xoroshevskij, D. Rutkovskaya, E. Bernat, L. Gil, K. M. Hamdia, K. Van Balen, H. Li, S. Shtovba, A. Rotshtein, O. Pankevich, V.A. Sokolov, T.N. Soldatenko and others.
In the technical diagnostics of the construction objects as well as in general construction sector, there are a lot of the national and international standards which must be taken into account since they form the basis for solving the ensuring safety problems of the construction facilities. This is a ready-made system model needs to be correctly represented in the formalized terminology to provide the user under certain structural rules which a computer can autonomously use in solving specific issues based on logical inference.
But at this stage, there is a certain gap between the technical development of general and detailed technical diagnostics building structures methods [4-7] and the intelligent systems methodology for organizing expert knowledge [8-14]. Consequently, there are the limitations in the possibilities of wider practical application of the intelligent systems in construction. The major difficulty is the interpretation of the entire volume of regulatory requirements into a machine-readable format.
In a certain sense the AI system simulates the intellectual activity of a person in general and the logic of the human reasoning in particular. Any intellectual person activity is inherently systemic. It involves the use of a set of the interrelated procedures on the way from setting a task and goals to finding and using solutions. And as the logic of the human expert reasoning in an intellectual system, machine processing should be easy to process; it should be structured, i.e. to represent the system of the subject area as a conceptual diagram. This requires: professional knowledge (theoretical and practical); understand the thinking process of a person – an experienced expert or a decision-maker in a specific type of activity, highlighting the main steps of this process; use a ready-made software tool that reproduces these actions.

2. ONTOLOGICAL MODELS IN THE KNOWLEDGE SYSTEM STRUCTURAL AND FUNCTIONAL ANALYSIS

Numerous models and knowledge representation methods have been proposed and tested with varying success for structuring information, such as: various logical models, semantic and neural networks, frames, production rules, etc. [13–18]. In recent years meaningful ontology works has been developed implementing new processing and presenting information methods at the junction of systems analysis, artificial intelligence and applied data analysis. Such works contain conceptual schemes for organizing the knowledge system of specific subject areas and ways of targeting this knowledge [19–22]. Ontology is widely used in all fields of natural language information processing. “Ontology” is a term in the artificial intelligence theory applied by specialists as one of the main formalisms of knowledge representation. It means formally presented knowledge in the form of anarray objects description as well as concepts and connections between them. In the simplest form the ontology composition is reduced to the basic concepts allocation for a specific subject area and the construction of links between themas the definition of relationships and interactions of basic concepts. An important feature of the ontological models in intelligent systems is that they are designed to be processed by agent software. All information needs to be presented in a formal form since a computer cannot understand the situation in the world as a person. For this purpose all concepts must be linked and patterns must be established, i.e. the expert's conclusions structure (frame) has to be identified. Domain ontology is usually built by domain experts or with their assistance. Meanwhile significant is not the concepts themselves as people's knowledge about these concepts and their exploitation by people. The logical and associative theories of thinking as two most popularapply as a basis forreconstruction an expert's reasoning. Traditional logic forms criteria that guarantee the accuracy, consistency of general concepts of reasoning and conclusions (classification, generalization, comparison, categorization, inference, abstraction, etc.). But a person rarely thinks in terms of mathematical logic. Thinking is a chain of ideas connected by general concepts. The main operations of such thinking are: associations acquired on the various connections basis, recalling past experiences, trial and error with random successes, etc. It all depends on the specific tasks, source material, and subject area complexity. Ontology allows restoring missing logical connections providing a systematic approach to the subject area study and makes inferences based on presented information. Ontological models are considered as knowledge bases of a special kind.

Any model is always a reality simplificationwith inherent fragmentation. In the process of identification of the various pieces of knowledge the main thing needs to be considered as follow: a basis for understanding processes and phenomena in relation with the development of knowledge about the subject area defines through the "center of the situation" as some holistic image, structure or important criterion. In meeting real challenges, the domain ontology can be constructed in various ways. It is important to know how to highlight the main lines.

3. THE CONFINEMENT MODELLING TECHNOLOGY AS AN APPLIED TOOL FOR THE MAIN SYSTEM COMPONENTS DETERMINATION

T.V. Gagin [23] proposed an original method for determination of the main system components which essential to achieve the stated objective and studying their influence on each other. This fundamentally stable model of the self-reproducing system developers called the Confinement® model. The “confinement” is the term taken from physics and literally means retention (confinement). The formation of ontology using confinement modelling is a system-cognitive analysis procedure. The domain model is represented in a sign oriented graph form with feedbacks (Fig. 2). Various events or key elements of the situation are located at the vertices of the graph. The arcs connecting the vertices represent the causal relationships between them. The essence of this approach lies in the presence of the fact that the system is closed in the form of a system loop, which contains the necessary and sufficient

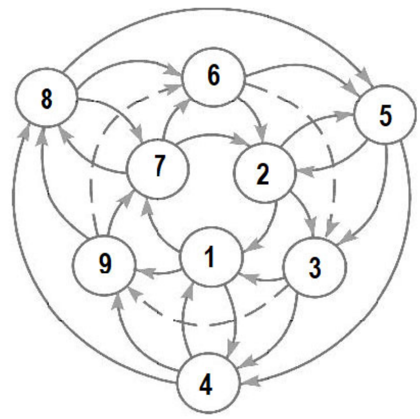

Figure 2. The confinement model.

direct and reverse cause-and-effect relationships. The elements of the system loop are the focus of attention where to look and where to search. Meanwhile is considered that the result is generally known, and the model is focused on the strategy for achieving this result.
Recently this technology actively used in various sectors scientific developments [24, 25]. The confinement system allows seeing key points and important relationships as well as helps to highlight the most important and necessary information for solving a certain issue, i.e., makes it possible to take a systematic look at the process. Filling of the confinement model at the level of common sense (the numbers of the elements correspond to the order of their determination) allows discovering not only the elements necessary for understanding the system, but also their most important relationships: each element in its place should be logically linked to six others: three should call it, and three - to follow from it.
Accordingly, each element is both an effect and a cause. (1) is the goal and the result of the system. (2), (3) and (4) are the immediate causes of this result. At the same time the cause (2) in addition to the result itself should strengthen the cause (3) which enhances the cause (4). The element (4) itself must call the result and be strengthened by it. (5) is the element which have to simultaneously calls all three previous reasons. It is important that all three reasons. Reason (5) is related to (2). (6) is the reason for (2) and (5) at the same time. (7) has to call (6) and (2). (8) is the element interconnected with (7) and calls (6) and (5). (9) is the important element which should close the system and this is a serious test of our discourse. It is the cause for (7) and (8), being simultaneously a consequence of both cause (4) and the main result of system (1). The links indicated in the diagram (fig. 2) by the dotted line are described by the word "calls". In the finished diagram they have to close the middle circle: (3) - (9) - (6). The confinement model is clearly layered into three circles: inner, middle and outer. The inner and outer circles are practically equal. The central one is considered to be the manager and serves as a connecting link, transferring the influence from the inner circle to the outer one and vice versa.
There are three sectors within the model. They are marked with different colours: "R" - red, "G" - green, "B" - blue. Elements in the "R" sector (1) - (4) - (9) describe the "key results" and "specifics". These are the most "real" and obvious parts of the system. The "G" sector (6) - (7) - (8) is an incentive for the transformation of the specifics of the "R" sector into actions and the conclusions of the "B" sector (2) - (3) - (5) which are the "procedural-thinking" sector covering the area of decisions, rules and conclusions. The logic of checks suggests the magical role of the element which is simultaneously located in the "control" central circle and the "transforming" G sector (the element (6) in the scheme). It often turns out to be significant and decisive.

4. CASE OF THE CONFINEMENT MODELLING APPLICATION TO CONSTRUCT THE CONCEPTUAL MODEL STRUCTURE FOR DIAGNOSING THE BUILDINGS AND STRUCTURES TECHNICAL STATE

Technical diagnostics of construction objects is carried out through a combination of mutually consistent and complementary practical, calculation, research and analytical procedures. Let's consider one example of the conceptual model forming process of an intelligent control system for the decision-making process on the buildings and structures technical condition (Fig. 3), using the technology of confinement modelling and a ready-made template [23].
The main criteria of damages type decisions and the requirement of the measures to bring the construction object to further safe exploitation is the technical condition category (CTC) of the building or structure (normative, operable, partially operable or emergency technical state). The technical condition category determinations of a construction object as a whole as the purpose of the construction expertise is the main component of the system (1). For this purpose the conduction

of a survey of the building structures is necessary (8). It includes a large list of works consisting of various sections of building science.

Regular engineering inspections of the construction objects (6) should be carried out at least once every 10 years and at least once every 5 years for buildings operating in adverse condition in accordance with the requirements of the "Rules for the Inspection of Building Structures" and other regulatory documents. There may be scope for conduction extraordinary checks of the operational suitability of structures after emergency impacts (earthquakes, fire, explosive impacts, etc.) or upon detection of significant defects, damage and deformations during maintenance (7).

The results of the building structures inspection make it possible to diagnose the technical condition of certain structures (5) and the construction object as a whole (1), i.e. carry out a set of surveys and necessary calculations to identify significant diagnostic parameters (2) and the defects and damage causes (3) used to judge the technical condition of the object. For this purpose, different methods of research and building elements examinations are applied as well as the survey the physical and mechanical properties of structures, foundations, soils (strength, deformation, physical and moral deterioration and other factors and parameters). This is necessary for a correct assessment and

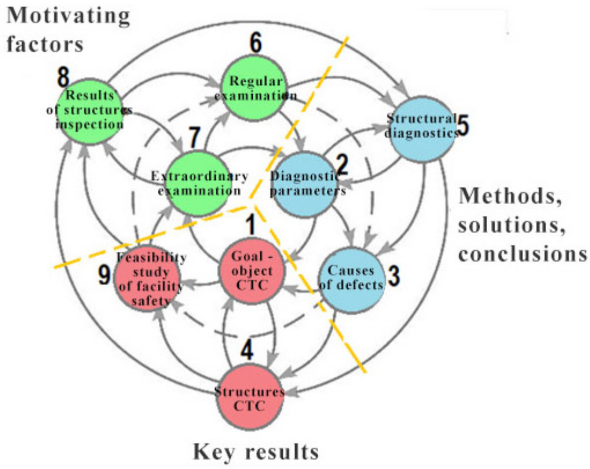

Figure 3. Ontograph of the definition of the CTC (the category of the technical condition) of a construction object.

classification of the true state of structures (4) based on the results of examinations (8) and diagnostics (5). As a result, the risk should be assessed of destruction and residual resource if identified category of the technical state decrease for some structural elements. Such assessment is conducted for a feasibility study of the relevance of carrying out measures for the repair, reconstruction or demolition of the facility (9).

Hence a conceptual confinement model of an intelligent system for diagnosing the technical condition of construction objects was formed. A specific system process operates within this system. It can be considered as a level 1 ontology that includes the required number of system processes.

Understanding the confinement model forming principles makes it possible next points: to unify the process of developing ontological knowledge bases and other subject areas, to facilitate lining of the structure in comparison with the known methods of constructing ontology [16, 20, 25], helps to filter out unimportant factors (which seemed important) and focus on those which are really affect the result.

At the same time, it should be understood that there is no single correct way to model the subject area. Always there are viable alternatives [6, 22]. Confinement modelling is an applied tool that allows facilitating the development of knowledge formalization models in intelligent systems, focusing on essentials further makes the right decisions by building action plans. Any element of this model can also be a sublevel. Creation of the sublevel structures both confinement models and other technologies and knowledge representation models can be used.

5. DIGITAL TECHNOLOGIES FOR RECEIVING AND PROCESSING INFORMATION FOR ARCHITECTURAL AND CONSTRUCTION OBJECTS WITH THE TERRITORY REFERENCE

Nowadays rapid digitalization of the architectural and construction industry is changing the technologies and possibilities of obtaining and processing information which is necessary for the construction objects diagnostics. Digital technologies are intensively implemented into the specialists' activities in the field of urban planning and architecture. Laser scanning, photogrammetry, and digital documentation methods make it possible to effectively collect three-dimensional and spatial data for architectural and construction objects into a database (DB) of point clouds [26, 27]. This is especially relevant for already existing and historical buildings or architectural complexes that can have defects and damage.

Parametric information models (BIM-models) are created [28] based on obtaining sufficiently accurate three-dimensional photogrammetric initial data for each building or structure after data processing. In figure 4 shows the results of such processing for a real architectural object St. Nicholas Church in Usolye of the Perm Krai as well as on figure 5 examples of the constructed BIM models in the cities of the Upper Kama region are presented.

Currently within the technical regulation framework is actively continuing implementation process of information modelling technologies at all stages of the life cycle of buildings and structures. This work carried out in creation the necessary regulatory and technical base process taking into account the needs of the construction process participants. Information models of each object are associated with the geographic information systems (GIS), i.e., with reference to the territory using an identification codes system assigned to each building (Fig. 6) [29].

Formalization and transformation into a machine-readable format and for the purpose of effective search, the obtained data is structured in the census form of the objects (Fig. 7).

The entire technologies set for architectural objects digital documentation not only in the form of three-dimensional point clouds, but also geospatial referenced data, as well as data about structures, defects and damage, provides researchers many options for subsequent work [30].

For instance, the historical and architectural heritage of Usolye database made it possible not only to

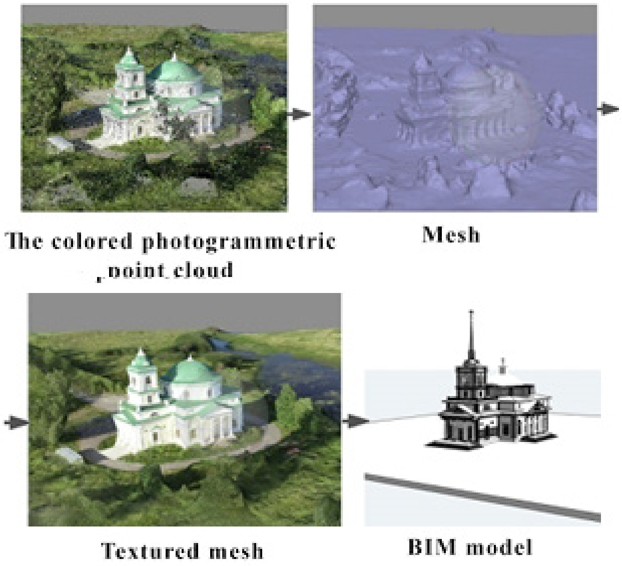

Figure 4. Photogrammetric data processing on the example of St. Nicholas Church in Usolye.

Figure 5. Examples BIM - models of churches and bell towers based on a point clouds and measured data in the cities of the Upper Kama region (Perm region).

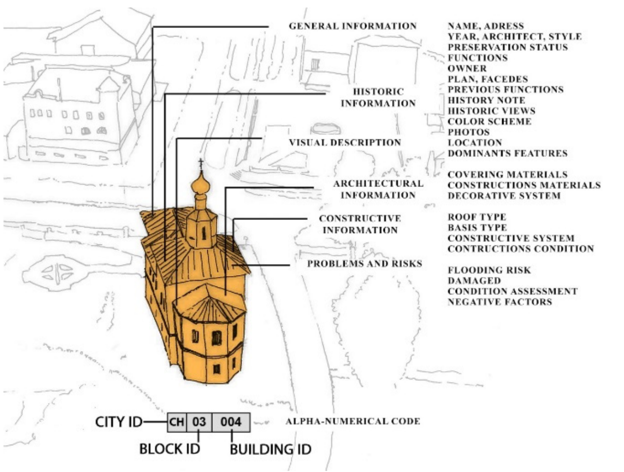

Figure 6. Information about an architectural object included in the database and object identifier

visualize the entire historical and architectural environment in three-dimensional form but also to present cartographic analytical materials necessary for management decisions. The strategy for the Usolye Stroganovskoye territory development was elaborated on the basis of such three-dimensional and cartographic materials.

6. CONCLUSION

Using the scan results, thermal imager data, non-destructive survey methods data, appropriate software as well as intelligent (for example, neural network) technologies, it is possible to more thoroughly diagnose certain units and parts and transmit digital information to determine the category of technical condition designs. Also, this information can be used for following situations: to control the dynamics of changes in the technical state of a construction object, to improve the accuracy of determining the scope of repair work, improve the quality of project documentation, improve methods for assessing the quality of restoration work and measures for the conservation of architectural monuments, etc.

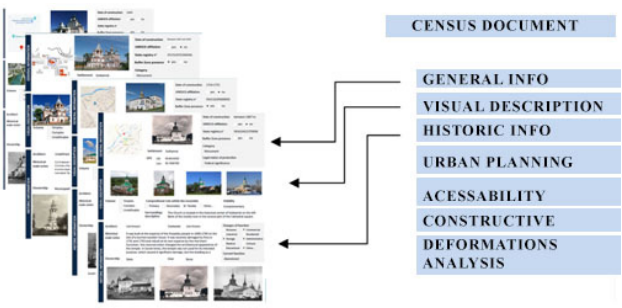

Figure 7. Census documents.

A well-structured database about the architectural and urban planning environment will allow not only effectively use existing information, but also add new information. In the presence of relevant and reliable input data, intelligent systems will be able to analyse not only the diagnostics of the technical condition and safety of construction objects, but also the urban planning situation.
The architecture represents a collection of not only numerical data, but also three-dimensional and semantic (ontology) data, which requires expert experience in data evaluation. Within architecture sector logical information often prevails over computational information and this can become a prospective field for intelligent systems.
Promotion of intelligent systems in relation to the functioning of each stage of the building "life cycle" will allow solving specific problems in the field of the architecture and constructions.

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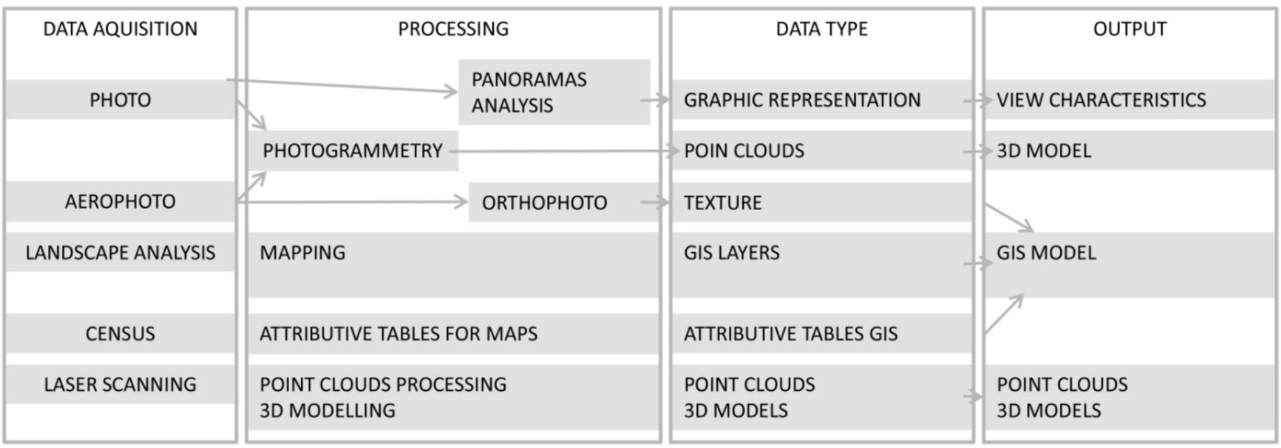

Figure 8. Scheme of processing data on the historical and architectural environment and the result

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SUSPENSION LARGE SPAN ROOFS STRUCTURES IN RUSSIA

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Abstract: Considered are large-span structures with suspended roof structures with a span of up to 200 m, erected in Russia over the past 40 years. Among them, there are different types of structures for covering sports facilities: cable-stayed systems, structures of the "bicycle wheel" type, combined systems, thin-sheet metal hanging shells, etc. The main technical characteristics of structures, principles of operation of structures, their advantages and disadvantages are given. The development of technologies in recent decades has determined the emergence of new forms, materials, design and construction methods. Unique large-span structures have an increased level of responsibility; their collapse can lead to severe economic and social consequences. In this regard, it is relevant to analyze the experience in the design and construction of large-span suspended structures.

Keywords: hanging large-span coatings in Russia, cable-stayed systems, structures of the "bicycle wheel" type, thin-sheet metal hanging shells, scientific and technical support

ВИСЯЧИЕ БОЛЬШЕПРОЛЕТНЫЕ КОНСТРУКЦИИ ПОКРЫТИЙ В РОССИИ

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Аннотация: Рассмотрены большепролетные сооружения с висячими конструкциями покрытий пролетом до 200 м, возведенных в России за последние 40 лет. В их числе, разные типы конструкций покрытий спортивных сооружений: вантовые системы, конструкции типа «велосипедное колесо», комбинированные системы, тонколистовые металлические висячие оболочки и т.д. Даны основные технические характеристики сооружений, принципы работы конструкций, их преимущества и недостатки. Развитие технологий в последние десятилетия определило появление новых форм, материалов, методов проектирования и строительства. Уникальные большепролетные сооружения имеют повышенный уровень ответственности, их обрушение может привести к тяжелым экономическим и социальным последствиям. В этой связи, актуальным является анализ опыта проектирования и возведения большепролетных висячих конструкций.

Ключевые слова: висячие большепролетные покрытия в России, вантовые системы, конструкции типа «велосипедное колесо», тонколистовые металлические висячие оболочки, научно-техническое сопровождение

INTRODUCTION

Hanging systems for covering structures were first proposed by the outstanding Russian engineer and scientist V.G. Shukhov. In 1896, at the All-Russian Exhibition in Nizhny Novgorod, he designed and built four pavilions (two – in plan dimensions 30 × 70 m, one 50 × 100 m and one round – 68 m in diameter) with hanging roofs with a total area of more than 10,000 m^2. The cover was made of thin crisscrossing steel rods and strips (Fig. 1). A pavilion 25 m in diameter, round in plan, was covered with a hanging shell made of riveted iron sheets 1.6 mm thick (Fig. 2). This was another step forward: from trusses with purlins to openwork mesh, and from it to a continuous thin sheet, which was used for the first time in the world and for many years was the only case of such a coating.

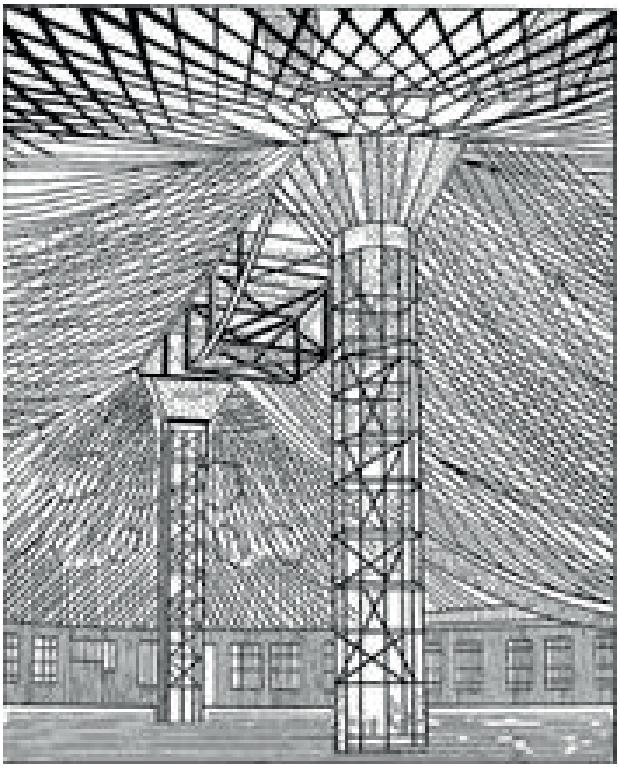

Figure 1. Hanging covering of the pavilion of the All-Russian Exhibition in N. Novgorod (1896).

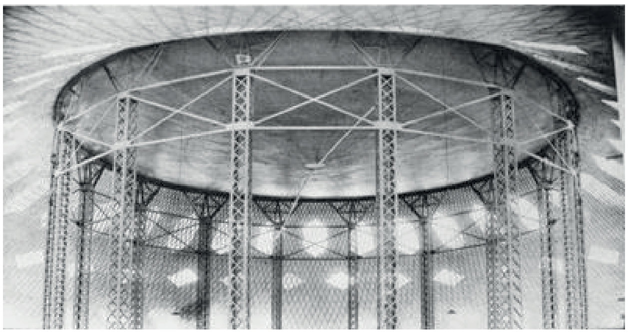

Figure 2. Membrane coating of the central part of the engineering and construction pavilion in N. Novgorod.

RESULTS

JSC "Research Center" Construction" developed new technical solutions, confirmed by copyright certificates and patents, carried out numerical and experimental studies of numerous hanging roofs of large-span structures. JSC "Research Center" Construction " took part in the design, provided scientific and technical support for their manufacture and installation. Below is a summary of the most interesting objects.

The most effective types of spatial structures of coatings include hanging shells made of a thin metal sheet stretched in two directions, combining bearing and enclosing functions in one material. Even in the case when a conventional roof is used, they do not require purlins, panels and other intermediate elements that are not involved in the spatial operation of the system [4, 5, 6].

The use of such structures allows: to reduce the consumption of materials; reduce the time, labor intensity and cost of construction; reduce the cost of underlying structures (columns, foundations) by reducing the weight of the coating; minimize transportation costs.

A thin-sheet (up to 6 mm) shell can cover structures with a span of more than 300 m with a plan outline from the simplest geometric shapes (square, rectangle, triangle, circle, oval, etc.) to a more complex combined outline. Thin-sheeted shells can have various surface shapes – cylindrical, spherical, hip-shaped, saddle-shaped, compound in the form of a combination of shells with the same or different surface shape. They are easy to manufacture and install, and have a minimum headroom. Thin-sheet panels with a width of up to 12 m and a span length are manufactured at the factory and delivered to the construction site rolled up. The tensile forces from the span structure are absorbed by the compressed-curved support contour, which works in conjunction with the membrane, which ensures its stability. The rational perception of these forces by the contour is one of the main tasks of designing thin-sheet pavements. This task is solved by the correct choice and mutual alignment of the surface shape and the outline of the coverage plan. Due to the lightness and low bending stiffness of the membranes, the problem arises of reducing their deformability. Stabilization of the coating can be provided in different ways: by its own weight, ballast weight, suspension of technological equipment; the introduction of the coating of reinforcing ribs with bending stiffness; prestressing the membrane. Experimental and theoretical studies of membrane shells were carried out, field surveys and monitoring of the erected structures were carried out, operating experience was generalized, recommendations for design and installation were developed. A number of structures with membrane coverings have been erected and are being successfully operated.

Indoor multifunctional stadium "Olympic" (Moscow, 1980). An oval building with plan dimensions of 224×183 m, covered with a 5 mm thick membrane shell (Fig. 3). The membrane is reinforced with a radial-ring system, which ensured the installation of the shell; during operation, it is used to fasten various technological equipment. The support contour with a section of 5×1.75 m is made of monolithic reinforced concrete in a metal formwork. The membrane is designed to withstand a load of 10 kPa in the center and 3.5 kPa at the periphery.

Cycling track "Krylatskoe" (Moscow, 1980) – dimensions in plan 168×138 m, shape close to an ellipse. The membrane covering 4 mm thick is made in the form of two saddle-shaped thin-sheet shells (Fig. 4), fixed on the arches with a span of 168 m. Internal arches do not have intermediate supports and are combined into a spatial block. The outer arches in the middle part of the span are supported by the consoles of the stands. Arches – steel box-section with dimensions of 3×2 m.

The universal sports complex “Izmailovo” (Moscow, 1980) consists of several volumes – the main hall measuring 66×72 m and two training halls – 36×36 m, each of which is covered with a 2 mm thick stainless-steel membrane (Fig. 5). The supporting contour with a section of 0.5×6.0 m is made of precast-monolithic reinforced concrete. The membrane was assembled by welding at ground level, from panels with a width of 9 ÷ 12 m, pre-enlarged at the factory from rolled tapes. The finished membrane was lifted by winches along the corner columns to the design position. For more than 35 years, the monitoring of structures has been carried out at all Olympic

Figure 3. Indoor stadium of the sports complex "Olympic" in Moscow.

Figure 5. Sports hall "Izmailovo" in Moscow.

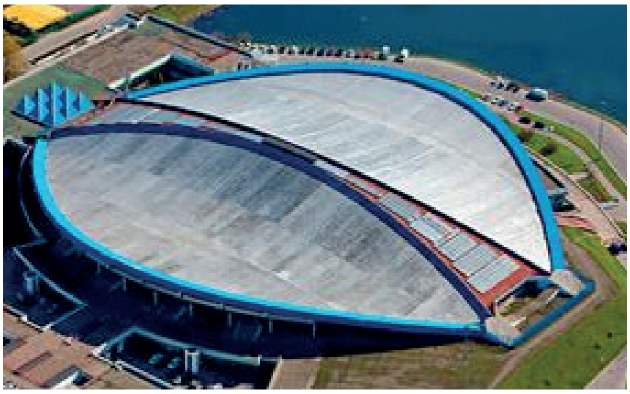

Figure 4. Cycling track in Krylatskoye in Moscow.

Figure 6. Ice Sports Palace in Angarsk.

facilities, ensuring their reliability, timely repair work, and making informed decisions on extending the period of trouble-free operation of facilities.

The membrane covering of the Ice Sports Palace (Angarsk, 2010) is made in the form of a cylindrical shell with dimensions in plan 90×87 m (Fig. 6). The closed support contour made of monolithic reinforced concrete consists of two rectilinear and two curved side elements. In the corners, the contour is reinforced with spacers.

Figure 7. Speed skating center in Kolomna (Moscow region).

Figure 8. Old Gostiny Dvor in Moscow.

Figure 9. Stadium "Lokomotiv" in Moscow

Indoor speed skating center (Moscow region, 2006), has an oval plan with dimensions of 200x110 m. The cover is designed in the form of a saddle membrane shell 4 mm thick (Fig. 7). The supporting contour of a rectangular section 1.2×2 m is made in the form of a steel box filled with concrete. The contour along the long sides' rests on two arches with a span of about 75 m, and in the rest of the sections on articulated posts with a pitch of about 12 m.

The overlaps of a number of structures are made in the form of combined spatial structures, which include stretched elements and elements working in compression and bending [7, 8, 9, 10, 11]. In combined systems, the design length of compressed-bent elements is reduced, stretched elements made of high-strength metal are rationally used, and the structure's performance for uneven loads is improved. There are many types of combined systems, even the simplest of them are distinguished by great freedom in choosing the design scheme, materials used, manufacturing and installation methods. Elementary circuits are combined in a variety of ways into complex spatial structures.

The building of the "Old Gostiny Dvor" (Moscow, 1998) [12] has the shape of an irregular trapezoid in plan. The courtyard with side dimensions of 56, 187, 84 and 163 m, with an area of 1500 m^2, is covered with a translucent glass shell (Fig. 8). The load-bearing combined structures of the coating form a cylindrical surface. They consist of a convex, compressed-curved upper chord made of I-beams, a stretched lower chord made of steel strips, and two V-shaped struts connecting the chords.

The football stadium "Lokomotiv-RZD Arena" (Moscow, 2001) was built oval in plan (205.7×157.3 m) for 29 thousand spectators (Fig. 9). The roof over the stands (canopy outreach 33 m) is suspended from four cable-stayed trusses. In the corners of the pavement, reinforced concrete pylons (about 50 m high) are installed, at the top of which there are hanging cables of two ropes (diameter 140 mm). To them are attached in pairs inclined cables with a diameter of 50 mm, to which the visor is suspended [13].

Ice Sports Palace (Moscow, 2005) – a building in the form of a cylinder with a diameter of 100 m and a height of 50 m (Fig. 10). The hanging mesh shell consists of an outer support contour, an inner ring, and a radial-annular system of flexurally rigid threads with diagonal ties.

Indoor speed skating center (Moscow, 2003). The construction plan is a segment of a circle with a radius of 117 m with a central angle of ~ 160º (Fig. 11).

The covering is formed by the same type of timber-metal trusses. A system of radial trusses (two spans 50.4 m each), with an annular beam in the middle of the covering, is suspended from 19 cables, which transmit the force to the foundation through a pylon (height 50 m) and two guys reinforcing it [3].

The Volgograd Arena is a round structure with a diameter of 290.0 m with a coating consisting of two parts: above the stands and above the foyer (Fig. 12). The covering over the stands (oval 243.8×206.7 m, with an opening above the playing area 123.0×85.9 m) is a "bicycle wheel" type system with one compressed outer contour and two stretched inner contours (from a package of cables) connected by radial cable-stayed trusses.

Football stadium in Nizhny Novgorod. The pavement of the round structure with a diameter of 290.0 m consists of two parts: the pavement above the stands and the pavement above the foyer (Fig. 13). The covering above the stands (oval 240.3×201.8 m, with an opening above the playing area 123.0×85.9 m) is a radial-circular rod shell of the "bicycle wheel" type, with load-bearing elements made of welded I-beams. "Rostov Arena" (Fig. 14) with an oval-shaped surface measuring 257.2×218.5.0 m with a rectangular cutout (130.5×91.8 m) above the football field. The main supporting structures are a system of radial cantilever beams with an outreach of 51.34 m, united by circular girders and ties. The cantilever beams are attached by two inclined cable-stayed suspensions to the top of the pylons located along the perimeter of the stadium, the forces from which are absorbed by the braces, which are locked onto the grillages.

Kaliningrad Stadium (Fig. 15). The structure in plan has the shape of a rectangle measuring 166.7 × 203.7 m with rounded corners with a

Figure 10. Ice Sports Palace.

Figure 12. "Volgograd Arena".

Figure 11. Indoor speed skating center.

Figure 13. Stadium in Nizhny Novgorod.

rectangular opening above the football field. The bearing structures of the pavement are a spatial core system that includes radial and ring trusses, united by ties. All main bar elements are designed from steel box sections. Cantilever radial trusses with an outreach of 38.2 m are suspended from the top of the pylons located along the perimeter of the stadium, the efforts from which are absorbed by the braces, which are closed on the frame of the stands.
Football stadium in Krasnodar. The covering above the stadium stands is a cable-stayed system, such as a "bicycle wheel" with two compressed steel outer contours and a stretched inner ring in the form of a set of cables connected by a system of radial cable-stayed trusses (Fig. 16).
The VTB-Arena stadium in Moscow has an oval shape in plan. The maximum dimensions are 300×187 m, the height of the structure is 66 m (Fig. 17). There is also a covered arena under the shell. The shell itself is made in the form of a spatial rod system, reinforced with trusses in the coverage area above the stands.
JSC "Research Center" Construction" took part in the design of the above objects, provided scientific and technical support for their manufacture and installation, which included [1, 2]:

– theoretical numerical research;
– testing of large-scale models in order to identify the actual operation of various systems, check the initial assumptions and conclusions of theoretical studies;
– carrying out field surveys and generalization of operating experience;
– development of practical recommendations for the calculation and design of large-span metal structures, taking into account the issues of their manufacture and installation.

These studies allowed solving numerous problems associated with identifying the actual operation of structures, additional reserves for increasing their efficiency and reliability. Based on the generalization of the experience of research, design, construction and operation of large-span spatial structures, a number of recommendations have been developed.

CONCLUSION

1. Hanging large-span structures have an increased level of responsibility, their failures can lead to severe economic and social consequences. In

Figure 14. Rostov Arena.

Figure 15. Kaliningrad Stadium.

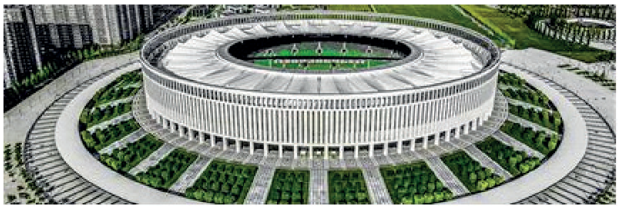

Figure 16. Stadium in Krasnodar.

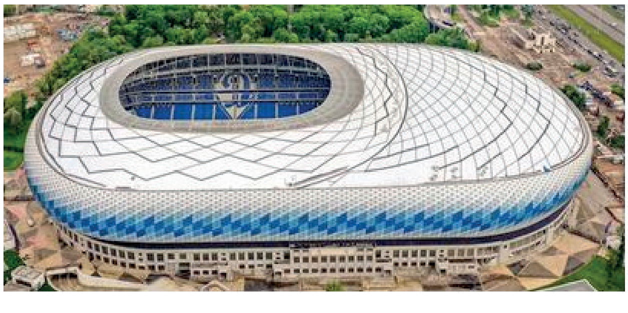

Figure 17. VTB-Arena.

this regard, additional requirements should be taken into account for the nomenclature and scope of surveys and design work, the manufacture and installation of structures, the rules for their acceptance and operation.
2. Design should be based on the choice of rational design solutions linked to the functional purpose, architecture, manufacturing and installation methods, operating conditions, the ideas put forward should be technically and economically justified.
3. When designing, problems arise that go beyond the normative documents, which require special knowledge and practical experience from the engineer. An important feature of the process is the generation of ideas based on the creativity of the designer.
4. To increase the reliability of the results, verification calculations should be carried out using various modern certified software systems. It should be borne in mind that using a computer has a downside, the risk of errors increases. A clear understanding of how the structure works, based on the rules of structural mechanics, is required.
5. Ensuring high reliability of hanging large-span structures requires mandatory scientific and technical support, which usually includes:
– development of recommendations for determining snow and wind loads based on the results of blowing a model of a structure in a specialized wind tunnel. Climatic loads are taken taking into account the service life of the structure;
– performing verification calculations;
– carrying out, in some cases, experimental studies of large-scale physical models and full-scale assemblies to assess the bearing capacity and reliability of structures; verification of the calculation model and calculation methods;
– development of "Specifications for the manufacture and installation of structures" with additional requirements that are not included in the current regulatory and technical documents.
– Carrying out technical monitoring at the stage of construction and the first years of operation.

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ASYMPTOTICS OF THE FILTRATION PROBLEM WITH ALMOST CONSTANT COEFFICIENTS

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Abstract: During the construction of hydraulic and underground structures, a grout solution is pumped into the ground to create waterproof partitions. The liquid grout is filtered in the porous rock and clogs the pores when hardened. The mathematical model of deep bed filtration describes the transfer of suspension particles and colloids by a fluid flow through the pores of a rock. For a one-dimensional filtration problem in a homogeneous porous medium with almost constant coefficients, an asymptotic solution is constructed. The asymptotics is compared with the numerical solution.

Keywords: deep bed filtration, suspensions and colloids, porous medium, suspended and retained particles, asymptotic solution.

АСИМПТОТИКА ЗАДАЧИ ФИЛЬТРАЦИИ С ПОЧТИ ПОСТОЯННЫМИ КОЭФФИЦИЕНТАМИ

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Аннотация: При строительстве гидротехнических и подземных сооружений для создания водонепроницаемых перегородок в грунт закачивается раствор укрепителя. Жидкий укрепитель фильтруется в пористой породе и при застывании закупоривает поры. Математическая модель фильтрации описывает перенос жидкостью частиц суспензий и коллоидов через поры горной породы. Для одномерной задачи фильтрации в однородной пористой среде с мало меняющимися коэффициентами построено асимптотическое решение. Асимптотика сравнивается с численным решением.

Ключевые слова: фильтрация, суспензии и коллоиды, пористая среда, взвешенные и осажденные частицы, асимптотическое решение.

1. INTRODUCTION

Filtration of suspensions and colloids in porous media occurs in many natural and technological processes: the spread of microorganisms in the aquatic environment, filtration of water in rocks, treatment of industrial and domestic wastewater, a decrease in oil production due to the deposition of small particles entrained in water near the well, and much more [1–3]. During the construction of tunnels and underground storage facilities for hazardous toxic and radioactive waste, a liquid grout is pumped into the rock under pressure to create watertight walls. The grout filters in the porous soil and clogs the pores after solidification [4].

The transport of micro- and nanoparticles in a porous medium is accompanied by the retention of particles and the formation of a deposit. Various retention mechanisms of particles carried by a fluid flow in a porous medium of complex structure are determined by electric, gravitational and hydrodynamic forces [5–7]. Filtration models take into account either a single prevailing retention mechanism, or several mechanisms acting simultaneously [8, 9].

The mathematical model of deep bed filtration includes the equation for the balance of the masses of suspended and retained particles and the kinetic equation of deposit growth, which form a quasilinear hyperbolic system of the first order partial differential equations [10]. To solve filtration problems, both numerical and analytical methods are used [11–16]. Analytical methods allow to obtain exact and asymptotic solutions and their dependence on parameters. This makes it possible to fine-tune experiments and to solve inverse filtration problems [17–19].
The classical filtration model assumes that the properties of the porous medium do not change with the formation of deposit. More sophisticated models take into account the dependence of porosity and permissible flow on the concentration of deposit [20]. In these models, it is assumed that a suspension or colloidal solution of constant volume concentration is injected at the inlet of a porous medium.
We consider a one-dimensional model for deep bed filtration of particles carried by a fluid flow in a homogeneous porous medium. It is assumed that the carrier fluid is incompressible; at the porous medium inlet the suspended particles concentration is variable. Experiments show that the coefficients of the filtration equations depending on the retained concentration do not change much. This allows us to construct an asymptotic solution to the filtration problem. The asymptotics is compared with the numerical solution.

2. MATHEMATICAL MODEL

In the domain $\Omega = \{0 \le x \le 1, t \ge 0\}$, consider the system of first-order differential equations

$$\frac{\partial}{\partial t}\left(g(S)C\right)+\frac{\partial}{\partial x}\left(f(S)C\right)+\frac{\partial S}{\partial t}=0\,, \qquad (1)$$

$$\frac{\partial S}{\partial t}=\Lambda(S)C\,. \qquad (2)$$

Here the blocking filtration function $\Lambda(S)$ is smooth and positive at $0 \le S < S_m$, $S_m > 0$; $\Lambda(S) = 0$ at $S \ge S_m$; the functions $g(S)$ and $f(S)$ are positive at $0 \le S \le S_m$; $C(x,t)$, $S(x,t)$ – the unknown volumetric concentrations of suspended and retained particles [21].
For the uniqueness of the solution to the system (1), (2), the initial and boundary conditions are set

$$C\big|_{x=0} = p(t),\;\; p(t) > 0\,, \qquad (3)$$

$$C\big|_{t=0} = 0, \quad S\big|_{t=0} = 0\,. \qquad (4)$$

Condition (3) means that a suspension of variable concentration is injected at the inlet of the porous medium; by condition (4), at the initial moment of time, the porous medium does not contain any suspended and retained particles. The concentrations front of the suspended and retained particles given by the formula $x = vt$, $v = f(0)/g(0)$ moves with a speed v from the inlet to the outlet of the porous medium. Ahead of the front in the domain $\Omega_0 = \{0 \le x \le 1, t < x\}$, the solution is zero; behind the front in the domain $\Omega_1 = \{0 \le x \le 1, t > x\}$, the decision is positive. Since conditions (3) and (4) do not matched at the origin, the solution C is discontinuous at the concentration front; the solution S is continuous in the whole domain Ω.
Consider the condition on the concentrations front

$$S\big|_{x=vt} = 0\,. \qquad (5)$$

In the domain $\overline{\Omega}_1$, the solution to problem (1)–(4) coincides with the solution to the Goursat problem (1)–(3), (5).
In characteristic variables $\tau = t - x/v$, $y = x$, the Goursat problem takes the form

$$\begin{gathered}\frac{\partial}{\partial \tau}\left(g(S)C\right)-\frac{1}{v}\frac{\partial}{\partial \tau}\left(f(S)C\right)+\\ +\frac{\partial}{\partial y}\left(f(S)C\right)+\frac{\partial S}{\partial \tau}=0\end{gathered}\,, \qquad (6)$$

$$\frac{\partial S}{\partial \tau}=\Lambda(S)C\,. \qquad (7)$$

Conditions (3) and (5) take the form

$$C\big|_{y=0} = p(\tau)\,, \tag{8}$$

$$S\big|_{\tau=0} = 0\,. \tag{9}$$

3. ASYMPTOTICS FOR ALMOST CONSTANT COEFFICIENTS

Assume that the coefficients of equations (1), (2) admit expansions

$$\begin{aligned} g(S) &= g_0 + \varepsilon g_1 S + ..., \\ f(S) &= f_0 + \varepsilon f_1 S + ..., \\ \Lambda(S) &= \Lambda_0 + \varepsilon \Lambda_1 S + ... \,. \end{aligned} \tag{10}$$

Here ε is a small positive parameter.
The solution to the system (6), (7) is obtained in the form [22, 23]

$$\begin{aligned} S(y,\tau) &= s_0(y,\tau) + \varepsilon s_1(y,\tau) + ..., \\ C(y,\tau) &= c_0(y,\tau) + \varepsilon c_1(y,\tau) + ... \,. \end{aligned} \tag{11}$$

Substitute the expansions (10), (11) into the equations (6), (7) and equate the terms at the same powers of ε. We obtain a recurrent system of differential equations

$$f_0 \frac{\partial c_0}{\partial y} + \Lambda_0 c_0 = 0\,, \tag{12}$$

$$\frac{\partial s_0}{\partial \tau} = \Lambda_0 c_0\,, \tag{13}$$

$$\begin{aligned} & f_0 \frac{\partial c_1}{\partial y} + \Lambda_0 c_1 + f_1 \frac{\partial s_0}{\partial y} c_0 + \\ & + f_0 \frac{\partial c_1}{\partial y} + \Lambda_1 s_0 c_0 = 0, \end{aligned} \tag{14}$$

$$\frac{\partial s_0}{\partial \tau} = \Lambda_0 c_1 + \Lambda_1 s_0 c_0\,. \tag{15}$$

Conditions for the equations (12)–(15) follow from (8) and (9)

$$c_0\big|_{y=0} = p(\tau),\ c_1\big|_{y=0} = 0\,. \tag{16}$$

$$s_0\big|_{\tau=0} = 0,\ s_1\big|_{\tau=0} = 0\,. \tag{17}$$

Solution to the system (12)–(15) with the conditions (16), (17)

$$c_0 = p(\tau) e^{-Ay},\ \ s_0 = \Lambda_0 P(\tau) e^{-Ay}, \tag{18}$$

$$c_1 = p(\tau)\big(B + DP(\tau)\big) e^{-Ay}\left(e^{-Ay} - 1\right), \tag{19}$$

$$\begin{aligned} s_1 = \Lambda_0 & \left(Bp_1(\tau) + 0.5DP^2(\tau)\right) e^{-Ay} \cdot \\ & \cdot \left(e^{-Ay} - 1\right) + 0.5\Lambda_0 \Lambda_1 P^2(\tau) e^{-2Ay}. \end{aligned} \tag{20}$$

Here

$$A = \frac{\Lambda_0}{f_0},\ \ P(\tau) = \int_0^\tau p(z)dz,$$
$$B = g_1 - f_1 / v,\ D = \Lambda_1 - \frac{f_1}{f_0}\Lambda_0.$$

Substituting the solutions (18)–(20) into the expansions (11) and passing to the Cartesian coordinates, we obtain an asymptotic solution to the problem (1)–(4) in the domain $\overline{\Omega}_1$

$$\begin{aligned} S(x,t) &= \Lambda_0 P(t - x/v) e^{-Ax} \left(1 + \varepsilon\left(\left(B + 0.5P(t - x/v)\right)\left(D(e^{-Ax} - 1) + \Lambda_1 e^{-Ax}\right)\right)\right), \\ C(x,t) &= p(t - x/v) e^{-Ax}\left(1 + \varepsilon\left(B + DP(t - x/v)\right)\left(e^{-Ax} - 1\right)\right). \end{aligned} \tag{21}$$

4. RESULTS OF NUMERICAL MODELLING

The numerical calculation was carried out for the coefficients of equations (1), (2) obtained from the results of experiments with particles of a suspension with a radius of 2.179 microns in the laboratory of the University of Adelaide, Australia [24]

$$g(S) = 0.9743 - 8.8818 \cdot 10^{-14} S,$$
$$f(S) = 0.9947 + 6.2733 \cdot 10^{-5} S,$$
$$\Lambda(S) = 0.5106 - 0.0060 S.$$

The calculation was made for a linearly increasing suspended concentration at the porous medium inlet $p(t) = 1 + 0.01t$. Figures 1–4 show the asymptotics at $\varepsilon = 0.01$ (yellow line) and the numerical solution (blue line).

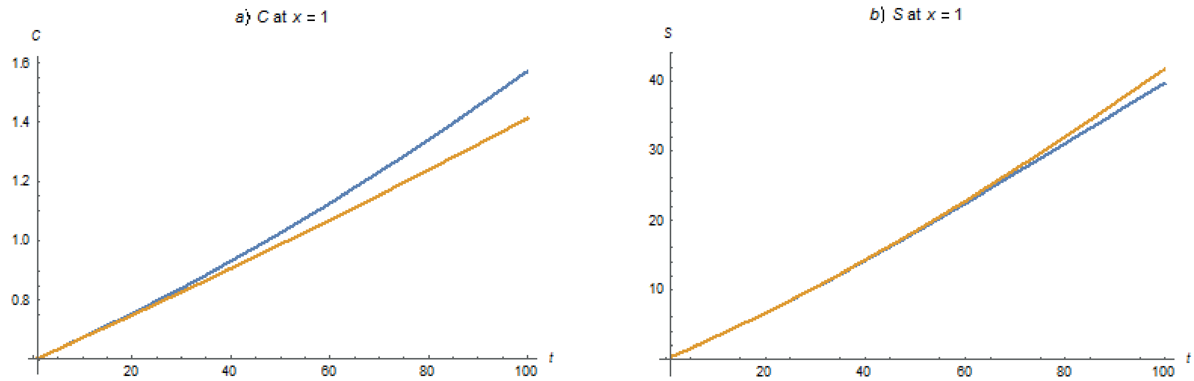

Figure 1. Concentrations at the porous medium outlet x=1 a) suspended C(1,t); b) retained S(1,t).

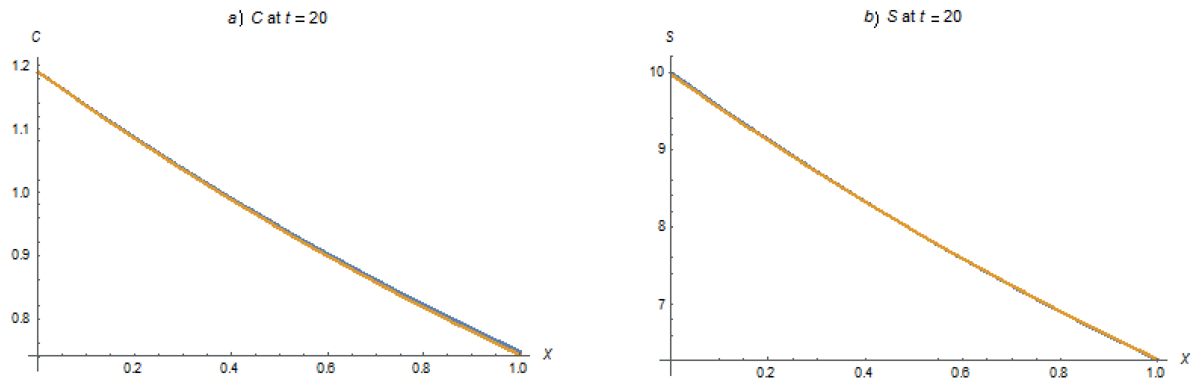

Figure 2. Concentrations at fixed time t=20 a) suspended C(x,20); b) retained S(x,20).

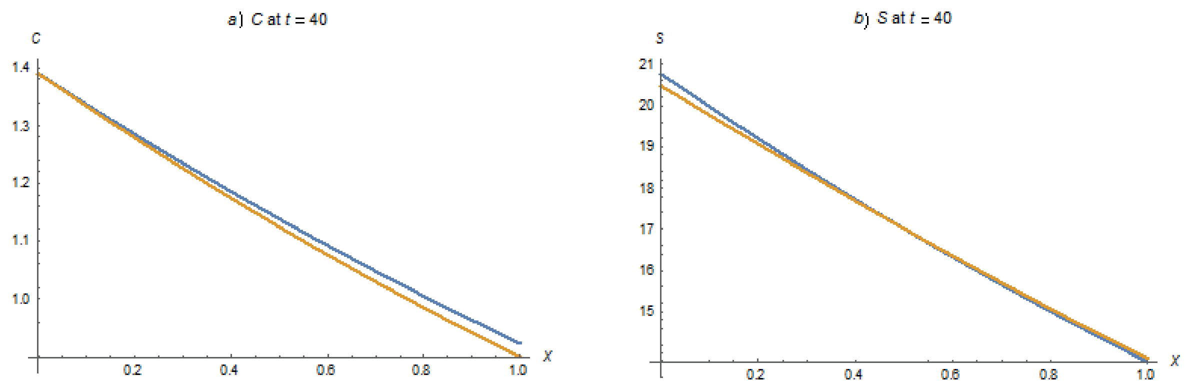

Figure 3. Concentrations at fixed time t=40 a) suspended C(x,40); b) retained S(x,40).

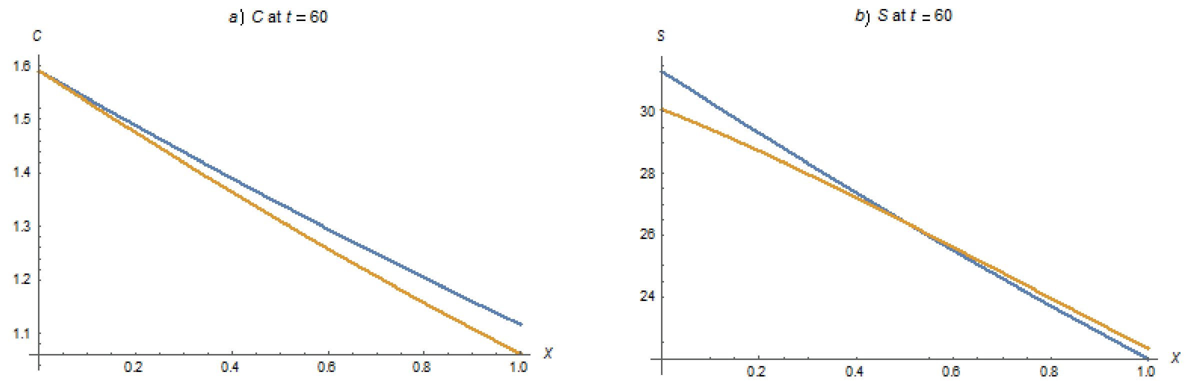

Figure 4. Concentrations at fixed time t=60 a) suspended C(x,60); b) retained S(x,60).

At the porous medium outlet $x = 1$, the relative error of the asymptotics increases with time. For the suspended and retained particles concentrations, the error reaches 2.5% and 0.1% at t = 40, 1% and 5% at t = 60, 5% and 10% at t = 100, respectively. For a fixed time, the relative error of the asymptotics throughout the whole porous medium does not exceed 1% at t = 20, 2% at t = 40, and 4% at t = 60 for both types of particles concentrations.

5. CONCLUSIONS

The study of the mathematical model of deep bed filtration of suspensions and colloids in a porous medium allows us to draw the following conclusions.

– An asymptotic solution to the filtration problem is constructed.

– The main term of the asymptotics coincides with the exact solution of the problem with constant coefficients.

– The asymptotics is close to the numerical solution.

– The asymptotic solution depending on the model parameters can be used to fine-tune laboratory experiments and to solve the inverse filtration problem.

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USING DISCRETE-CONTINUOUS APPROACH FOR THE SOLUTION OF UNSTEADY-STATE MOISTURE TRANSFER EQUATION FOR MULTILAYER BUILDING WALLS

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Abstract: Moisture regime of enclosing structures is one of the most complicated and controversial directions in construction industry. Temporary climate impact on enclosing structures and low moisture inertia of building materials lead to the situation in which it is impossible to calculate the steady-state moisture regime. Numerical methods are usually used to assess the moisture behaviour of the enclosing structures. In the current paper, a differential equation of moisture transfer is formulated. The solution of the unsteady-state equation of moisture transfer was obtained using the discrete-continuous approach. Thus, a formula which allows scientists to calculate unsteady-state moisture transfer in multilayer walls of buildings was obtained. A two-layer building enclosing structure with aerated concrete base and mineral wool insulation was calculated.

Keywords: moisture regime, mathematical model, discrete-continuous method, moisture potential, multilayer enclosing structure.

ИСПОЛЬЗОВАНИЕ ДИСКРЕТНО-КОНТИНУАЛЬНОГО ПОДХОДА К РЕШЕНИЮ УРАВНЕНИЯ НЕСТАЦИОНАРНОГО ВЛАГОПЕРЕНОСА В МНОГОСЛОЙНЫХ СТЕНАХ ЗДАНИЙ

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Аннотация: Влажностный режим ограждающих конструкций зданий является одним из наиболее сложных и противоречивых направлений в строительстве. Постоянное изменение климатических воздействий на ограждающую конструкцию и низкая влажностная инерция строительных материалов приводят к ситуации, в которой невозможно производить расчеты стационарного влажностного режима. Для оценки влажностного состояния ограждающей конструкции используются численные методы. В настоящей работе сформулировано дифференциальное уравнение влагопереноса. Получено решение нестационарного уравнения влагопереноса с помощью дискретно-континуального подхода. В результате была получена формула, которая позволяет рассчитывать нестационарный влагоперенос в многослойных стенах зданий. Была рассчитана двухслойная ограждающая конструкция с основанием из газобетона и утеплителем из минеральной ваты.

Ключевые слова: влажностный режим, математическая модель, дискретно-континуальный метод, потенциал влажности, многослойная ограждающая конструкция.

1. INTRODUCTION

Heat and moisture transfer inside enclosing structures is a vital problem in modern construction industry [1–7].

In nowadays, there are many multilayers walls that are used in building, so it is crucial to assess heat-conductivity coefficients under various climate conditions [8, 9], durability of building materials [10–13] and influence of the moisture content inside enclosures on human health [14–18].

Calculations of moisture transfer are based on a transfer potential. For instance, it can be gradient of water vapor partial pressure [19]. Moreover, moisture transportation can be described by some moisture transfer potentials. For example, gradient of water vapor partial pressure and gradient of capillary pressure [20] or liquid content pressure [21]. The most convenient method is a moisture potential theory, which allows scientists to solve only one moisture transfer equation using the moisture potential [22]. A huge number of moisture potentials exist but in Russian Federation the moisture potential *F*, which is included in regulatory documents, was developed by V.G. Gagarin and V.V. Kozlov [23].

The moisture potential *F* can be written as a function of moisture and temperature [23]:

$$F(w,t) = E_t(t)\cdot\varphi(w) + \frac{1}{\mu}\int_0^w \beta(\zeta)\,d\zeta. \qquad (1)$$

where *F* – moisture potential, Pa; E_t – saturated water vapor pressure, Pa; φ – relative air humidity, %; μ – vapor permeability coefficient, *kg/(m·s·Pa)*; β – moisture conductivity coefficient, k*g/(m· s· kg/kg),* which depends on moisture, t – temperature, $^\circ C$; w – material moisture , % by weight (1 kg/kg = 100 % by weight).

Moisture transfer differential equation based on the moisture potential *F* can be formulated as [23]:

$$\frac{\partial F(w,t)}{\partial \tau} = \left(\frac{1}{\mu}\beta(w) + \frac{\partial \varphi(w)}{\partial w}E_t(t)\right)\frac{\mu}{\gamma_0}\frac{\partial^2 F(w,t)}{\partial x^2}. \qquad (2)$$

where γ_0 – enclosing structure dry material density, kg/m^3, τ – time, *s*; *x* – coordinate, *m*.

In 2010, the new discrete-continuous approach was developed by Zolotov A.B., Akimov P.A., Sidorov V.N. and Mozgaleva M.L. This approach gives an opportunity to find an analytical solution of the unsteady-state heat transfer equation [24, 25].

The heat transfer equation can be formulated as [24, 25]:

$$\frac{\partial t}{\partial \tau} = a\cdot\frac{\partial^2 t}{\partial x^2}. \qquad (3)$$

where a – thermal diffusivity coefficient, m^2/s.

First-order boundary conditions for the heat transfer equation can be written as [24, 25]:

$$t_{x=0} = t_{ext}. \qquad (4)$$

$$t_{x=l} = t_{in}. \qquad (5)$$

where $t_{x=0}$ – temperature in *x*=0, $^\circ C$; $t_{x=0}$ – temperature in *x*=*l*, $^\circ C$; t_{ext} – temperature of outside air, $^\circ C$; t_{in} – temperature of inside air, $^\circ C$; *l* – thickness of researched enclosing structure, *m*.

If inside and outside temperatures do not change during time, it is possible to use discrete-continuous formula:

$$\bar{U}(\tau) = e^{A\cdot\tau}\cdot\bar{U}_0 - A^{-1}\left(E - e^{A\cdot\tau}\right)\cdot\bar{S}. \qquad (6)$$

where $\bar{U}$ – temperature distribution column vector; $\bar{U}_0$ – initial temperature distribution column vector; A – coefficient matrix; $\bar{S}$ – boundary conditions column vector.

Opportunities of the formula (6) has been developed by V.N. Sidorov and S.M. Matskevich [26–28]. First-order boundary conditions varied with time, and temperature distribution was described by the following expression at any moment of time:

$$\bar{U}(\tau) = e^{A\cdot\tau} \cdot \bar{U}_0 + \int_0^{\tau} e^{A\cdot(\tau-\sigma)} \cdot \bar{S}(\sigma) d\sigma. \quad (7)$$

The integral in equation (7) can be determined by method of trapezoidal.

2. THE PROBLEM

To obtain analytical solution of the unsteady-state moisture transfer equation (2) for multilayer building walls using discrete-continuous method.

3. MATERIALS AND METHODS

The formula (2) was reformulated as [29,30]:

$$\frac{\partial F(w,t)}{\partial \tau} = \kappa_{F_0} \cdot E_t(t) \frac{\partial^2 F(w,t)}{\partial x^2}. \quad (8)$$

where κ_{F_0} – average material heat-humidity characteristic coefficient, $m^2/(s\cdot Pa)$.

Thus, saturated water vapor pressure E_t depends on temperature and can be calculated by the following expression:

$$E_t(t) = 1.84 \cdot 10^{11} \cdot \exp(-5330/(273+t)). \quad (9)$$

In order to simplify equation (8) let us consider the steady-state heat-transfer equation with third order boundary conditions:

$$\frac{\partial^2 t}{\partial x^2} = 0. \quad (10)$$

$$-\lambda \left.\frac{\partial t}{\partial x}\right|_{i=1} = \alpha_{ext}\left(t_{ext} - t_1\right). \quad (11)$$

$$\lambda \left.\frac{\partial t}{\partial x}\right|_{i=N} = \alpha_{in}\left(t_{in} - t_N\right). \quad (12)$$

where t_1 – temperature of the enclosing structure surface which contacts with outside air, °C; α_{ext} – heat exchange coefficient of outside air and enclosing structure section, $W/(m^2\cdot°C)$; t_N – temperature of the enclosing structure surface which contacts with inside air, Pa; α_{in} – heat exchange coefficient of inside air and enclosing structure section, $W/(m^2\cdot°C)$.

Third-order boundary condition for moisture transfer equation can be written as:

$$-\mu \left.\frac{\partial F}{\partial x}\right|_{i=1} = \beta_{ext}\left(F_{ext} - F_1\right). \quad (13)$$

$$\mu \left.\frac{\partial F}{\partial x}\right|_{i=N} = \beta_{in}\left(F_{in} - F_N\right). \quad (14)$$

where F_{ext} – outside air moisture potential equal to partial pressure of outside air water vapor, Pa; F_{in} – inside air moisture potential equal to partial pressure of inside air water vapor, Pa; F_1 – moisture potential of the enclosing structure surface which contacts with outside air, Pa; F_N – moisture potential of the enclosing structure surface which contacts with inside air, Pa; β_{in} – moisture exchange coefficient of inside air and enclosing structure section, $kg/(m^2\cdot s\cdot Pa)$; β_{ext} – moisture exchange coefficient of outside air and enclosing structure section, $kg/(m^2\cdot s\cdot Pa)$.

According to the analytical expressions (9) – (14), there is a possibility to find discrete-continuous solution of the moisture-transfer equation for the multi-layer enclosing structure:

$$\begin{aligned}\bar{F} = p \cdot (&(G + K\cdot E_t \cdot A)^{-2} \cdot e^{(G+K\cdot E_t \cdot A)\cdot\tau} - \\ &-\tau\cdot(G + K\cdot E_t \cdot A)^{-1} - (G + K\cdot E_t \cdot A))^{-2} \cdot \bar{L} + \\ &+(G + K\cdot E_t \cdot A)^{-1}\left(e^{(G+K\cdot E_t \cdot A)\cdot\tau} - E\right)\cdot \bar{B} + \\ &+e^{(G+K\cdot E_t \cdot A)\cdot\tau} \cdot \bar{F}_0.\end{aligned} \quad (15)$$

where G – matrix of coefficients for materials joint; K – matrix, which takes into account the differences in the thermal and moisture

properties of the materials of the calculating enclosing structure; A – matrix of coefficients for a multilayer enclosing structure; $\overline{L}$ – a column vector, the first element of which is equal to one, other elements are equal to 0 for a multilayer enclosing structure $\overline{B}$ – a column vector, the first and last elements of which describe the boundary conditions on the outer and inner surfaces of the enclosing structure, other elements are equal to 0 for a multi-layer enclosing structure; E_t – matrix of the saturated water vapour pressure; p – the coefficient of the external boundary condition for a multilayer building enclosing structure, Pa/s^2.
A computer program based on formula (15) has been created. It was made by MATHLAB application, which is able to use an engineer's work.

4. RESULTS AND DISCUSSION

The new discrete-continuous formula was used for calculation of the moisture regime of the building wall with aerated concreate base and mineral wool insulation. The climate data of Moscow (Russian Federation) for temperature and moisture field was taken as initial conditions for moisture behaviour assessment.
The results of the moisture behaviour calculation in the building wall with aerated concreate base and mineral wool insulation in January are given at (Figure 1).

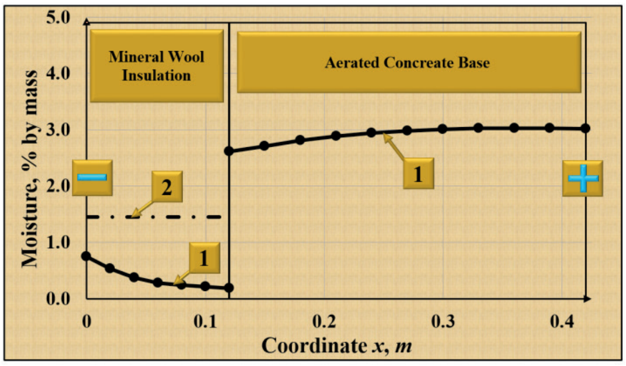

Figure 1. Result of the moisture behaviour calculation in building wall with aerated concreate base and mineral wool insulation in January (1 – a moisture behaviour inside enclosing structure; 2 – a maximum sorption zone).

5. CONCLUSIONS

The new efficient method was proposed for HVAC (heating, ventilation and air conditioning) engineers. This method is based on the discrete-continuous approach, which allows scientists calculate unsteady-state moisture transfer by final formula (15).

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ANALYSIS AND DESIGN OF STRUCTURAL STEEL JOINTS AND CONNECTION: SOFTWARE IMPLEMENTATION

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Abstract. The paper presents COMET software which enables to design steel structural joints widely used in civil and industrial engineering. Algorithm for designing each joint prototype has been presented as a set of operations implementing the rules for determining the interrelated values of the joint parameters. Each prototype is developed as an independent program that performs a full cycle of designing the joint and verification of the joint parameters according to the specified design codes.

Searching of unknown joint parameters has been transformed to a decision making problem based on analysis of the joint mathematical model. Automatic searching of unknown joint parameters has been implemented as a multiple targeted improvement of a certain initial joint design in order to satisfy load-carrying capacity constraints taking into account the structural and assortment-based constraints. Multiple improvement of current joint design is performed on the basis of sensitivity analysis relative to variation of governing joint parameters.

Keywords: structural steel joint, decision making problem, sensitivity analysis, software implementation.

О ПРОГРАММНОЙ РЕАЛИЗАЦИИ РАСЧЕТА И ПРОЕКТИРОВАНИЯ УЗЛОВ И СОЕДИНЕНИЙ СТАЛЬНЫХ КОНСТРУКЦИЙ

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Аннотация. В статье представлена программа КОМЕТА, позволяющая рассчитывать и проектировать узлы металлических конструкций, широко используемые в промышленном и гражданском строительстве. Алгоритм проектирования каждого прототипа узла представлен в виде набора операций, реализующих правила определения значений взаимосвязанных параметров узлов. Для каждого прототипа узла разработан независимый модуль программы, выполняющий полный цикл проектирования узла и проверки его параметров на соответствие выбранным нормам проектирования.

Задача поиска неизвестных параметров узлов стальных конструкций представлена как проблема принятия решений на основе анализа математической модели узла. Автоматический поиск неизвестных значений параметров узла реализован как многократное целенаправленное улучшение некоторого исходного проектного решения с целью удовлетворения ограничений несущей способности, а также с учетом конструктивных и сортаментных ограничений. Многократное улучшение текущего проектного решения узла выполняется на основе анализа чувствительности по отношению к варьированию управляющих параметров узла.

Ключевые слова: узел стальной конструкции, проблема принятия решений, анализ чувствительности, программная реализация.

INTRODUCTION

Design and analysis of joints is one of the most important stages of design of steel structures. Unlike the stress-strain state analysis on the basis of the design model, which follows the strict rules of structural mechanics, “algorithms” for the analysis of joints use the traditional methods (taking into account the previous experience) of approximate solutions, which are based on a simplified representation of the behavior of joints. These methods are usually closely related to the set of proven designs of joints (prototypes) used for this type of structures.

Despite the variety of prototypes of joints of one type (for example, columns bases), the number of parameters that have to be determined for each of them in the design process is limited. Taking into account the peculiarities of the behavior of a certain prototype of the joint in the structure (e.g., the cross sections of structural elements coming into the joint, loads taken by the joint etc.), and the requirements of various design codes, the algorithm for designing each joint prototype should be presented as a set of operations implementing the rules for determining the interrelated values of the parameters. Each prototype is developed as an independent procedure that performs a full cycle of designing the joint, check of the parameters according to the specified design codes, as well as the generation of a drawing of the designed joint. Taking into account the fact that regardless of the selected prototype most of the parameters of the same purpose of the joints of the same type are determined according to the same rules, the software implementation of parametric prototypes comes down to the organization of information exchange between different software modules that serve to determine the specific parameters. The focus on the use of parametric prototypes of joints that meet the above requirements has been adopted in the first programs developed by SCAD Soft Ltd. since the mid-1990s and implemented in software COMET [3]. A similar approach to the solution of the problem of designing steel structural joints has been also used by other developers of CAD-CAE systems, for example, RFEM and RSTAB modules of Dlubal Software [1], Connections modules of Autodesk Robot Structural Analysis Professional [5] or STK-SAPR and ESPRI of LIRA-SAPR.

PARAMETRIC PROTOTYPES ORIENTATION

The current version of COMET, which will be further considered, enables to design steel structural joints widely used in civil and industrial engineering [3], [4]. The application is also used to perform a structural appraisal of a steel joint according to the requirements of Ukrainian codes (DBN B 2.6-163: 2010 or DBN B 2.6-198: 2014), Russian codes (SNiP II-23-81*, SP 53-103-2004 or SP 16.13330.2011) and European codes (EN 1993-1-8, EN 1993-1-1). Among other things, the selection of codes defines the set of prototypes of joints proposed for the analysis, which includes only the joints that are reflected in the text of the codes.

The COMET software provides the following groups of prototypes for steel structural joints: nominally pinned and rigid column bases, beam and rafter splices, hinged and rigid joints between columns and rafters, and truss joints.

The set of parametric prototypes for each type of joint has been determined on the basis of different requirements, the consideration of which has affected not only the selected designs,

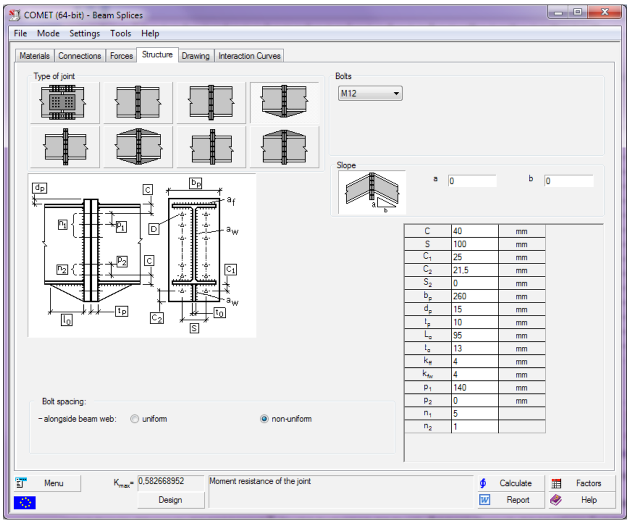

Figure 1. Set of prototypes of the design of beam splices.

but also the parameters necessary for their implementation.
A set of prototypes implemented for beam splices is given as an example in Fig. 1.
The joint prototype is selected by the designer. Only formal checks are performed at the stage of the data input (e.g. correspondence of the set of force factors to the selected joint prototype). Once the calculation is completed, it is up to the designer whether to accept or reject the analyzed design. Given that the time of the calculation of joint is comparable with the time it takes to press a "button" invoking this operation, it becomes possible to analyze other options and make a justified decision.

SOFTWARE IMPLEMENTATION

Conceptual provisions and flowchart

The conceptual basis of the project was the idea of creating a program with the following functions:
• automatic determination of all parameters of the joint which formally satisfies the requirements of design codes for the given internal forces combinations;
• automatic determination of some parameters of the joint, taking into account the fact that other parameters are specified by the user and can not be changed;

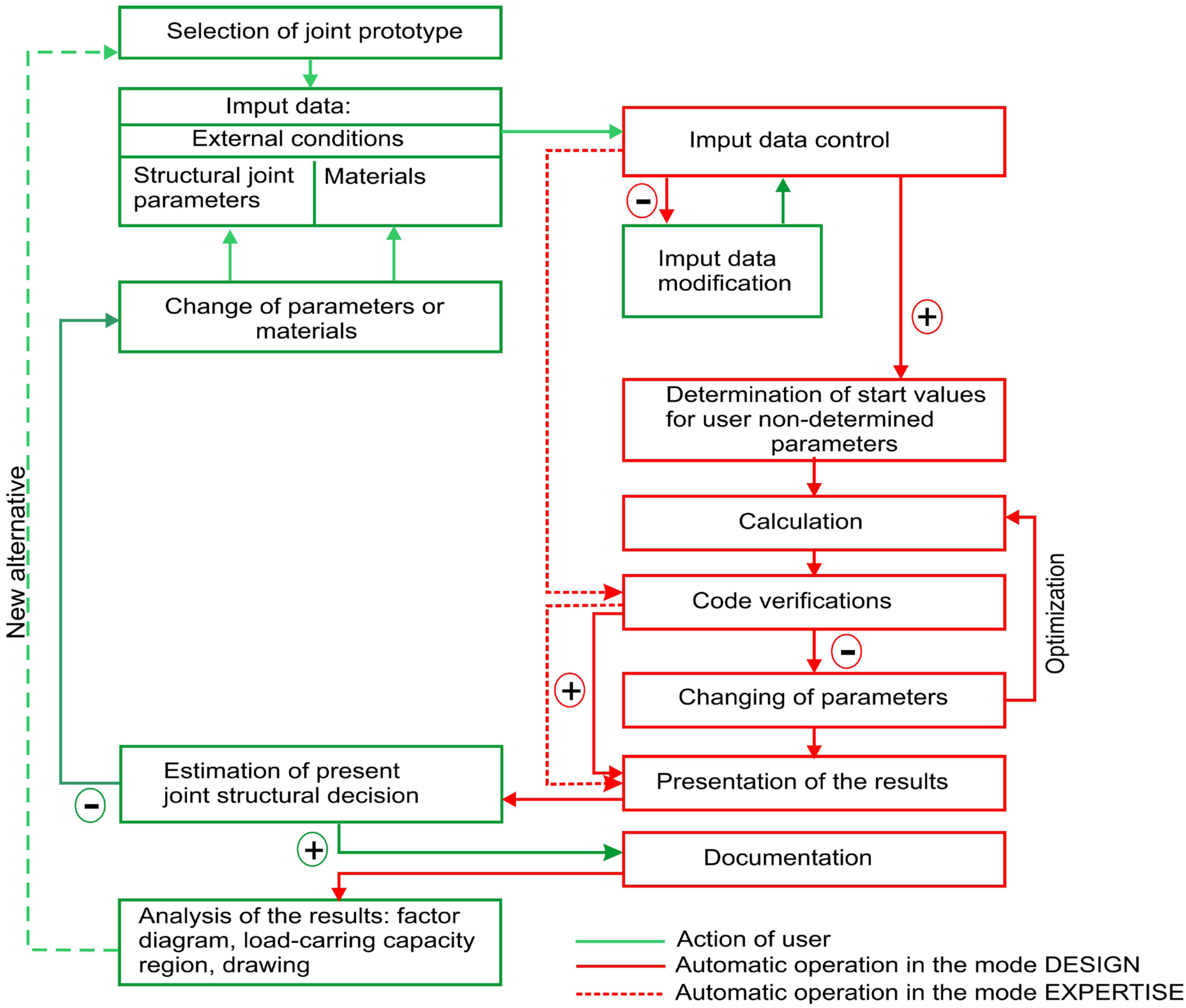

Figure 2. Generalized flow chart in the designer-software interaction mode.

• implementation of all control verifications of whether the load-bearing capacity constraints as well as structural constraints are satisfied in the cases when all parameters of the joint are specified and can not be changed.

A generalized flowchart for solving the problem in the designer-software interaction mode corresponding to this concept is shown in Fig. 2. In fact, the role of the program can change from the "generator of all parameters" of the design specified by the user, taking into accounts the codes and external factors to the "simple check" of the capacity of the joint in accordance with the codes (the check of the parameters specified by the user). In cases when the values of some parameters have to be taken as fixed (user-defined), and all others are determined by the conditions of compliance with the codes, the program works according to the second variant. The program considers the parameters specified by the user as the same kind of initial data like the class of concrete or steel grade.

A role of the structural engineer is an important feature of the concept of searching/checking the parameters of the joint adopted in the program. He is responsible for the choice of the design, completeness and correctness of the loads taken into account and acting on the structure with the considered joint, as well as the analysis of the applicability of the obtained solution. If in the result of the analysis the designer changes some parameters, the program will perform the check and the search for such values of other parameters that would ensure the operation of the joint and would not contradict the codes.

Input and control of the initial data

Initial data include the information about the structural members connected in the considered joint, their sections and steel grades, joint's type (prototype), set of internal forces acting in adjacent sections of the connected structural members (see Fig. 3) as well as the data allowing to select the properties of the used bolted and welded connections (see Fig. 4).

Taking into account the fact that the joint has to work in different design situations, the program enables to specify the necessary number of internal forces combinations. These combinations can be specified by the user or be the result of the calculation performed by structural analyzer SCAD. The main requirement is the simultaneous action of forces included in one combination.

The check of the initial data is performed by the program both in the process of their specification

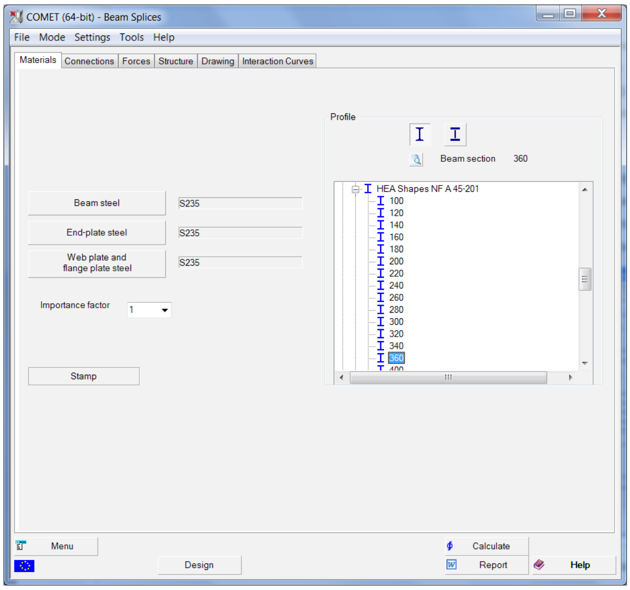

Figure 3. Dialog boxes for specifying the main initial data for beam splices.

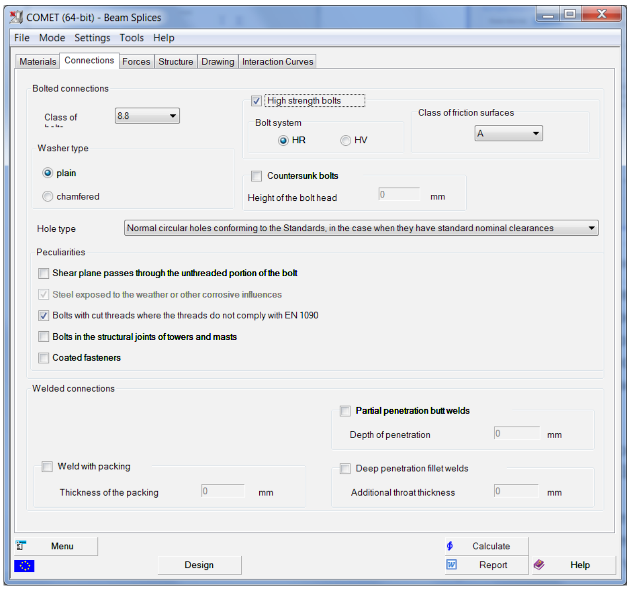

Figure 4. Dialog boxes to specify initial data for welded and bolted connections in structural joints

(detection of formal errors) and in the design process. Diagnostic messages are generated on the basis of the check results.

Software modes

As noted above, the program enables to select the parameters of the joint on the basis of the design selected by the user and the conditions of its reliability and operability under the given operating conditions and materials (see Fig. 5).

If the joint is already operating in the real structure, i.e. all its parameters and operating conditions are known, the check of the joint can be performed, the results of which enable to make a justified decision about the possibility of the operation of the joint in the new conditions (for example, at high loads on the structure).

Thus, two fundamentally different modes have been implemented in the program – DESIGN and EXPERTISE, which are invoked by the respective buttons.

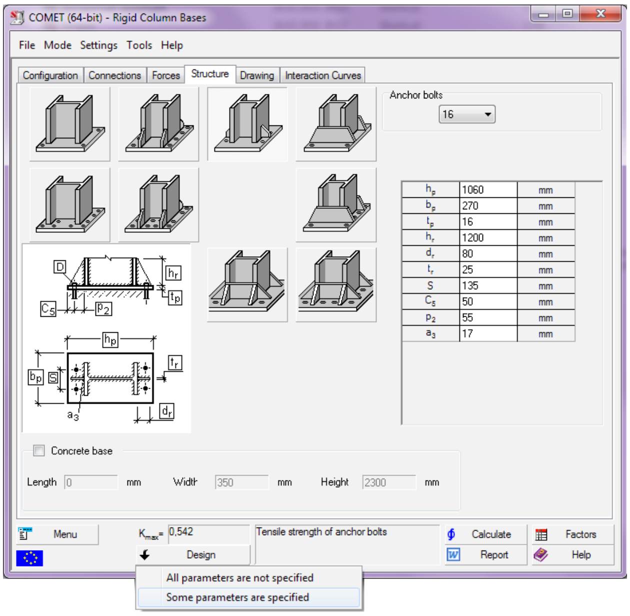

Figure 5. Control of the software mode: button «Design» – searching for structural joint's design decision (mode DESING: calculation of variable parameters and verification of the design decision), botton «Calculate» – verification of the design decision (mode EXPERTISE)

If in the result of the design the parameters of the prototype provide the operability of the joint, but some of them can not be accepted because of the (possibly not formalized) requirements the designer knows, the list of the initial data can be extended by specifying the desired values for such parameters. At the same time, all other parameters not subject to the above requirements are set to zero and the design mode repeats.

When setting the parameters the program analyzes their values and reports detected violations of the design requirements. These can be strict guidelines on the need to change the specified values, or warnings about the violation of the recommendations of codes that can be ignored.

Thus, the technology implemented in COMET supports the mode of active user decision making. Such a mode can both satisfy an experienced designer allowing him to achieve the necessary solution, and allow the beginner to solve a design problem with minimal interference in the decision-making process.

Solution optimization

If the design check indicates the need to change the design parameters, this change is performed in the program on the basis of the sensitivity analysis. The idea of this approach is considered below.

Inequalities of the type $E_{d,i} \le R_{d,i}$ should be satisfied in all checks between the design values of the action effects $E_{d,i}$ and the design strength $R_{d,i}$ according to the limit state calculation method adopted in EN 1990. It is more convenient to represent these inequalities in the following form:

$$k_i = E_{d,i}/R_{d,i} \le 1{,}0 \ (i = 1,..., n) \qquad (1)$$

where k_i is the utilization factor of the i^{th} constraint, it is the reciprocal of the factor of safety. The value of the factors k_i is a function of the governing design parameters X_j $(j = 1,..., p)$.

It should be noted that not all design parameters are independent. Some of them can be considered as governing, while others are unambiguously calculated at the known values of the governing parameters, and are not considered further. Moreover, some values of the parameters can be forced by the user; they are fixed and are not considered in the following description as well.

Automatic selection of the unknown values of the internal parameters of a joint design is implemented as a multiple targeted improvement of a certain initial design of the joint in order to satisfy the bearing capacity constraints taking into account the structural and assortment constraints [6, 7]. Multiple improvement of the design is performed on the basis of the analysis of its sensitivity to variation of the controlled parameters of the joint design. The response of the system, the values of the utilization factors of the load-bearing capacity constraints, is evaluated at each variation of a certain controlled parameter. Let's consider the case when it is necessary to improve the design that does not satisfy the requirements of the codes, since its check has shown that some of the inequalities (1) are violated and the utilization factor of the constraints is greater than one. If an increment ΔX_s is given to one of the parameters, for example X_s, all utilization factors can change obtaining new values $k_{is} = k_i + \Delta k_{is}$. It is logical to first use the change of the parameter ΔX_r for which the value of the greatest utilization factor of restrictions improves the most, i.e. $k_{jr} = \min\limits_{s=1,\dots,p} \max\limits_{i=1,\dots,n} (k_i + \Delta k_{is})$.

PRESENTATION OF RESULTS

Results of the design and verification are given as a diagram of checked factors, drawing with a preliminary design of the considered joint, a family of graphs bounding the region of the load-bearing capacity of the joint in the coordinate system of the selected internal forces and the report in RTF-format.

Factors diagrams

Results of the load-bearing capacity verifications for compliance with the requirements of design codes are given in the form of a factors diagram. Each factor is accompanied by a reference to the respective section of the code which regulates these requirements. The values of the factors are given in the form of the constraint utilization factors (see Fig. 6).

Detail drawing

A graphical representation of the designed joint is given in the form of a simplified drawing, which describes the structure completely and in detail, including the specification (see Fig. 7), but it does not take into account the manufacturer's technical requirements. In order to correct the drawing it

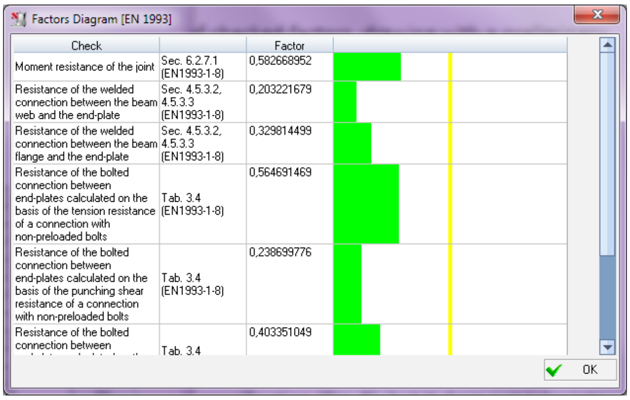

Figure 6. Factors diagram.

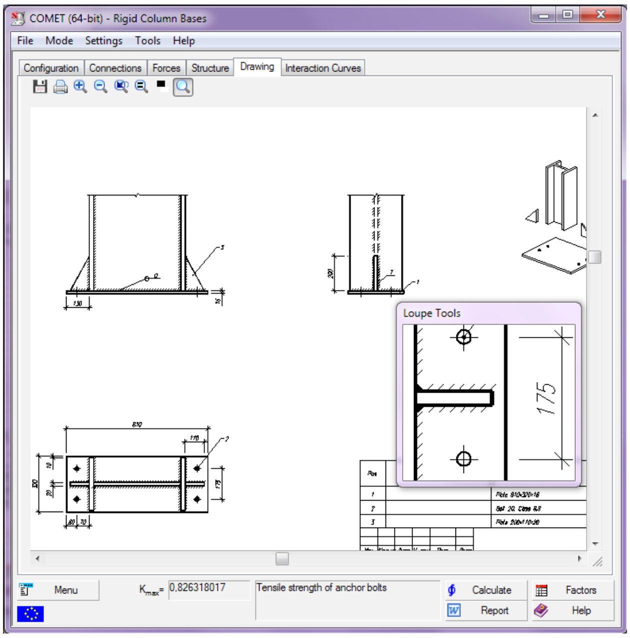

Figure 7. Design drawing on the screen.

can be presented in DXF-format, a file that can be used by various graphic editors.

Load-bearing capacity regions for structural joints

Load-bearing capacity region for the structural joint is a family of graphs in the coordinate system of the selected internal forces bounding the region where all utilization factors are less or equal to one or, by other words, where all inequalities (1) are satisfied (see Fig. 8). Such family of graphs gives us a representation of the load-bearing capacity of the designed joint in terms of the selected design code [2].

Plotting each variant of such a region deals with a design verification of hundreds of internal forces combinations. It seems to us that such a large-scale verification has never been performed before.

The program also enables to show the position of points corresponding to the internal forces and to plot a convex shell on the basis of these points thus bounding the part of the load-bearing capacity region (see Fig. 8), which corresponds to any linear combination of design internal forces in the considered joint.

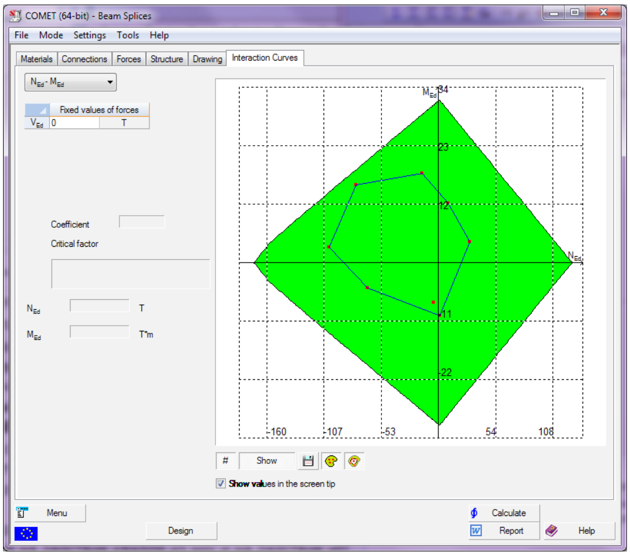

Figure 8. Plotting the load-bearing capacity region together with points corresponded to the acting internal forces in the joint and convex shell of internal forces.

CONCLUSIONS

Algorithm for designing each joint prototype has been presented as a set of operations implementing the rules for determining the interrelated values of the joint parameters. Searching of unknown joint parameters has been transformed to a decision making problem based on an analysis of mathematical model of the joint. Automatic searching of unknown joint parameters has been implemented as a multiple targeted improvement of a certain initial joint design in order to satisfy load-bearing capacity constraints taking into account the structural and assortment-based constraints. Multiple improvement of current joint design is performed on the basis of sensitivity analysis relative to variation of governing joint parameters.

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INTERACTION OF LARGE PILES WITH A MULTILAYER SOIL MASS, TAKING INTO ACCOUNT HARDENING AND SOFTENING

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Abstract: This article discusses the formulation and solution of the problem of the interaction of a long pile with the surrounding multilayer and underlying soils, taking into account the rheological properties of the surrounding soil mass. The creep process is considered taking into account hardening and softening. The problem was considered in a linear setting. The solution is presented by analytical method. To describe the creep process, the rheological parameters of hardening and softening were used. An expression is obtained for finding the reduced shear modulus for a multilayer soil mass. A dependence is obtained for determining the force on the pile heel on time, taking into account the rheological parameters of hardening and softening. Analytical solutions in the article are supported by a graphical part. The graphs of the dependence of the settlement of the pile, the force on the heel of the pile cutting through alternating layers, on time for various parameters of viscosity, as well as for variable parameters of hardening and softening are given. The solutions obtained can be used for preliminary determination of the movement of long piles with the surrounding multilayer and underlying soils.

Keywords: pile interaction, multilayer and underlying soils, analytical method, reduced modulus, rheological properties, settlement rate, hardening factor, softening factor.

ВЗАИМОДЕЙСТВИЕ СВАИ БОЛЬШОЙ ДЛИНЫ С МНОГО-СЛОЙНЫМ МАССИВОМ ГРУНТА С УЧЕТОМ УПРОЧНЕНИЯ И РАЗУПРОЧНЕНИЯ

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Аннотация: В данной статье рассмотрена постановка и решение задачи о взаимодействии сваи большой длины с окружающим многослойным и подстилающим грунтами с учетом реологических свойств окружающего массива грунта. Процесс ползучести рассматривается с учетом упрочнения и разупрочнения. Задача рассматривалась в линейной постановке. Решение изложено аналитическим методом. Для описания процесса ползучести использовались реологические параметры упрочнения и разупрочнения. Получено выражение для нахождения приведенного модуля сдвига для многослойного массива грунта. Получена зависимость для определения усилия на пяту сваи от времени с учетом реологических параметров упрочнения и разупрочнения. Аналитические решения в статье подкреплены графической частью. Приведены графики зависимости осадки сваи, усилия на пяту сваи, прорезающей чередующиеся слои, от времени при различных параметрах вязкости, а также при переменных параметрах упрочнения и разупрочнения. Полученные решения могут быть использованы для предварительного определения перемещения сваи большой длины с окружающим многослойным и подстилающим грунтами.

Ключевые слова: взаимодействие сваи, многослойный и подстилающий грунты, аналитический метод, приведенный модуль, реологические свойства, скорость осадки, коэффициент упрочнения, коэффициент разупрочнения.

INTRODUCTION

Construction sites are characterized mainly by difficult engineering and geological conditions, represented by the presence of several layers at the base, including weak clayey water-saturated soils. Under these conditions, as a rule, it is used: soil consolidation [1–5], soil reinforcement [6–7], significant deepening of the underground part of buildings, etc. But the pile foundation is considered as the main type of foundation on such sites [8–14]. When the basement of weak clayey soils occurs, the settlement of the building can continue for a long period of time. There are cases when the settlements of buildings and structures did not subside for several decades. The most famous example of the Leaning Tower of Pisa, the slope of which developed over several centuries. In the design of foundations on such soils, the forecast of settlement over time is of great importance. The strength and stability of structures will depend both on the rate of development of the settlement over time and on the final settlement of the structure. Therefore, the approach to the description of the process of foundation settlement should be considered as rheological [15–22]. It is known that when a long pile interacts with the surrounding multilayer and underlying soils, a complex inhomogeneous stress-strain state arises. In this paper, we consider the problem of the interaction of a long pile with a multilayer soil massif in a linear formulation, which has rheological properties, as well as the problem of determining the reduced shear modulus for a soil massif.

Studies of the operation of a long pile show that the effect of the length of the pile on the surrounding soil mass extends to a distance of no more than 6–7 pile diameters, and of the same order in depth under its lower end [23]. The distance between the piles less than six diameters ensures the displacement of the pile and the soil in the inter-pile space at the same time. These studies allow us to consider the displacement of the pile foundation and soil as a single massif, and also allow the problem of the interaction of a long pile with a soil massif to be considered as the problem of the interaction of a pile with a soil massif of limited dimensions in the form of a cylinder with a diameter $2b$ and height $L > l$, where l is the length of the pile (Fig. 1).

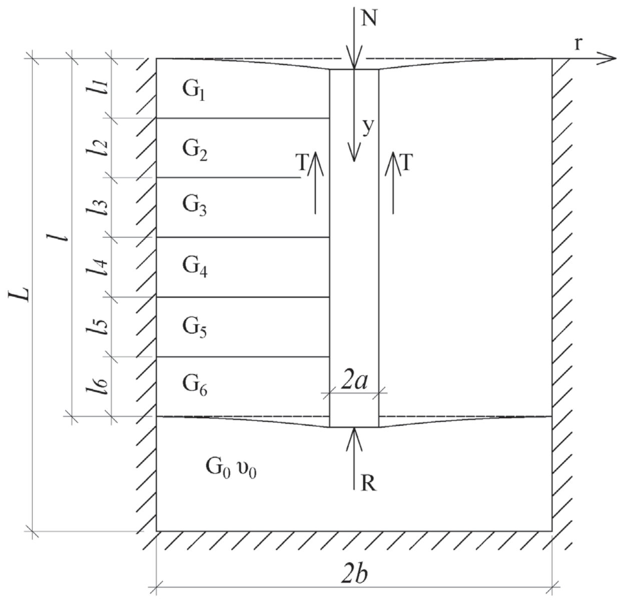

Figure 1. Design model of interaction of pile with multilayer soil column.

MATERIALS AND METHODS

Analysis of the stress-strain state of soils around the pile and under its end showed that shear deformations prevail when the pile interacts with the soil, volumetric deformations can be ignored [24]. The solution to the problem will be considered for a round pile. We also assume that the stiffness of the pile significantly exceeds the stiffness of the soil.

$$E_{св.} >> E_{гр.}.$$

Let us write down the equilibrium equation for the considered case (Fig. 1):

$$N = T + R, \quad (1)$$

where

$$N = \pi a^2 p_1 , \quad (2)$$

$$T = 2\pi a l \tau \ , \qquad (3)$$

$$R = \pi a^2 p_2 . \qquad (4)$$

Substituting equations (2), (3), (4) into equation (1), we obtain expressions for τ :

$$\tau = (p_1 - p_2)\frac{a}{2l} . \qquad (5)$$

Since $E_{св.}$ >> $E_{гр.}$ the settlement of the pile of each layer under consideration is equal, i.e.

$$S_1 = S_2 = S_i = S , \qquad (6)$$

where S_i is a pile settlement for *i*-for layer; *S* is – a total pile settlement.

Shear deformation of the elementary soil layer around the pile can be determined by the following relationship:

$$\gamma_i(r) = -\frac{\tau_i(r)}{G_i} , \qquad (7)$$

where G_i is shear modulus for *i* -th layer; *i* = *1, 2, ..., n* is layer number.

Based on condition (6), we can write an expression for the tangential stresses of the i-th layer:

$$\tau_i = \frac{\tau}{G} G_i , \qquad (8)$$

where G is a reduced shear modulus for multilayer soil mass.

Based on the condition for the distribution of shear stresses along the length of the pile, we obtain:

$$\tau l = \tau_1 l_1 + \tau_2 l_2 + \tau_i l_i , \qquad (9)$$

Considering (9) and (8) together, we obtain an expression for determining the reduced shear modulus for a multilayer soil mass:

$$G = \frac{l_1 G_1 + l_2 G_2 + l_i G_i}{l} , \qquad (10)$$

where *l* is a pile length; *i* = *1, 2, ..., n* is layer number.

Let us write the dependence for the rate of change in shear deformation around the pile, taking into account the rheological properties of the surrounding soil mass:

$$\dot{\gamma} = -\frac{\dot{\tau}_\alpha}{G} + \frac{\tau_\alpha}{\eta(t)} , \qquad (11)$$

where $\dot{\tau}_\alpha$ is shear stress rate; $\tau_\alpha = T / 2\pi a l$; $\eta(t)$ – weighted average viscosity index.

Because the forces transferred to the pile are constant ($p_1 = const$), the pressure rate at the pile head does not change ($\dot{p}_1 = 0$). Based on this, we determine the rate of change in shear stresses:

$$\dot{\tau}_\alpha = -\dot{p}_2 \frac{a}{2l} , \qquad (12)$$

The rate of settlement of the pile from the action of shear stresses on the lateral surface, taking into account the elastic-viscous characteristics of the surrounding soil mass:

$$\dot{V}_T = \frac{a\tau_\alpha}{\eta(t)} \ln\left(\frac{b}{a}\right) + \frac{a\dot{\tau}_\alpha}{G} \ln\left(\frac{b}{a}\right), \qquad (13)$$

where *G* was obtained from (10).

Let us determine the rate of settlement of the pile due to the deformation of the soil under the lower end of the pile, assuming that the pile acts as a flat round stamp. The equation is:

$$\dot{V}_R = \dot{p}_2 \frac{\pi a(1-\nu_0)K}{4G_0} , \qquad (14)$$

where $\dot{p}_2$ is pressure rate under the pile heel; ν_0 and G_0 is deformation parameters of the soil under the lower end of the pile; $K \leq 1$ – coefficient taking into account the depth of application of the load on the pile heel.

Based on the fact that $E_{св.} >> E_{гр.}$ the rate of settlement from forces on the lateral surface is equal to the rate of settlement from the action of forces at the level of the lower end of the pile. Equating (13) and (14), as well as taking into account (5) and (12), we obtain:

$$(p_1 - p_2)\frac{a^2}{2l\eta(t)}\ln\left(\frac{b}{a}\right) - \dot{p}_2\frac{a^2}{2lG}\ln\left(\frac{b}{a}\right) = = \dot{p}_2\frac{\pi a(1-\nu_0)K}{4G_0}, \quad (15)$$

After performing certain transformations, we get the following differential equation:

$$\dot{p}_2 + p_2\frac{1}{\eta(t)A} = \frac{p_1}{\eta(t)A}, \quad (16)$$

где

$$A = \frac{\pi(1-\nu_0)Kl}{2G_0 a\ln\left(\frac{b}{a}\right)} + \frac{1}{G}. \quad (17)$$

The general solution of the differential equation (16) is found by the formula [25]:

$$p_2(t) = e^{-\int\frac{dt}{\eta(t)A}}\left(\int\frac{p_1}{\eta(t)A}e^{\int\frac{dt}{\eta(t)A}}dt + C\right). \quad (18)$$

To describe the creep process, we use the rheological parameters of hardening and softening. Let us consider the solution of Eq. (18) taking into account hardening, when $\eta(t) = \eta_0 e^{\alpha t}$. In this case:

$$p_2(t) = e^{-\int\frac{dt}{\eta_0 e^{\alpha t}A}}\left(\int\frac{p_1}{\eta_0 e^{\alpha t}A}\cdot e^{\int\frac{dt}{\eta_0 e^{\alpha t}A}}dt + C\right), \quad (19)$$

where η_0 is initial coefficient of soil viscosity; α is soil hardening factor.

$$p_2(t) = e^{\frac{e^{-\alpha t}}{\alpha\eta_0 A}}\left(\int\frac{p_1}{\eta_0 e^{\alpha t}A}e^{-\frac{e^{-\alpha t}}{\alpha\eta_0 A}}dt + C\right) = = e^{\frac{e^{-\alpha t}}{\alpha\eta_0 A}}\left(p_1 e^{-\frac{e^{-\alpha t}}{\alpha\eta_0 A}} + C\right) = p_1 + Ce^{\frac{e^{-\alpha t}}{\alpha\eta_0 A}}. \quad (20)$$

Integration constant C is determined from the initial condition at t= 0. Then:

$$C = (p_2(0) - p_1)\Big/ e^{\frac{1}{A\alpha\eta_0}}. \quad (21)$$

Finally, we obtain:

$$p_2(t) = p_1 + \left(p_2(0) - p_1\right)e^{\frac{e^{-\alpha t}-1}{A\alpha\eta_0}}. \quad (22)$$

The settlement of the pile at a certain point in time t can be determined by the formula:

$$V_R(t) = p_2(t)\frac{\pi a(1-\nu_0)K}{4G_0}, \quad (23)$$

where $p_2(t)$ we find by the formula (22).

Consider the solution to Eqs. (22) and (23) with the initial condition $p_2(0) = 0$ with variable values $\eta_1 = 1\cdot 10^{11}\,П$, $\eta_2 = 5\cdot 10^{11}\,П$, $\eta_3 = 1\cdot 10^{12}\,П$, $\eta_4 = 5\cdot 10^{12}\,П$, а также $\alpha = 0.05$; $l = 30$ m; $a = 0.5$ m; $b = 6.5\cdot a$; *E₁* = *30 MPa, E₂ = 10 MPa, E₃ = 25 MPa, E₀ = 50 MPa*; $\nu_1 = \nu_2 = \nu_3 = \nu_0 = 0.35; K = 0.7$

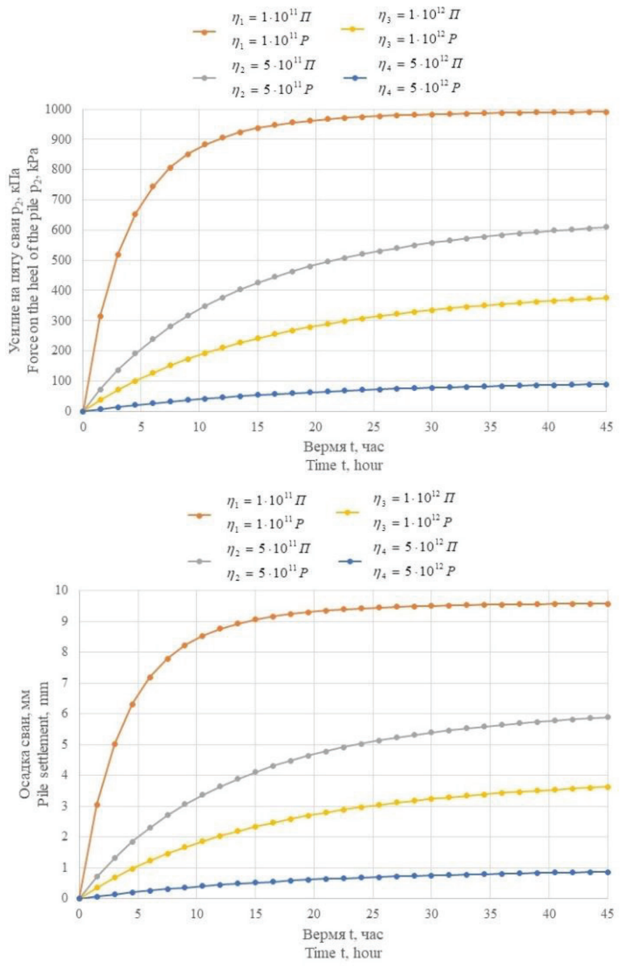

Figure 2. Dependency graphs $p_2(t)$ (up) and $V_R(t)$ (down) at various parameters of the viscosity of the surrounding soil.

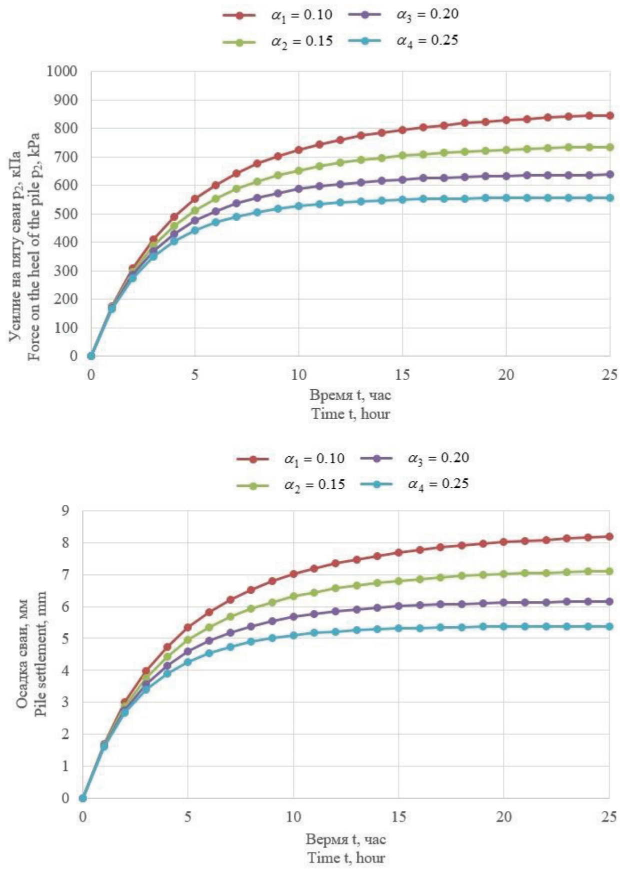

Figure 2. Dependency graphs $p_2(t)$ (up) and $V_R(t)$ (down) at different values of the hardening coefficient

Consider the solution to Eqs. (22) and (23) for variable values $\alpha_1 = 0.10, \alpha_2 = 0.15, \alpha_3 = 0.20, \alpha_4 = 0.25$, as well as $\eta_1 = 1\cdot10^{12}\,П$; $l = 30$ m; $a = 0.5$ m; $b = 6.5\cdot a$; $E_1 = 30$ *MPa*, $E_2 = 10$ *MPa*, $E_3 = 25$ *MPa*, $E_0 = 50$ *MPa*; $\nu_1 = \nu_2 = \nu_3 = \nu_0 = 0.35$; $K = 0.7$

Let us consider the solution of Eq. (18) taking into account the softening, when $\eta(t) = \eta_0 e^{-\beta t}$. In this case:

$$p_2(t) = e^{-\int \frac{dt}{\eta_0 e^{-\beta t} A}} \left(\int \frac{p_1}{\eta_0 e^{-\beta t} A} \cdot e^{\int \frac{dt}{\eta_0 e^{-\beta t} A}} dt + C \right), \quad (24)$$

where η_0 is initial coefficient of soil viscosity; β is soil softening factor.

$$p_2(t) = e^{-\frac{2e^{\beta t}}{\eta_0 A}} \left(p_1 e^{\frac{2e^{\beta t}}{\eta_0 A}} + C \right) = p_1 + Ce^{-\frac{2e^{\beta t}}{\eta_0 A}}. \quad (25)$$

Integration constant C is determined from the initial condition at. Then:

$$C = (p_2(0) - p_1) \Big/ e^{-\frac{2}{\eta_0 A}} \, . \qquad (26)$$

Finally, we obtain:

$$p_2(t) = p_1 + \left(p_2(0) - p_1\right) e^{\frac{-2(e^{\beta t}-1)}{A\eta_0}} \, . \qquad (27)$$

Consider the solution to Eqs. (27) and (23) with the initial condition $p_2(0) = 0$ with variable values $\eta_1 = 1 \cdot 10^{11}\,\Pi$, $\eta_2 = 5 \cdot 10^{11}\,\Pi$, $\eta_3 = 1 \cdot 10^{12}\,\Pi$, $\eta_4 = 5 \cdot 10^{12}\,\Pi$, as well as $\beta = 0.05$; $l = 30$ *м*; $a = 0.5$*м*; $b = 6.5 \cdot a$; *E₁ = 30 MPa, E₂ = 10 MPa, E₃ = 25 MPa, E₀ = 50 MPa*; $\nu_1 = \nu_2 = \nu_3 = \nu_0 = 0.35; K = 0.7$

Consider the solution to Eqs. (27) and (23) for variable values $\beta_1 = 0.10$, $\beta_2 = 0.15$, $\beta_3 = 0.20$, $\beta_4 = 0.25$, as well as $\eta_1 = 1 \cdot 10^{12}\,\Pi$; $l = 30$ m; $a = 0.5$ m; $b = 6.5 \cdot a$; *E₁ = 30 MPa, E₂ = 10 MPa, E₃ = 25 MPa, E₀ = 50 MPa*; $\nu_1 = \nu_2 = \nu_3 = \nu_0 = 0.35; K = 0.7$

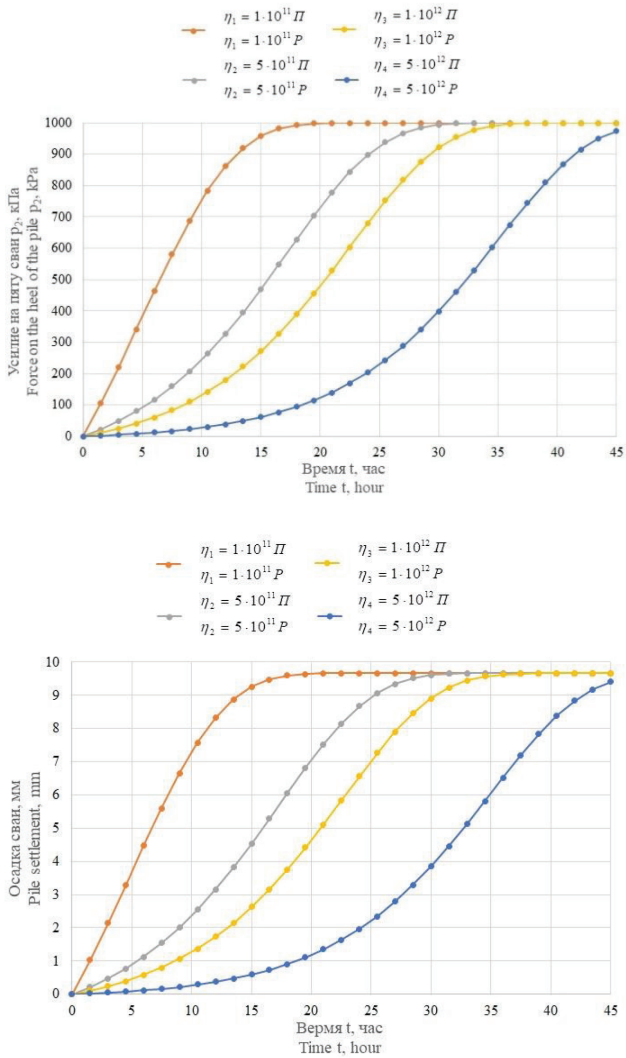

Figure 4. Dependency graphs $p_2(t)$ (up) and $V_R(t)$ (down) at various parameters of the viscosity of the surrounding soil

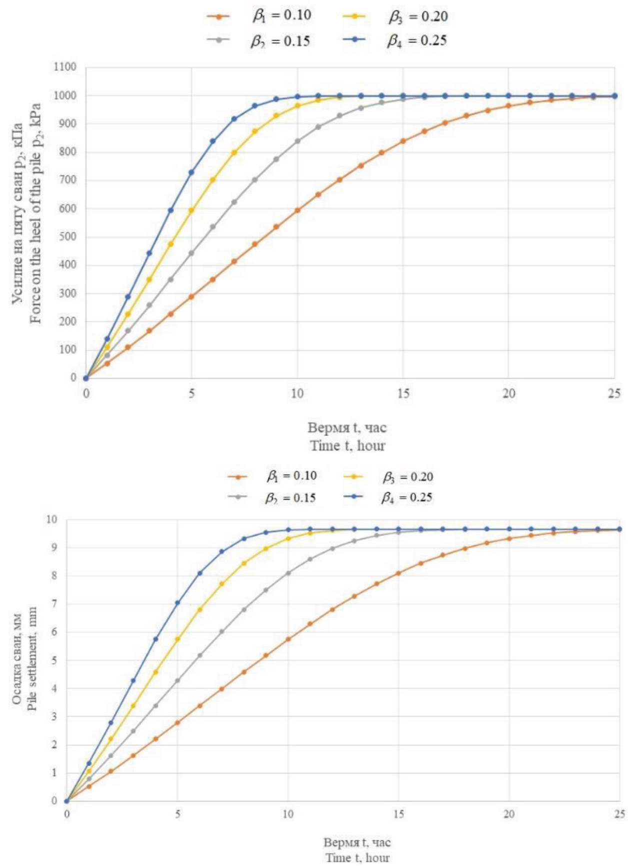

Figure 5. Dependency graphs $p_2(t)$ (up) and $V_R(t)$ (down) at different values of the hardening coefficient

RESULTS

Analysis of the obtained dependences shows that over time, the stress under the pile base and the pile settlement change at different rates and tend to a constant value (for $t \to \infty$, $p_2(t) \to p_{2\infty} = const$, $V_R(t) \to V_{R\infty} = const$). Therefore, based on condition (5), the shear stresses on the lateral surface of the pile decrease with time. The obtained dependencies make it possible to predict the development of precipitation in time. According to the data obtained, with an increase in the viscosity coefficient, a decrease in the rate of stress change under the pile base, as well as pile settlement, is observed. According to the obtained dependences (Fig. 3, 5), with an increase in the hardening coefficient, the final pressure under the pile base and the pile settlement decreases. Taking into account the rheological coefficient of softening, a complex nature of the time redistribution of efforts between the head and the fifth pile is observed.

CONCLUSION AND DISCUSSION

When the pile interacts with the surrounding multilayer soil massif, which has elastic-viscous characteristics, a complex stress-strain state arises, in which the stress under the pile base changes over time. According to the obtained dependencies, over time, the stresses on the heel of the pile increase, while the shear stresses decrease. The obtained dependencies allow predicting the development of pile settlement in time. The rheological properties of a multilayer soil massif have a significant impact on the nature of the redistribution of forces on the pile between the lateral surface and the lower end. The results obtained indicate that the formation of stress-strain state around the pile in time depends on the form of the function of changing the viscosity parameters. Analysis of the graphs obtained shows that the time of stabilization of the sediment, as well as the time of stabilization of the pressure under the heel of the pile, significantly depends on the rheological coefficients of hardening and softening.

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AIMED CONTROL OF THE FREQUENCY SPECTRUM OF EIGENVIBRATIONS OF ELASTIC PLATES WITH A FINITE NUMBER OF DEGREES OF FREEDOM OF MASSES BY SUPERIMPOSING ADDITIONAL CONSTRAINTS

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Abstract: As is known, for some elastic systems with a finite number of degrees of freedom of masses, for which the directions of motion of the masses are parallel and lie in the same plane, methods have been developed for creating additional constraints that purposefully change the spectrum of natural frequencies. In particular, theory and algorithm for the formation of aimed additional constraints have been developed for the rods, the introduction of each of which does not change any of the modes of natural vibrations, but only increases the value of only one frequency, without changing the values of the remaining frequencies. The distinctive paper is devoted to the method of forming a matrix of additional stiffness coefficients corresponding to such aimed constraint in the problem of natural vibrations of rods. This method can also be applied to solving a similar problem for elastic systems with a finite number of degrees of freedom, in which the directions of motion of the masses are parallel, but not lie in the same plane. In particular, such systems include plates. However, the algorithms for the formation of aimed additional constraints, developed for rods and based on the properties of rope polygons, cannot be used without significant changes in a similar problem for plates. The method for the formation of design constraint schemes that purposefully change the spectrum of frequencies of natural vibrations of elastic plates with a finite number of degrees of freedom of masses, will be considered in the next work.

Keywords: frequency of natural vibrations, form of natural vibrations, aimed constraint, additional constraint, stiffness coefficients

ПРИЦЕЛЬНОЕ РЕГУЛИРОВАНИЕ СПЕКТРА ЧАСТОТ СОБСТВЕННЫХ КОЛЕБАНИЙ УПРУГИХ ПЛАСТИН С КОНЕЧНЫМ ЧИСЛОМ СТЕПЕНЕЙ СВОБОДЫ МАСС ПУТЁМ НАЛОЖЕНИЯ ДОПОЛНИТЕЛЬНЫХ СВЯЗЕЙ

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Аннотация: К настоящему времени для некоторых упругих систем с конечным числом степеней свободы масс, у которых направления движения масс параллельны и лежат в одной плоскости, разработаны методы создания дополнительных связей прицельно изменяющих спектр частот собственных колебаний. В частности, для стержней разработаны теория и алгоритм формирования прицельных дополнительных связей, введение каждой из которых не изменяет ни одну из форм собственных колебаний, а лишь увеличивает значение только одной частоты, не изменяя величин остальных частот. В данной статье показано, что метод формирования матрицы дополнительных коэффициентов жесткости, характеризующих такую прицельную связь в задаче о собственных колебаниях стержней, может быть применен и при решении аналогичной задачи для упругих систем с конечным числом степеней свободы, у которых направления движения масс параллельны, но не лежат в одной плоскости. В частности, к таким системам относятся пластины. Однако, алгоритмы формирования прицельных дополнительных связей, разработанные для стержней на основе свойств верёвочных многоугольников, не могут быть без существенных изменений использованы в аналогичной задаче для пластин. Метод формирования расчётных схем связей, прицельно изменяющих спектр частот собственных колебаний упругих пластин с конечным числом степеней свободы масс, будет рассмотрен в следующей работе.

Ключевые слова: частота собственных колебаний, форма собственных колебаний, прицельная связь, дополнительная связь, коэффициенты жесткости

When designing structures, in some cases it becomes necessary to deduce one or several natural frequencies from a certain frequency interval. As is known (see, for instance, [1-5]), one of the methods for solving such a problem is the creation of aimed generalized constraints. Each of aimed generalized constraints increases the value of only one selected natural frequency to a predetermined value in advance, without changing the rest of the natural frequencies and all natural modes.

In the above-mentioned works for elastic rods carrying a finite amount of concentrated masses (corresponding sample is presented in Figure 1*a*), a method was developed for creating aimed constraints. The principal system of the displacement method was chosen (Figure 1*b*). The equations of the displacement method were written in the form traditional for systems with a finite number of degrees of freedom (n)

$$\begin{aligned}
&(r[1,1]+m[1]\omega^2)v[1,j]+r[1,2]v[2,j]+\ldots\\
&\quad +r[1,q]v[q,j]+\ldots+r[1,n]v[n,j]=0\\
&r[2,1]v[1,j]+(r[2,2]+m[2]\omega^2)v[2,j]+\ldots\\
&\quad +r[2,q]v[q,j]+\ldots+r[2,n]v[n,j]=0\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&r[n,1]v[1,j]+r[n,2]v[2,j]+\ldots+r[n,q]v[q,j]+\\
&\quad +\ldots+(r[n,n]+m[n]\omega^2)v[n,j]=0
\end{aligned} \tag{1}$$

It should be noted that elements $r[i,k]$ form a matrix of stiffness coefficients, $A=\|r[i,k]\|$; $m[i]$ are masses, which form a diagonal matrix, $M=\|m[i]\|$; ω is the natural frequency of the system; $v[k,j]$ are displacements in the direction of motion of the masses in the j-th ($(j=1,2,..,q,,..,n)$) natural mode.

Roots of the equation

$$\left|A-\omega^2 M\right|=0 \tag{2}$$

determine the spectrum of natural frequencies of the system

$$\omega[1],\omega[2],\ldots,\omega[q-1],\omega[q],\omega[q+1],\ldots,\omega[n]\,. \tag{3}$$

The problem of creation of aimed constraint, which increases only one natural frequency (for instance, $\omega[q]$) to a predetermined value (ωs) in advance and does not change any of the natural modes and the values of the remaining frequencies of the spectrum is under consideration. It is shown that the creation of such constraint is based on the formation of a matrix of additional stiffness coefficients

$$A_0=A_{SO}A_S=A_{SO}\left\|a_0\,[i,k]\right\|_{i,k=1}^{n}, \tag{4}$$

where $$A_S=\left\|a_0\,[i,k]\right\|_{i,k=1}^{n}. \tag{5}$$

The matrix A_0 must have special properties. If the introduced constraint is “aimed” at the q-th natural frequency, then the stiffness coefficients $\left\|a_0\,[i,k]\right\|_{i,k=1}^{n}$ should be orthogonal to the coordinates of the natural modes corresponding to the remaining $(n-1)$ frequencies of the spectrum.

Thus, we have

$$\sum_{k=1}^{n}a_0[i,k]v[k,j]=0, \quad (i=1,2,..,n,\ j=1,2,..,(q-1),(q+1),..,n) \tag{6}$$

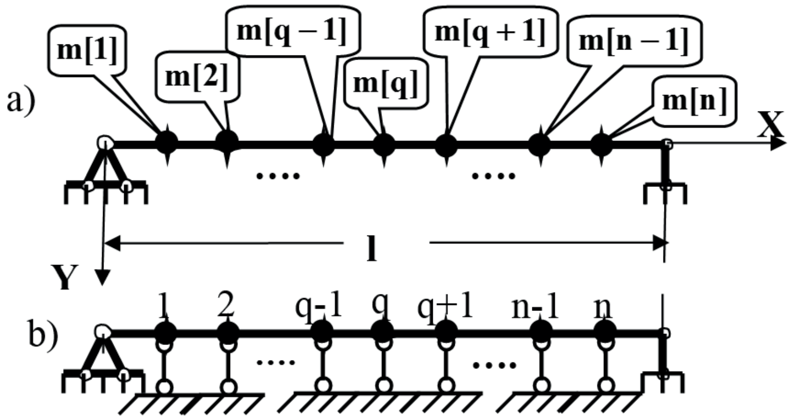

Figure 1. Sample of analysis.

The coefficients are not orthogonal with respect to the q-th natural frequency, at which the introduced constraint is "aimed". We have

$$\sum_{k=1}^{n} a_0[i,k]v_\omega[k,q] \neq 0, \quad (i = 1,2,..,n). \qquad (7)$$

It is shown (for instance, [2-5]) that conditions (6) and (7) will be satisfied by the coefficients

$$a_0[i,k] = m[i]m[k]v_\omega[i,q]v_\omega[k,q]. \qquad (8)$$

The value of the factor A_{SO} is defined as the root of the equation

$$\left|(A - \omega_S^2 M) + A_{SO}A_S\right| = 0. \qquad (9)$$

Since the q th natural mode of the original system remains natural mode after the introduction of the aimed constraint and at the frequency ω_S, the factor A_{SO} can be found as

$$A_{SO} = \frac{-\sum_{i=1}^{n}\sum_{k=1}^{n}(a[i,k] - \omega_S^2 m[i,k])v_\omega[i,q]v_\omega[k,q]}{\sum_{i=1}^{n}\sum_{k=1}^{n} a_0[i,k]v_\omega[i,q]v_\omega[k,q]}. \qquad (10)$$

The result of solving the equation

$$\left|(A + A_{SO}A_S) - \omega^2 M\right| = 0. \qquad (11)$$

must confirm that the natural modes have not changed, and the aimed frequency has increased to ω_S.

The supporting device, which will correspond to the matrix of additional stiffness coefficients, must ensure the relationship between the nodal displacements is the same as between the coordinates of the q-th form of natural modes of the original system. It has been shown (see, for instance, [2-5]) that such a relationship will be realized if the additional support system transfers forces to the nodes of the main system of the bar. The relationships between which are proportional to the values

$$R_0[i] = m[i]v[i,(q)]. \qquad (12)$$

An example of such a generalized aimed constraint for a rod is a sprengel, the outline of which is determined by a rope polygon built in the plane of motion of masses by forces (see, for instance, [2-5]).

This constraint is once statically undefined. Vertical posts of constraint are installed in the nodes of the main system in the direction of mass movement. The prestressing of any one rod will cause such forces in the truss struts, the ratios between which will be the same as the ratios between the forces $R_0[i]$.

The distinctive paper is devoted to the problem of creating of aimed constraint for elastic plates carrying a finite number of concentrated masses.

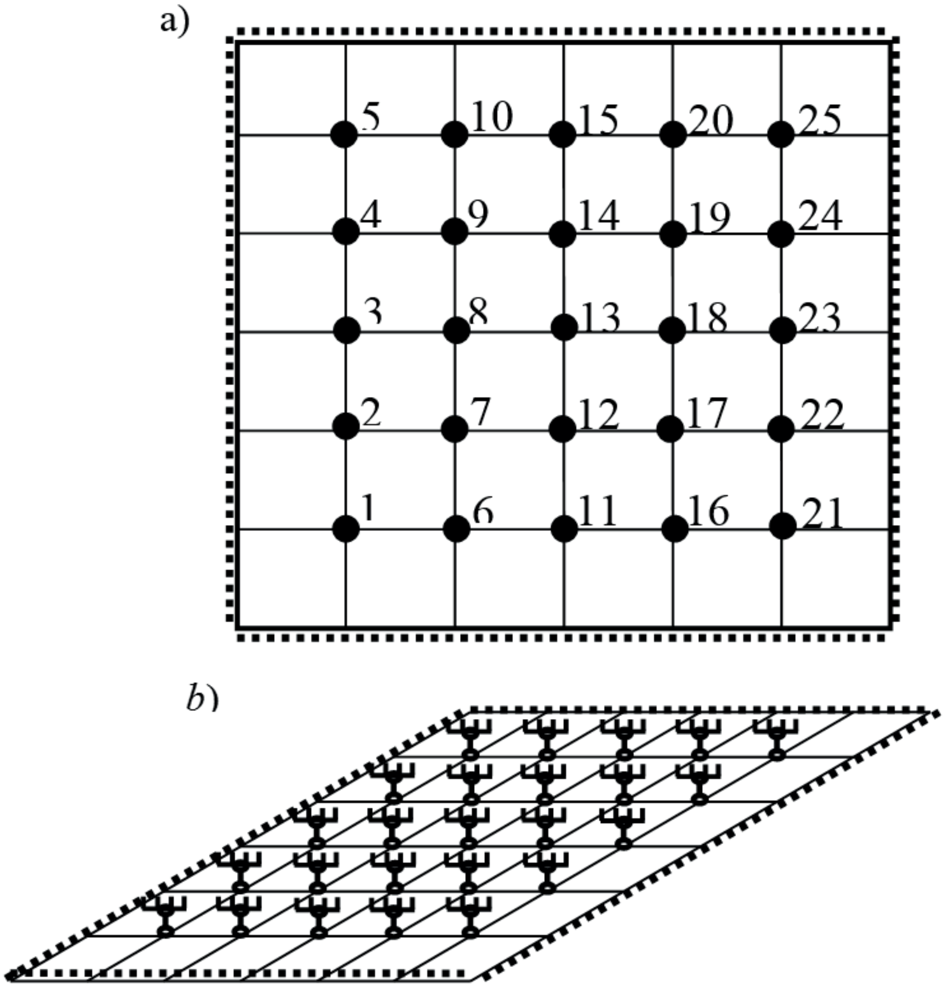

Figure 2. Sample of plate and principal system of the displacement method.

As in a similar problem for the rods, it is proposed to use the principal system of the displacement method. Examples of such a plate and the corresponding principal system of the displacement method are shown in Figures 2*a* and 2*b*.

If we accept one-dimensional numbering in the method of displacements for plates, then the equations in this case will also be written in the form (1). Additional constraints in the principal system and displacements $v[k,j]$ in the equations correspond to the direction of motion of the masses.

The derivation of the expressions for the coefficients of the matrix of additional stiffnesses (4) for the rods is based on the properties of the natural modes.

Since the natural modes of the plates, as well as for rods, are orthogonal,

$$\sum_{k=1}^{n} m[k] v_{\omega}[k,q] v_{\omega}[k,j] = 0, \quad (q \neq j), \quad (13)$$

then the problem for elastic plates carrying a finite number of concentrated masses, as well as for rods, will be based on the formation of a matrix of additional stiffness coefficients (4). The coefficients $\left\| a_0[i,k] \right\|_{i,k=1}^{n}$ must also satisfy conditions (6) and (7) and be determined by dependencies (8).

Let us give an example that confirms that the matrix of additional stiffness coefficients (4) serves as the basis for creating aimed constraints also for elastic plates carrying a finite number of concentrated masses.

Let us consider a plate carrying 25 concentrated masses (Figure 1*a*).

Table 1. The values of the first five frequencies of natural modes of the plate and the coordinates of their corresponding natural modes.

	Initial frequencies and modes					**Modified frequencies and modes**				
w	36.6583	91.0084	92.7466	146.834	178.911	91.0084	92.7466	100.00	146.8337	178.911
1	0.0830	0.1995	0.0499	-0.2495	0.1547	-0.1995	-0.0499	0.0830	-0.2495	0.1547
2	0.1434	0.2935	-0.0043	-0.2433	-0.0063	-0.2935	0.0043	0.1434	-0.2433	-0.0063
3	0.1649	0.2568	-0.1468	0.0129	-0.1657	-0.2568	0.1468	0.1649	0.0129	-0.1657
4	0.1420	0.1514	-0.2494	0.2641	-0.0063	-0.1514	0.2494	0.1420	0.2641	-0.0063
5	0.0818	0.0579	-0.1971	0.2624	0.1548	-0.0579	0.1971	0.0818	0.2624	0.1548
6	0.1441	0.2533	0.1398	-0.2517	0.2788	-0.2533	-0.1398	0.1441	-0.2517	0.2788
7	0.2492	0.3484	0.0840	-0.2447	0.0003	-0.3484	-0.0840	0.2492	-0.2447	0.0003
8	0.2867	0.2601	-0.1502	0.0123	-0.2788	-0.2601	0.1502	0.2867	0.0123	-0.2788
9	0.2468	0.1025	-0.3415	0.2593	0.0003	-0.1025	0.3415	0.2468	0.2593	0.0003
10	0.1423	0.0090	-0.2912	0.2642	0.2789	-0.0090	0.2912	0.1423	0.2642	0.2789
11	0.1672	0.1467	0.2455	-0.0058	0.3359	-0.1467	-0.2455	0.1672	-0.0058	0.3359
12	0.2895	0.1491	0.2411	-0.0017	0.0088	-0.1491	-0.2411	0.2895	-0.0017	0.0088
13	0.3336	0.0082	-0.0119	0.0070	-0.3237	-0.0082	0.0119	0.3336	0.0070	-0.3237
14	0.2877	-0.1313	-0.2630	0.0124	0.0088	0.1313	0.2630	0.2877	0.0124	0.0088
15	0.1657	-0.1331	-0.2605	0.0131	0.3359	0.1331	0.2605	0.1657	0.0131	0.3359
16	0.1454	0.0007	0.2856	0.2417	0.3025	-0.0007	-0.2856	0.1454	0.2417	0.3025
17	0.2522	-0.0908	0.3339	0.2415	0.0119	0.0908	-0.3339	0.2522	0.2415	0.0119
18	0.2915	-0.2485	0.1317	-0.0018	-0.2941	0.2485	-0.1317	0.2915	-0.0018	-0.2941
19	0.2513	-0.3314	-0.1103	-0.2446	0.0118	0.3314	0.1103	0.2513	-0.2446	0.0118
20	0.1446	-0.2398	-0.1592	-0.2432	0.3025	0.2398	0.1592	0.1446	-0.2432	0.3025
21	0.0842	-0.0528	0.1956	0.2437	0.1811	0.0528	-0.1956	0.0842	0.2437	0.1811
22	0.1461	-0.1445	0.2454	0.2416	0.0151	0.1445	-0.2454	0.1461	0.2416	0.0151
23	0.1688	-0.2486	0.1349	-0.0059	-0.1580	0.2486	-0.1349	0.1688	-0.0059	-0.1580
24	0.1457	-0.2827	-0.0135	-0.2517	0.0151	0.2827	0.0135	0.1457	-0.2517	0.0151
25	0.0838	-0.1910	-0.0633	-0.2494	0.1811	0.1910	0.0633	0.0838	-0.2494	0.1811

In node 9, the mass is equal to 600 kg, in node 18 it is equal to 1000 kg, and in the rest of the nodes it is equal to 800 kg. The dimensions of the plate in the plan are equal to 6 and 6 meters (m), the thickness is equal to 0.12 meters. The modulus of elasticity of the plate material is equal to $E = 24000000000\, N/m^2$, Poisson's ratio $\nu_0 = 0.2$. With the principal system of the displacement method (Figure 2*b*) and one-dimensional numbering of values in accordance with Figure 2*a*, the spectrum of natural frequencies is determined as the roots of equation (2). The values of the first five frequencies of natural modes of the plate and the coordinates of their corresponding natural modes are given in Table 1 (columns are initial frequencies and modes).

Suppose now it is required to increase the value of the first natural frequency to 100 sec^{-1}. For this, in accordance with (8)-(10), we will form a matrix of additional stiffness coefficients. All data necessary for using dependencies (8)-(10) are given in Table 1 (columns are initial frequencies and modes).

After the formation of the matrix of additional stiffness coefficients with allowance for their influence, we determine from equation (11) the modified spectrum of natural frequencies and the corresponding natural modes.

The first five natural frequencies and their corresponding natural modes are shown in Table 1 (columns are modified frequencies and modes).

It can be seen from the table that taking into account additional stiffness coefficients did not change any of the natural modes of the plate, but only increased the value of one of the frequencies from 36.6583 s-1 to a given value of 100 s-1. This result clearly illustrates the possibility of using dependencies (5), (8)-(10) for solving the problem of aimed constraints for elastic plates with a finite number of degrees of freedom of masses.

The generalized aimed constraint for the plate, as well as for the rod, should create additional stiffness, ensuring the aiming of the constraint.

As noted above, the properties of the aimed constraints for the rods are based on the properties of the natural modes. The properties used here can be also applied to elastic plates. This circumstance serves as a justification for using the results of formulating the properties of aimed constraints for rods and in a similar problem for plates.

Thus, the matrix of additional stiffness coefficients (4) must correspond to the generalized aimed constraint in case of the plate. If the design scheme of the constraint is represented by a variant of the hinge-rod system, then it must be statically indeterminate once.

In the nodes of the plate where the masses are located, posts are installed in the direction of movement of the masses, and the preliminary stress of any one element of constraint causes such forces in the posts of the system, the relations between which are proportional to the relationship between the stresses (12). At the same time, there should be no connections with the plate in the structure of the constraint, except for the posts installed in the nodes of the plate, where the masses are located.

Design schemes of generalized aimed constraints that meet the above requirements are multivariate and depend on the shape of the plate, the locations of the masses and some other features of the original object. Taking this circumstance into account, the approaches and algorithms for the formation of design schemes of constraint that purposefully change the spectrum of natural frequencies of elastic plates with a finite number of degrees of freedom of masses represent a separate problem and will be considered in the next research work.

So, this paper shows that the method of forming a matrix of additional stiffness coefficients that determine the aimed constraint in the problem of natural vibrations of rods can also be used to solve a similar problem for elastic systems with a finite number of degrees of freedom of masses, for which the directions of motion of the masses are parallel, but do not lie in the same plane.

The paper also substantiates and formulates the properties and requirements to which the design schemes of aimed constraints in the problem under consideration must correspond.

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LOCALIZATION OF SOLUTION OF THE PROBLEM OF TWO-DIMENSIONAL THEORY OF ELASTICITY WITH THE USE OF B-SPLINE DISCRETE-CONTINUAL FINITE ELEMENT METHOD

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Abstract: Localization of solution of the problem of two-dimensional theory of elasticity with the use of B-spline discrete-continual finite element method (specific version of wavelet-based discrete-continual finite element method) is under consideration in the distinctive paper. The original operational continual and discrete-continual formulations of the problem are given, some actual aspects of construction of normalized basis functions of a B-spline are considered, the corresponding local constructions for an arbitrary discrete-continual finite element are described, some information about the numerical implementation and an example of analysis are presented.

Keywords: localization, wavelet-based discrete-continual finite element method, B-spline discrete-continual finite element method, discrete-continual finite element method, finite element method, B-spline, numerical solution, two-dimensional theory of elasticity, structural analysis.

ЛОКАЛИЗАЦИЯ РЕШЕНИЯ ДВУМЕРНОЙ ЗАДАЧИ ТЕОРИИ УПРУГОСТИ НА ОСНОВЕ ВЕЙВЛЕТ-РЕАЛИЗАЦИИ ДИСКРЕТНО-КОНТИНУАЛЬНОГО МЕТОДА КОНЕЧНЫХ ЭЛЕМЕНТОВ С ИСПОЛЬЗОВАНИЕМ B-СПЛАЙНОВ

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Аннотация: В настоящей статье рассматривается локализация решения двумерной задачи теории упругости на основе вейвлет-реализации дискретно-континуального метода конечных элементов с использованием B-сплайнов. Приведены исходные операторные континуальная и дискретно-континуальная постановки задачи, рассмотрены некоторые актуальные вопросы построения нормализованных базисных функций B-сплайна, описаны соответствующие локальные построения для произвольного дискретно-континуального конечного элемента, представлены некоторые сведения о численной реализации и пример расчета.

Ключевые слова: локализация, вейвлет-реализация метода конечных элементов, дискретно-континуальный метод конечных элементов, метод конечных элементов, B-сплайны, численное решение, двумерная задача теории упругости, расчеты конструкций.

INTRODUCTION

As we have already mentioned [1, 2], the B-spline in a given simple knot sequence can be constructed by employing piecewise polynomials between the knots and joining them together at the knots [1-3].

For instance, compared with commonly used Daubechies wavelets [4-8] B-spline wavelet on interval (BSWI) has explicit expressions, facilitating the calculation of coefficient integration and differentiation [1-3]. Besides, the multiresolution and localization properties of BSWI can also supply some superiority for engineering structural analysis [1-3]. The early applications of spline can be found in papers of H. Antes [9], J.G. Han [10, 11, 27], Y. Huang [10, 11], W.X. Ren [10, 11]. The spline wavelet finite element method was further developed in papers of D.P. Chen [28], X.F. Chen [12, 13, 15-18, 23, 24, 26], H.B. Dong [23], J.G. Han [25], Y.M. He [17], Z.H. He [18], Z.J. He [12, 13, 15-17, 23, 24, 26], Y. Huang [25, 27], Z.S. Jiang [22], B. Li [13, 15, 17, 23], M. Liang [19, 21], J.Q. Long [20], G. Ma [20], T. Matsumoto [20, 22], S.T. Mau [30], H.H. Miao [15], Q,M. Mo [18], T.H.H. Pian [28-30], K.Y. Qi [23], W.X. Ren [25, 27], K. Sumihara [29], P. Tong [30], Y.W. Wang [22], J.W. Xiang [12-14, 17-22], Z.B. Yang [15, 16, 24], X.W. Zhang [16, 24, 26], Y.H. Zhang [12], Y.T. Zhong [14].

As is known, generally the structural analysis normally require accurate computer-intensive calculations using numerical (discrete) methods. The field of application of discrete-continual finite element method (DCFFEM), proposed by A.B. Zolotov [33] and P.A. Akimov [31-33] comprises structures with regular (in particular, constant or piecewise constant) physical and geometrical parameters in some dimension (so-called "basic" direction (dimension)). Considering problems remain continual along "basic" direction while along other directions DCFEM presupposes finite element approximation. Solution of corresponding resultant multipoint boundary problems [34] for systems of ordinary differential equations with piecewise constant coefficients and immense number of unknowns is the most time-consuming stage of the computing, especially if we take into account the limitation in performance of personal computers, contemporary software and necessity to obtain correct semianalytical solution in a reasonable time.

High-accuracy solution at all points of the model is not required normally, it is necessary to find only the most accurate solution in some pre-known domains. Generally the choice of these domains is a priori data with respect to the structure being modelled. Designers usually choose domains with the so-called edge effect (with the risk of significant stresses that could potentially lead to the destruction of structures, etc.) and regions which are subject to specific operational requirements. It is obvious that the stress-strain state in such domains is of paramount importance. Specified factors along with the obvious needs of the designer or researcher to reduce computational costs by application of DCFEM cause considerable urgency of constructing of special algorithms for obtaining local solutions (in some domains known in advance) of boundary problems. Wavelet analysis provides effective and popular tool for such researches. Solution of the considering problem within multilevel wavelet analysis is represented as a composition of local and global components. Wavelet-based DCFEM is presented in papers of P.A. Akimov [35-42], M. Aslami [38-40], T.B. Kaytukov, M.L. Mozgaleva [35-42] and O.A. Negrozov [38-40].

The distinctive paper is devoted to numerical solution of the problem of two-dimensional theory of elasticity with the use of B-spline DCFEM.

1. FORMULATIONS OF THE PROBLEM

In accordance with [1] let the constancy of the parameters of the problem be in the direction corresponding to x_2 (main direction). The operational formulation of the problem with the use of so-called method of extended domain [43], taking into account the selection of the main direction, is determined by the equation:

$$L\,\bar{u} = \bar{\bar{F}}\,, \quad 0 \le x_1 \le \ell_1\,,\; 0 \le x_2 \le \ell_2\,, \quad (1.1)$$

where we have

$$L = -L_{vv}\,\partial_2^2 + L_{uv}\,\partial_2 + L_{uu}\,; \quad (1.2)$$

$$L_{vv} = D_2^T C D_2\,; \quad (1.3)$$

$$L_{uv} = \partial_1^* D_1^T C D_2 - D_2^T C D_1 \partial_1\,; \quad (1.4)$$

$$L_{uu} = \partial_1^* D_1^T C D_1 \partial_1\,; \quad (1.5)$$

$$\overline{\overline{F}} = \theta \overline{F} + \delta_{\Gamma} \bar{f} \; ; \qquad (1.6)$$

$$\theta(x_1, x_2) = \begin{cases} 1, & (x_1, x_2) \in \Omega \\ 0, & (x_1, x_2) \notin \Omega; \end{cases} \qquad (1.7)$$

$$\delta_{\Gamma}(x_1, x_2) = \partial\theta / \partial\overline{n} \; ; \qquad (1.8)$$

Ω is the domain, occupied by structure;

$$\Omega = \{ (x_1, x_2) : \; 0 < x_1 < \ell_1 ; \; 0 < x_2 < \ell_2 \} \; ; \qquad (1.9)$$

ℓ_1, ℓ_2 are corresponding dimensions of extended domain (linear dimensions of considering structure); $x = (x_1, x_2)$; x_1, x_2 are Cartesian coordinates; $\theta(x_1, x_2)$ is characteristic function of domain Ω; $\delta_{\Gamma} = \delta_{\Gamma}(x_1, x_2)$ is the delta function of boundary $\Gamma = \partial\Omega$; $\overline{n} = [\, n_1 \;\; n_2 \,]^T$ is boundary normal vector; $\overline{u}$ is the vector of displacements (unknown vector function),

$$\overline{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}; \qquad (1.10)$$

$$D_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}; \quad D_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}; \qquad (1.11)$$

$$C = \begin{bmatrix} 2\mu + \lambda & \lambda & 0 \\ \lambda & 2\mu + \lambda & 0 \\ 0 & 0 & \mu \end{bmatrix}; \qquad (1.12)$$

μ and λ are Lame parameters; $\overline{F}$ is the load vector in domain Ω; $\bar{f}$ is the corresponding boundary load vector; $\partial_s = \partial / \partial x_s$, $s = 1, 2$; $\partial_s^* = -\partial / \partial x_s$, $s = 1, 2$.

Let us introduce the following notations

$$\overline{v} = \partial_2 \overline{u} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}; \qquad (1.13)$$

$$\overline{u}' = \partial_2 \overline{u} \; ; \quad \overline{v}' = \partial_2 \overline{v} \, . \qquad (1.14)$$

Thus we can rewrite (1.1):

$$L_{uu} \, \overline{u} + L_{uv} \overline{v} - L_{vv} \partial_2 \overline{v} = \overline{\overline{F}} \, . \qquad (1.15)$$

Finally we obtain system of differential equations with operational coefficients:

$$\begin{cases} \overline{u}' = \overline{v} \\ \overline{v}' = L_{vv}^{-1} L_{uu} \overline{u} + L_{vv}^{-1} L_{uv} \overline{v} - L_{vv}^{-1} \overline{\overline{F}} \end{cases} \qquad (1.16)$$

or

$$\overline{z}' = \widetilde{L} \overline{z} + \overline{\widetilde{F}} \, , \qquad (1.17)$$

where

$$\widetilde{L} = \begin{bmatrix} 0 & E \\ L_{vv}^{-1} L_{uu} & L_{vv}^{-1} L_{uv} \end{bmatrix}; \quad \overline{\widetilde{F}} = \begin{bmatrix} 0 \\ -L_{vv}^{-1} \overline{\overline{F}} \end{bmatrix}; \qquad (1.18)$$

$$\overline{z} = \begin{bmatrix} \overline{u} \\ \overline{v} \end{bmatrix}; \quad E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \qquad (1.19)$$

The system of equations (1.17) is supplemented by boundary conditions, which are set in sections with coordinates $x_2^1 = 0$ and $x_2^2 = \ell_2$.

2. SOME ASPECTS OF THE CONSTRUCTION OF NORMALIZED BASIS FUNCTIONS OF THE B-SPLINE

The construction of B-spline basic functions is determined by the recursive Cox-de Boer formulas [1]:

$$k = 1: \quad \varphi_{i,1}(t) = \begin{cases} 1 \, , & x_i \le t < x_{i+1} \\ 0 \, , & t < x_i \vee t \ge x_{i+1} \end{cases} , \qquad (2.1)$$

$$k \ge 2: \quad \varphi_{i,k}(t) = \frac{(t - x_i)\varphi_{i,k-1}(t)}{x_{i+k-1} - x_i} + \frac{(x_{i+k} - t)\varphi_{i+1,k-1}(t)}{x_{i+k} - x_{i+1}} . \qquad (2.2)$$

We will consider such a construction for the case $x_i = i$ are integers. Let us note that,

$$\varphi_{i,k}(t) = \varphi_{0,k}(t - i)$$

and therefore, recursive formulas (2.1)-(2.2) can be represented in the form

$$k=1: \qquad \varphi_{0,1}(t)=\begin{cases}1, & 0\le t<1\\ 0, & t<0\vee t\ge 1;\end{cases} \qquad (2.3)$$

$$k\ge 2: \qquad \varphi_{0,k}(t)=\frac{1}{k-1}[t\cdot\varphi_{0,k-1}(t)+ +(k-t)\varphi_{0,k-1}(t-1)]. \qquad (2.4)$$

The function $\varphi_{0,1}(t)$ can be represented by formula

$$\varphi_{0,1}(t)=\frac{1}{2}[\mathrm{sign}(t)-\mathrm{sign}(t-1)]. \qquad (2.5)$$

Let us denote by Δ_1 the operator of the first difference. Then we have

$$\varphi_{0,1}(t)=-\frac{1}{2}\Delta_1\,\mathrm{sign}(t). \qquad (2.6)$$

We can substitute formula (2.5) into (2.4) in order to determine $\varphi_{0,2}(t)$:

$$\varphi_{0,2}(t)=1\cdot[t\cdot\varphi_{0,1}(t)+(2-t)\varphi_{0,1}(t-1)]= =\frac{1}{2}\{t\cdot[\mathrm{sign}(t)-\mathrm{sign}(t-1)]+ (2-t)[\mathrm{sign}(t-1)-\mathrm{sign}(t-2)]\}= =\frac{1}{2}[t\,\mathrm{sign}(t)-2(t-1)\,\mathrm{sign}(t-1)+ (t-2)\,\mathrm{sign}(t-2)]=\frac{1}{2}[|t|-2|t-1|+|t-2|.$$

Let us denote by Δ_2 the operator of the second difference. Then we have

$$\varphi_{0,2}(t)=\frac{1}{2}[|t|-2|t-1|+|t-2|=\frac{1}{2}\Delta_2\,|t-1|. \qquad (2.7)$$

We can define function $\varphi_{0,3}(t)$:

$$\varphi_{0,3}(t)=\frac{1}{2}[t\cdot\varphi_{0,2}(t)+(3-t)\varphi_{0,2}(t-1)].$$

Omitting intermediate calculations, we get

$$\varphi_{0,3}(t)=\frac{1}{4}[t\cdot|t|-3(t-1)|t-1|+ +3(t-2)|t-2|-(t-3)|t-3|]= =-\frac{1}{2!}\frac{1}{2}\Delta_1\Delta_2((t-1)|t-1|). \qquad (2.8)$$

Based on formulas (2.8) and (2.4), we can define the function

$$\varphi_{0,4}(t)=\frac{1}{3}[t\cdot\varphi_{0,3}(t)+(4-t)\varphi_{0,3}(t-1)].$$

Omitting intermediate calculations, as a result we get

$$\varphi_{0,4}(t)= =\frac{1}{2\cdot 3}\cdot\frac{1}{2}[t^2\cdot|t|-4(t-1)^2|t-1|+ +6(t-2)^2|t-2|-4(t-3)^2|t-3|+ +(t-4)^2|t-4|]= =\frac{1}{3!}\frac{1}{2}(\Delta_2)^2((t-2)^2|t-2|). \qquad (2.9)$$

It can be proved that for even $k=2m$ we have

$$\varphi_{0,k}(t)=\frac{1}{(2m-1)!}\frac{1}{2}(\Delta_2)^m((t-m)^{2m-2}|t-m|) \qquad (2.10)$$

and for odd (uneven) $k=2m+1$ we have

$$\varphi_{0,k}(t)=-\frac{1}{(2m)!}\frac{1}{2}\Delta_1(\Delta_2)^m((t-m)^{2m-1}|t-1|). \qquad (2.11)$$

Note that $\varphi_{0,k}(t)$ is a polynomial of degree $k-1$ with bounded support and, as follows from the difference operator, this support is equal to the interval $[0,k]$.

In addition, we should note the following property of B-spline basis functions:

$$\sum_i\varphi_{0,k}(t-i)\equiv 1 \text{ for arbitrary } t. \qquad (2.12)$$

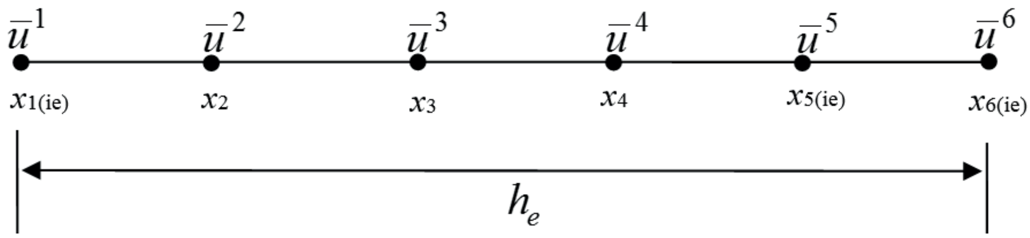


Figure 3.1. Finite element discretication for $N_k = 5$ (sample).

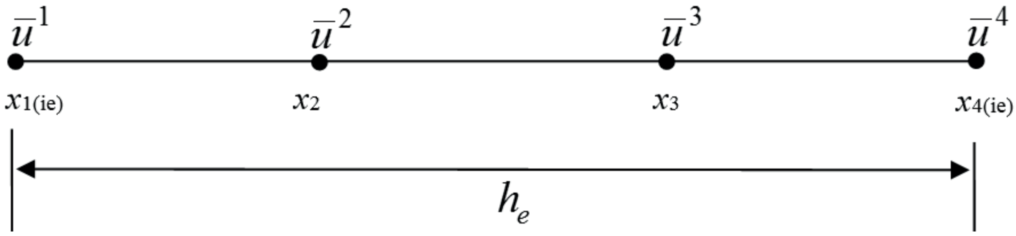


Figure 3.2. Finite element discretication for $N_k = 3$ (sample).

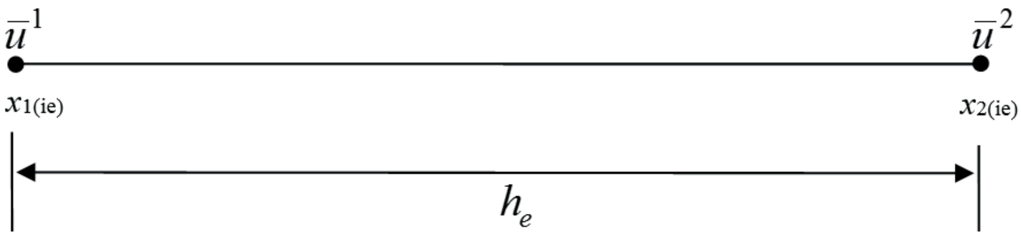


Figure 3.3. Finite element discretication for $N_k = 1$ (sample).

3. SOME GENERAL ASPECTS OF FINITE ELEMENT APPROXIMATION

The discrete component of the numerical solution is represented by the direction along the axis corresponding to x_1. The fulfillment within an element (interval) for all components of a vector functions $\bar{u}$ and $\bar{v}$ (see (1.10), (1.13)) is the same. Therefore, let us use the following notation for simplicity:

$$x = x_1, \ \ell = \ell_1, \ y = y(x), \qquad (3.1)$$

where $y = y(x)$ is unknown function (component of vector function).

Let us divide the interval $(0, \ell)$ segment into N_e parts (elements). Therefore $h_e = \ell / N_e$ is the length of the element. Besides, let us also divide each element into N_k parts. It should be noted that on the elements of the localization of the solution, parameter N_k is of greater importance than on the other elements. For example, on localization elements, we can set $N_k = 5$, i.e. unknown functions will be represented by polynomials (B-splines) of the 5th degree (Figure 3.1).

Let us use the following notation system: i_e is the element number; $N_p = N_k + 1 = 6$ is the number of nodes within the element; $x_1(i_e)$ is the coordinate of the starting point of the i_e-th element; $x_6(i_e)$ is the coordinate of the end point of the i_e-th element. Thus, the number of unknowns per element with such approximation is equal to

$$N_{ie} = 2N_p = 12\,.$$

For the elements of localization we can take reduced number of N_k. For instance, if we take

$N_k = 3$ (Figure 3.2) we get $N_p = N_k + 1 = 4$ and the number of unknowns per element with such approximation is equal to

$$N_{ie} = 2N_p = 8;$$

$x_1(i_e)$ is the coordinate of the starting point of the i_e-th element; $x_4(i_e)$ is the coordinate of the end point of the i_e-th element.

Besides, let us consider the case with $N_k = 1$ (Figure 3.3). Therefor we have $N_p = N_k + 1 = 2$ and the number of unknowns per element with such approximation is equal to

$$N_{ie} = 2N_p = 4,$$

where $x_1(i_e)$ is the coordinate of the starting point of the i_e-th element; $x_2(i_e)$ is the coordinate of the end point of the i_e-th element.

4. LOCAL CONSTRUCTIONS FOR ARBITRARY FINITE ELEMENT

Let us introduce local coordinates:

$$t = (x - x_{1(ie)}) / h_e, \quad x_{1(ie)} \le x \le x_{N_p(ie)}, \; 0 \le t \le 1. \tag{4.1}$$

In this case, we have the following relations:

$$x = x_i \Rightarrow t_i = (x_i - x_{1(ie)}) / h_e, \quad i = 1,..., N_p; \tag{4.2}$$

$$\frac{d^p}{dx^p} = \frac{1}{h_e^p} \frac{d^p}{dt^p}; \quad dx = h_e \cdot dt. \tag{4.3}$$

Since the number of unknowns on the element is equal to $N_{ie} = 12$, we use a B-spline of the fifth degree in order to represent the unknown deflection function.

Let us use the following notation:

$$\varphi(t) = \varphi_{0,6}(t+3);$$

$$\begin{aligned}\varphi(t) &= \frac{1}{5!}\frac{1}{2}(\Delta_2)^3 (t^4 |t|) = \\ &= \frac{1}{5!\cdot 2}[(t+3)^4 |t+3| - 6(t+2)^4 |t+2| + \\ &+ 15(t+1)^4 |t+1| - 20t^4 |t| + \\ &+ 15(t-1)^4 |t-1| - 6(t-2)^4 |t-2| + \\ &+ (t-3)^4 |t-3|].\end{aligned} \tag{4.4}$$

This function is a B-spline, symmetric with respect to $t = 0$ and its support is defined by an interval $[-3,3]$ (Figure 4.1).

We take the following eight functions as basis functions on the unit interval (Figures 4.2, 4.3):

$$\begin{gathered}\varphi_1(t) = \varphi(t+2), \quad \varphi_2(t) = \varphi(t+1), \\ \varphi_3(t) = \varphi(t), \quad \varphi_4(t) = \varphi(t-1), \\ \varphi_5(t) = \varphi(t-2), \quad \varphi_6(t) = \varphi(t-3), \\ 0 \le t \le 1.\end{gathered} \tag{4.5}$$

Since the number of unknowns on the element is equal to $N_{ie} = 8$, we use a B-spline of the third degree in order to represent the unknown deflection function.

Let us use the following notation:

$$\varphi(t) = \varphi_{0,4}(t+4);$$

$$\begin{aligned}\varphi(t) &= \frac{1}{3!}\frac{1}{2}(\Delta_2)^2 (t^2 |t|) = \\ &= \frac{1}{3!\cdot 2}[(t+2)^2 |t+2| - 4(t+1)^2 |t+1| + \\ &+ 6t^2 |t| - 4(t-1)^2 |t-1| + \\ &+ (t-2)^2 |t-2|].\end{aligned} \tag{4.6}$$

This function is a B-spline, symmetric with respect to $t = 0$ and its support is defined by an interval $[-2,2]$ (Figure 4.4).

We take the following four functions as basis functions on the unit interval (Figures 4.5):

$$\begin{gathered}\varphi_1(t) = \varphi(t+1), \quad \varphi_2(t) = \varphi(t), \\ \varphi_3(t) = \varphi(t-1), \quad \varphi_4(t) = \varphi(t-2), \\ 0 \le t \le 1.\end{gathered} \tag{4.7}$$

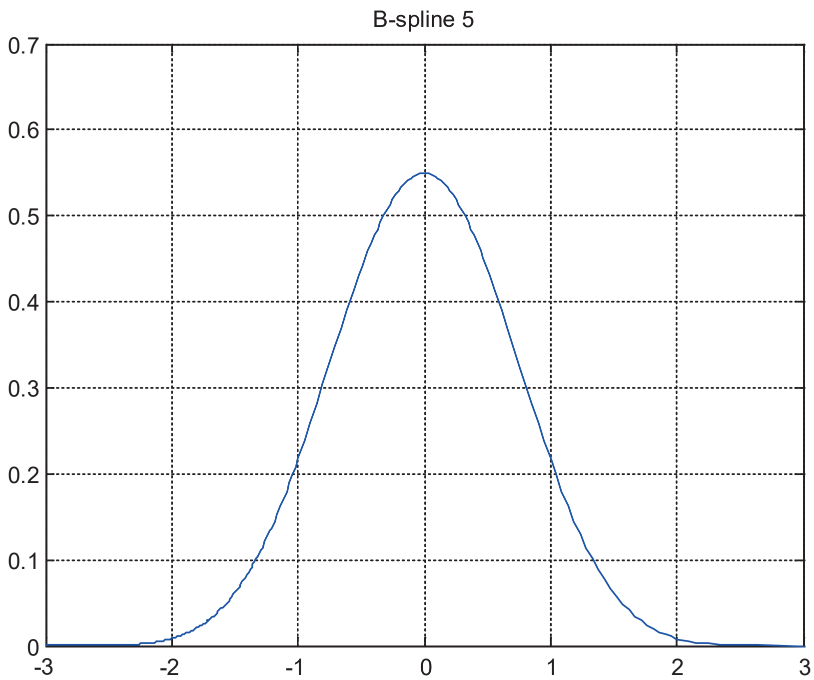


Figure 4.1. B-spline of the fifth order $\varphi(t)=\varphi_{0,6}(t+3)$.

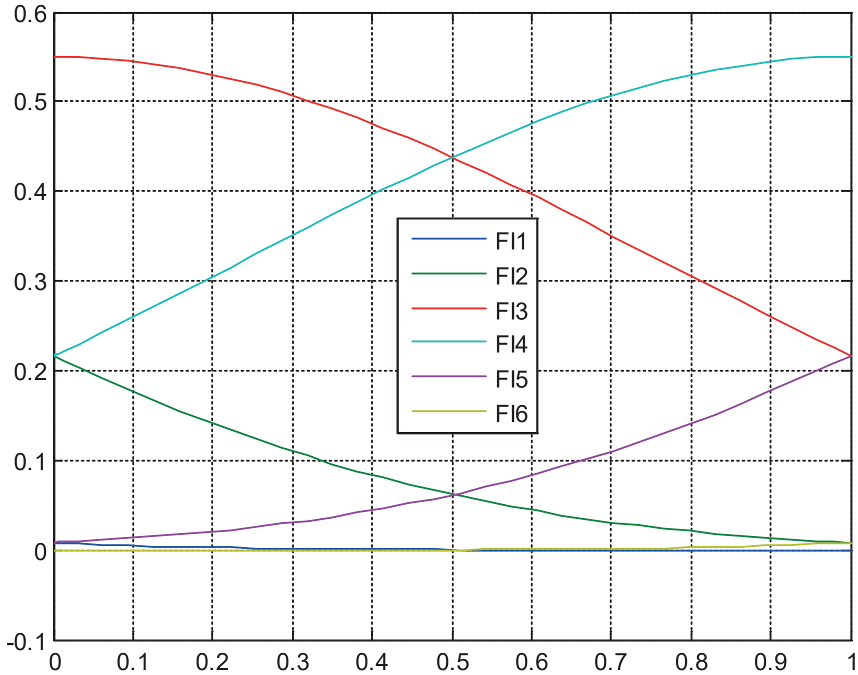


Figure 4.2. Basis functions $\varphi_k(t)$, $k=1,2,\ldots,6$.

Since the number of unknowns on the element is equal to $N_{ie}=4$, we use a B-spline of the first degree in order to represent the unknown deflection function.

Let us use the following notation:

$$\varphi(t)=\varphi_{0,2}(t+1);$$

$$\varphi(t)=\frac{1}{2}\Delta_2\,|\,t\,|=\frac{1}{2}[\,|\,t+1\,|-2\,|\,t\,|+|\,t-1\,|\,]. \quad (4.8)$$

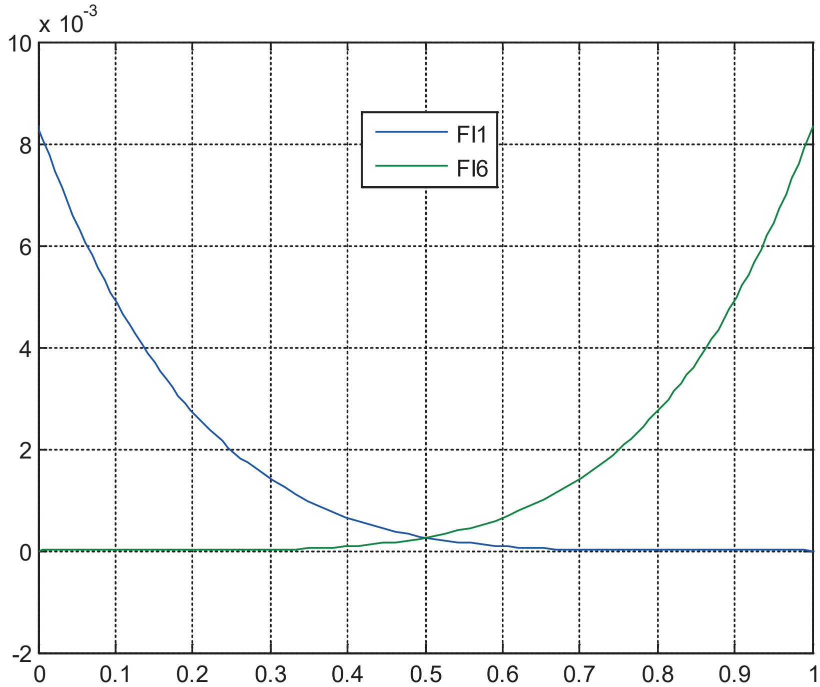


Figure 4.3. Basis functions $\varphi_1(t)$ and $\varphi_6(t)$.

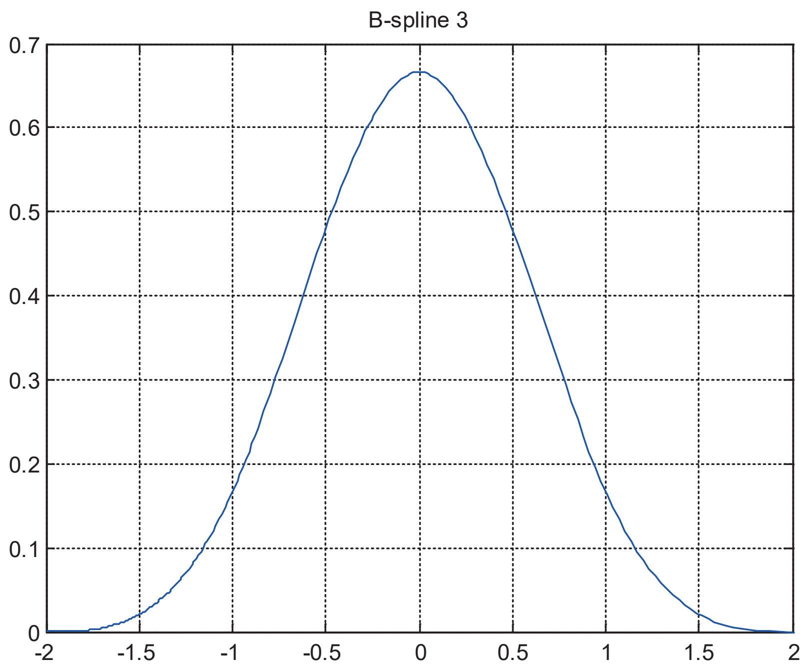


Figure 4.4. B-spline of the third order $\varphi(t) = \varphi_{0,4}(t+2)$.

This function is a B-spline, symmetric with respect to $t = 0$ and its support is defined by an interval $[-1,1]$ (Figure 4.6).

We take the following four functions as basis functions on the unit interval (Figures 4.7):

$$\varphi_1(t) = \varphi(t), \quad \varphi_2(t) = \varphi(t-1), \quad 0 \le t \le 1. \quad (4.9)$$

We represent the unknown function $y(x)$ within the element number i_e in the form

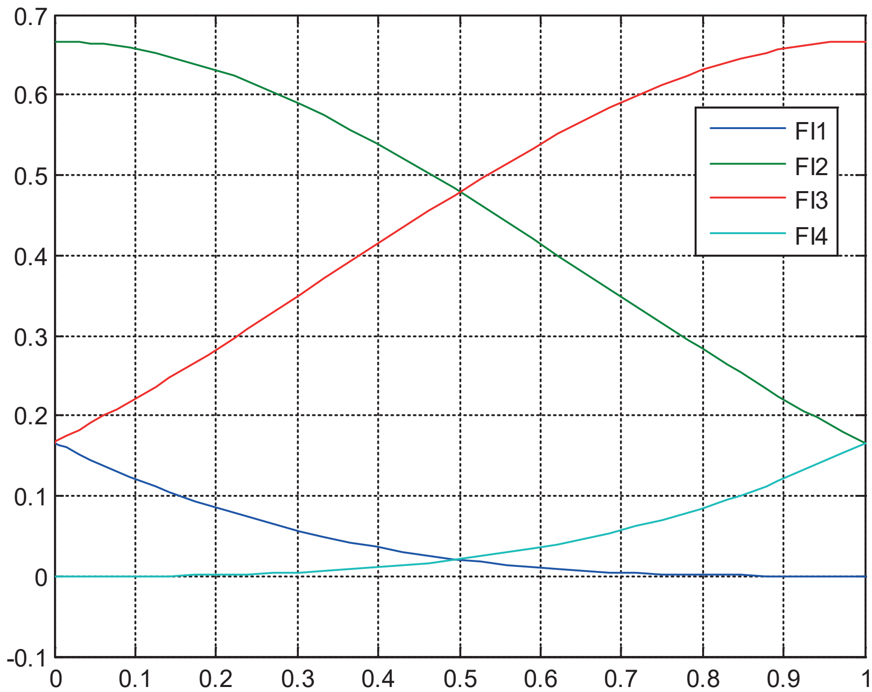


Figure 4.5. Basis functions $\varphi_k(t)$, $k=1,2,3,4$.

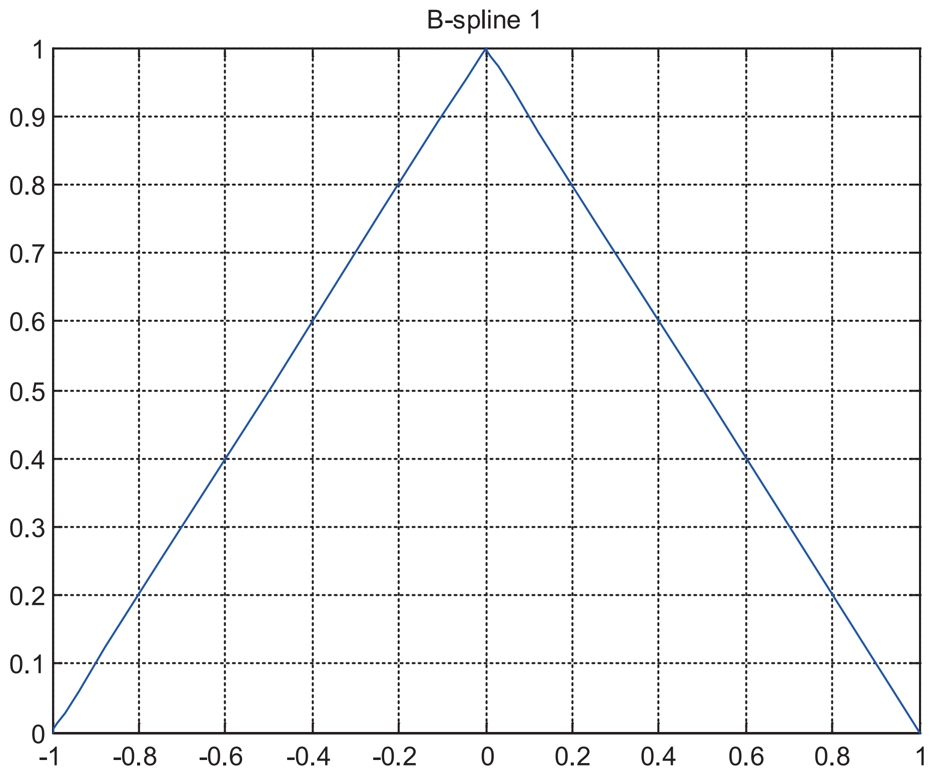


Figure 4.6. B-spline of the first order $\varphi(t)=\varphi_{0,4}(t+2)$.

$$y(x)=w(t)=\sum_{k=1}^{N_p}\alpha_k\varphi_k(t),\quad x_{1(i_e)}\le x\le x_{N_p(i_e)},\quad 0\le t\le 1. \quad (4.10)$$

We can define parameters α_k with the use of nodal unknowns of the element:

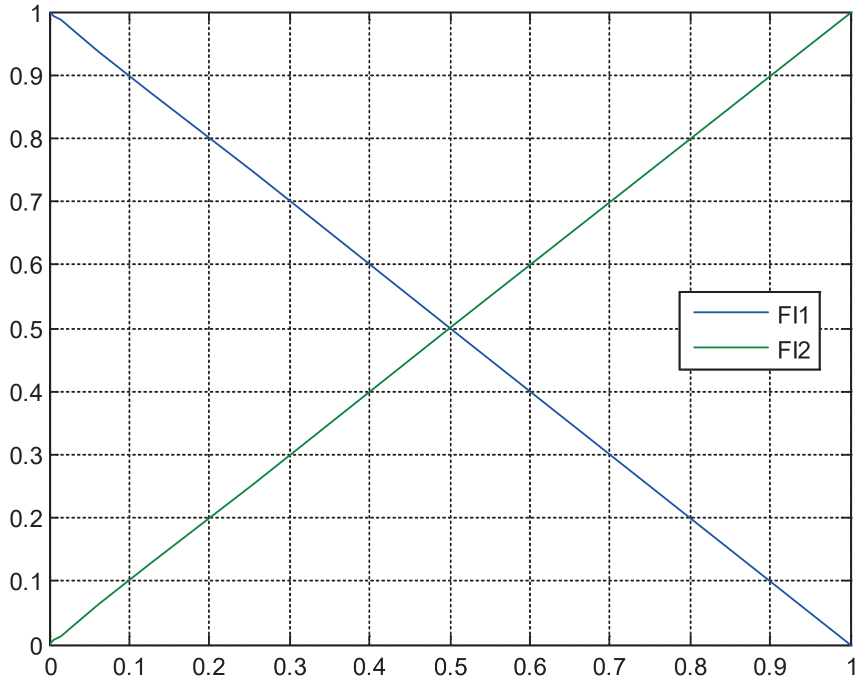


Figure 4.7. *Basis functions* $\varphi_k(t)$, $k = 1,2$.

$$T_6 = 1\begin{bmatrix} \varphi_1(0) & \varphi_2(0) & \varphi_3(0) & \varphi_4(0) & \varphi_5(0) & \varphi_6(0) \\ \varphi_1(0.2) & \varphi_2(0.2) & \varphi_3(0.2) & \varphi_4(0.2) & \varphi_5(0.2) & \varphi_6(0.2) \\ \varphi_1(0.4) & \varphi_2(0.4) & \varphi_3(0.4) & \varphi_4(0.4) & \varphi_5(0.4) & \varphi_6(0.4) \\ \varphi_1(0.6) & \varphi_2(0.6) & \varphi_3(0.6) & \varphi_4(0.6) & \varphi_5(0.6) & \varphi_6(0.6) \\ \varphi_1(0.8) & \varphi_2(0.8) & \varphi_3(0.8) & \varphi_4(0.8) & \varphi_5(0.8) & \varphi_6(0.8) \\ \varphi_1(1) & \varphi_2(1) & \varphi_3(1) & \varphi_4(1) & \varphi_5(1) & \varphi_6(1) \end{bmatrix}$$

Figure 4.8. *Matrix* T_6.

$$y_i = w(t_i) = \sum_{k=1}^{N_p} \alpha_k \varphi_k(t_i), \quad x_{1(i_e)} \le x \le x_{N_p(i_e)}, \quad 0 \le t \le 1. \quad (4.11)$$

In case $N_p = 6$ we have (Figure 4.8)

$$\bar{y}^{i_e} = T_6 \bar{\alpha}, \quad (4.12)$$

$$\bar{y}^{i_e} = [y_1 \ \ y_2 \ \ y_3 \ \ y_4 \ \ y_5 \ \ y_6]^T; \quad (4.13)$$

$$\bar{\alpha} = [\alpha_1 \ \ \alpha_2 \ \ \alpha_3 \ \ \alpha_4 \ \ \alpha_5 \ \ \alpha_6]^T. \quad (4.14)$$

In case $N_p = 4$ we have

$$\bar{y}^{i_e} = T_4 \bar{\alpha}, \quad (4.15)$$

$$\bar{y}^{i_e} = [y_1 \ \ y_2 \ \ y_3 \ \ y_4]^T; \quad (4.16)$$

$$\bar{\alpha} = [\alpha_1 \ \ \alpha_2 \ \ \alpha_3 \ \ \alpha_4]^T; \quad (4.17)$$

$$T_4 = \begin{bmatrix} \varphi_1(0) & \varphi_2(0) & \varphi_3(0) & \varphi_4(0) \\ \varphi_1(1/3) & \varphi_2(1/3) & \varphi_3(1/3) & \varphi_4(1/3) \\ \varphi_1(2/3) & \varphi_2(2/3) & \varphi_3(2/3) & \varphi_4(2/3) \\ \varphi_1(1) & \varphi_2(1) & \varphi_3(1) & \varphi_4(1) \end{bmatrix}. \quad (4.18)$$

In case $N_p = 2$ we have

$$\bar{y}^{i_e} = T_2 \bar{\alpha}, \quad (4.19)$$

$$\bar{y}^{i_e} = [y_1 \ \ y_2]^T; \quad \bar{\alpha} = [\alpha_1 \ \ \alpha_2]^T; \quad (4.20)$$

$$T_2 = \begin{bmatrix} \varphi_1(0) & \varphi_2(0) \\ \varphi_1(1) & \varphi_2(1) \end{bmatrix}. \quad (4.21)$$

Using (4.12)-(4.21), we get

$$\bar{\alpha} = T_{N_p}^{-1} \bar{y}^{i_e}, \quad (4.22)$$

where

$$T_{N_p} = \{T_{ij}\}_{i,j=1,..,N_p}\,; \quad T_{ij} = \varphi_j(t_i)\,. \quad (4.23)$$

Let us introduce the following notation system:

$$\overline{u}^{ie} = \begin{bmatrix} \overline{u}^{1} \\ \vdots \\ \overline{u}^{N_p} \end{bmatrix} \quad (4.24)$$

is nodal vector-function of element number i_e;

$$\overline{u}^{i} = \begin{bmatrix} u_1^{i} \\ u_2^{i} \end{bmatrix} \quad (4.25)$$

is vector-unction in node number i with the element number i_e;

$$\overline{u}_k^{ie} = \begin{bmatrix} u_k^{1} \\ \vdots \\ u_k^{N_p} \end{bmatrix}, \quad k = 1,2 \quad (4.26)$$

is nodal component number k of vector-function of element number i_e.

Let P be permutation matrix,

$$\overline{u}^{ie} = P\begin{bmatrix} \overline{u}_1^{ie} \\ \overline{u}_2^{ie} \end{bmatrix}; \quad (4.27)$$

$$P = \underbrace{\left[\begin{array}{cccc|cccc} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & 0 & \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & 1 & \vdots & \vdots & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & & 1 \end{array}\right]}_{2\cdot N_p}; \quad (4.28)$$

Due to $P^{-1} = P^T$ we have

$$\begin{bmatrix} \overline{u}_1^{ie} \\ \overline{u}_2^{ie} \end{bmatrix} = P^T \overline{u}^{ie}. \quad (4.29)$$

We have to consider bilinear forms with allowance for relations (4.2)-(4.3) in order to construct local stiffness matrices corresponding to the operators L_{uu}, L_{uv}, L_{vv} (see (1.3)-(1.5)):

$$(L_{uu}\overline{u},\overline{z}) = (\partial^* D_1^T C D_1 \partial\overline{u},\overline{z}) = (D_1^T C D_1 \partial\overline{u},\partial\overline{z}) =$$
$$= \int_{x_{1(ie)}}^{x_{Np(ie)}} (\begin{bmatrix} 2\mu+\lambda & \\ & \mu \end{bmatrix}\partial\overline{u},\partial\overline{z})dx\,; \quad (4.30)$$

$$(L_{uv}\overline{u},\overline{z}) = (\partial^* D_1^T C D_2 \overline{u} - D_2^T C D_1 \partial\overline{u},\overline{z}) =$$
$$= (D_1^T C D_2 \overline{u},\partial\overline{z}) - (D_2^T C D_1 \partial\overline{u},\overline{z}) =$$
$$= \int_{x_{1(ie)}}^{x_{Np(ie)}} (\begin{bmatrix} & \lambda \\ \mu & \end{bmatrix}\overline{u},\partial\overline{z})dx - \int_{x_{1(ie)}}^{x_{Np(ie)}} (\begin{bmatrix} & \mu \\ \lambda & \end{bmatrix}\partial\overline{u},\overline{z})dx\,;$$
$$(4.31)$$

$$(L_{vv}\overline{u},\overline{z}) = (D_2^T C D_2 \overline{u},\overline{z}) =$$
$$= \int_{x_{1(ie)}}^{x_{Np(ie)}} (\begin{bmatrix} \mu & \\ & 2\mu+\lambda \end{bmatrix}\overline{u},\overline{z})dx \quad (4.32)$$

for the following type of functions

$$y(x) = w(t) = \sum_{j=1}^{N_p} \varphi_j(t)\overline{\alpha}^j\,,$$

$$z(x) = \overline{q}(t) = \sum_{j=1}^{N_p} \varphi_j(t)\overline{\beta}^j\,, \quad (4.33)$$

where we have

$$x_{1(ie)} \le x \le x_{N_p(ie)}, \quad 0 \le t \le 1\,;$$

$$\overline{\alpha}^j = \begin{bmatrix} \alpha_1^j \\ \alpha_2^j \end{bmatrix}, \quad \overline{\beta}^j = \begin{bmatrix} \beta_1^j \\ \beta_2^j \end{bmatrix}, \quad j = 1,...,N_p\,; \quad (4.34)$$

Let us substitute (4.33) sequentially into (4.30)-(4.32), changing the variable of integration (see (4.2)-(4.3)). Let us consider (4.30):

$$\int_{x_{1(ie)}}^{x_{Np(ie)}} (\begin{bmatrix} 2\mu+\lambda & \\ & \mu \end{bmatrix} \partial\overline{u}, \partial\overline{z}) dx$$

$$= \frac{h_e}{h_e^2} \int_0^1 (\begin{bmatrix} 2\mu+\lambda & \\ & \mu \end{bmatrix} \sum_{j=1}^{N_p} \varphi_j'(t)\overline{\alpha}^j, \sum_{j=1}^{N_p} \varphi_j'(t)\overline{\beta}^j) dt =$$

$$= \sum_{i=1}^{N_p}\sum_{j=1}^{N_p} (\frac{1}{h_e}\int_0^1 \varphi_i'(t)\varphi_j'(t)dt \begin{bmatrix} 2\mu+\lambda & \\ & \mu \end{bmatrix} \overline{\alpha}^j, \overline{\beta}^i) =$$

$$= \sum_{i=1}^{N_p}\sum_{j=1}^{N_p} b_{ij} (\begin{bmatrix} 2\mu+\lambda & \\ & \mu \end{bmatrix} \overline{\alpha}^j, \overline{\beta}^i), \quad (4.35)$$

where we have

$$b_{ij} = \frac{1}{h_e}\int_0^1 \varphi_i'(t)\varphi_j'(t)dt. \quad (4.36)$$

It should be noted, in particular, that, $b_{ji} = b_{ij}$ i.e. if $B = \{b_{ij}\}_{i,j=1,..,N_p}$, then $B^T = B$.

For further transformations, we use the representation

$$\overline{\alpha}^j = \begin{bmatrix} \alpha_1^j \\ \alpha_2^j \end{bmatrix} = \alpha_1^j \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \alpha_2^j \begin{bmatrix} 0 \\ 1 \end{bmatrix};$$

$$\overline{\beta}^j = \begin{bmatrix} \beta_1^j \\ \beta_2^j \end{bmatrix} = \beta_1^j \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \beta_2^j \begin{bmatrix} 0 \\ 1 \end{bmatrix};$$

$$\begin{bmatrix} 2\mu+\lambda & \\ & \mu \end{bmatrix} \overline{\alpha}^j = \alpha_1^j (2\mu+\lambda) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \alpha_2^j \mu \begin{bmatrix} 0 \\ 1 \end{bmatrix};$$

$$(\begin{bmatrix} 2\mu+\lambda & \\ & \mu \end{bmatrix} \overline{\alpha}^j, \overline{\beta}^i) = (2\mu+\lambda)\alpha_1^j\beta_1^i + \mu\alpha_2^j\beta_2^i. \quad (4.37)$$

We can substitute (4.37) in (4.35). Taking into account (4.22)-(4.29) and the adopted notation we get

$$\sum_{i=1}^{N_p}\sum_{j=1}^{N_p} b_{ij}((2\mu+\lambda)\alpha_1^j\beta_1^i + \mu\alpha_2^j\beta_2^i) =$$

$$= (2\mu+\lambda)\sum_{i=1}^{N_p}\sum_{j=1}^{N_p} b_{ij}\alpha_1^j\beta_1^i + \mu\sum_{i=1}^{N_p}\sum_{j=1}^{N_p} b_{ij}\alpha_2^j\beta_2^i =$$

$$= (2\mu+\lambda)(B\overline{\alpha}_1, \overline{\beta}_1) + \mu(B\overline{\alpha}_2, \overline{\beta}_2) =$$

$$= (2\mu+\lambda)(BT_{N_p}^{-1}\overline{u}_1^{ie}, T_{N_p}^{-1}\overline{z}_1^{ie}) +$$

$$+ \mu(BT_{N_p}^{-1}\overline{u}_2^{ie}, T_{N_p}^{-1}\overline{z}_2^{ie}) =$$

$$= (2\mu+\lambda)((T_{N_p}^{-1})^T BT_{N_p}^{-1}\overline{u}_1^{ie}, \overline{z}_1^{ie}) +$$

$$+ \mu((T_{N_p}^{-1})^T BT_{N_p}^{-1}\overline{u}_2^{ie}, \overline{z}_2^{ie}) =$$

$$= (2\mu+\lambda)(A_{11}\overline{u}_1^{ie}, \overline{z}_1^{ie}) + \mu(A_{11}\overline{u}_2^{ie}, \overline{z}_2^{ie}) =$$

$$= \left(\left[\begin{array}{c|c} (2\mu+\lambda)A_{11} & 0 \\ \hline 0 & 0 \end{array} \right] \begin{bmatrix} \overline{u}_1^{ie} \\ \overline{u}_2^{ie} \end{bmatrix}, \begin{bmatrix} \overline{z}_1^{ie} \\ \overline{z}_2^{ie} \end{bmatrix} \right) +$$

$$+ \left(\left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & \mu A_{11} \end{array} \right] \begin{bmatrix} \overline{u}_1^{ie} \\ \overline{u}_2^{ie} \end{bmatrix}, \begin{bmatrix} \overline{z}_1^{ie} \\ \overline{z}_2^{ie} \end{bmatrix} \right) =$$

$$= \left(\left[\begin{array}{c|c} (2\mu+\lambda)A_{11} & 0 \\ \hline 0 & \mu A_{11} \end{array} \right] \begin{bmatrix} \overline{u}_1^{ie} \\ \overline{u}_2^{ie} \end{bmatrix}, \begin{bmatrix} \overline{z}_1^{ie} \\ \overline{z}_2^{ie} \end{bmatrix} \right) =$$

$$= \left(\left[\begin{array}{c|c} (2\mu+\lambda)A_{11} & 0 \\ \hline 0 & \mu A_{11} \end{array} \right] P^T\overline{u}^{ie}, P^T\overline{z}^{ie} \right) =$$

$$= \left(P \left[\begin{array}{c|c} (2\mu+\lambda)A_{11} & 0 \\ \hline 0 & \mu A_{11} \end{array} \right] P^T\overline{u}^{ie}, \overline{z}^{ie} \right) =$$

$$= (K_{uu}^{ie}\overline{u}^{ie}, \overline{z}^{ie}). \quad (4.38)$$

Thus, an expression is obtained for the local stiffness matrix corresponding to the operator L_{uu} within the element number i_e:

$$K_{uu}^{ie} = P \left[\begin{array}{c|c} (2\mu+\lambda)A_{11} & 0 \\ \hline 0 & \mu A_{11} \end{array} \right] P^T, \quad (4.39)$$

where we have

$$A_{11} = (T_{N_p}^{-1})^T BT_{N_p}^{-1}. \quad (4.40)$$

Then we can consider (4.31) in a similar way:

$$\int_{x_{1(ie)}}^{x_{Np(ie)}} (\begin{bmatrix} & \lambda \\ \mu & \end{bmatrix} \overline{u}, \partial\overline{z}) dx - \int_{x_{1(ie)}}^{x_{Np(ie)}} (\begin{bmatrix} & \mu \\ \lambda & \end{bmatrix} \partial\overline{u}, \overline{z}) dx =$$

$$= \sum_{i=1}^{N_p}\sum_{j=1}^{N_p} r_{ij} (\begin{bmatrix} & \lambda \\ \mu & \end{bmatrix} \overline{\alpha}^j, \overline{\beta}^i) - \sum_{i=1}^{N_p}\sum_{j=1}^{N_p} s_{ij} (\begin{bmatrix} & \mu \\ \lambda & \end{bmatrix} \overline{\alpha}^j, \overline{\beta}^i), \quad (4.41)$$

where we have

$$r_{ij} = \int_0^1 \varphi_i'(t)\varphi_j(t)dt\,; \quad s_{ij} = \int_0^1 \varphi_i(t)\varphi_j'(t)dt\,. \quad (4.42)$$

We should note that if

$$R = \{r_{ij}\}_{i,j=1,..,N_p}\,, \quad S = \{s_{ij}\}_{i,j=1,..,N_p}\,,$$

we get

$$R^T = S\,.$$

Let us define the elements of the sums (4.41):

$$\begin{bmatrix} & \lambda \\ \mu & \end{bmatrix}\overline{\alpha}^j = \alpha_2^j\lambda\begin{bmatrix}1\\0\end{bmatrix} + \alpha_1^j\mu\begin{bmatrix}0\\1\end{bmatrix};$$

$$(\begin{bmatrix} & \lambda \\ \mu & \end{bmatrix}\overline{\alpha}^j, \overline{\beta}^i) = \mu\alpha_1^j\beta_2^i + \lambda\alpha_2^j\beta_1^i\,; \quad (4.43)$$

$$\begin{bmatrix} & \mu \\ \lambda & \end{bmatrix}\overline{\alpha}^j = \alpha_2^j\mu\begin{bmatrix}1\\0\end{bmatrix} + \alpha_1^j\lambda\begin{bmatrix}0\\1\end{bmatrix};$$

$$(\begin{bmatrix} & \mu \\ \lambda & \end{bmatrix}\overline{\alpha}^j, \overline{\beta}^i) = \lambda\alpha_1^j\beta_2^i + \mu\alpha_2^j\beta_1^i\,. \quad (4.44)$$

Substituting (4.43) into (4.41) and, taking into account (4.22)-(4.29) and the accepted notation, we obtain

$$\sum_{i=1}^{N_p}\sum_{j=1}^{N_p} s_{ij}(\lambda\alpha_1^j\beta_2^i + \mu\alpha_2^j\beta_1^i) =$$
$$= \lambda\sum_{i=1}^{N_p}\sum_{j=1}^{N_p} s_{ij}\alpha_1^j\beta_2^i + \mu\sum_{i=1}^{N_p}\sum_{j=1}^{N_p} s_{ij}\alpha_2^j\beta_1^i =$$
$$= \lambda(S\overline{\alpha}_1, \overline{\beta}_2) + \mu(S\overline{\alpha}_2, \overline{\beta}_1) =$$
$$= \lambda(ST_{N_p}^{-1}\overline{u}_1^{ie}, T_{N_p}^{-1}\overline{z}_2^{ie}) + \mu(ST_{N_p}^{-1}\overline{u}_2^{ie}, T_{N_p}^{-1}\overline{z}_1^{ie}) =$$
$$= \lambda((T_{N_p}^{-1})^T ST_{N_p}^{-1}\overline{u}_1^{ie}, \overline{z}_2^{ie})$$
$$+ \mu((T_{N_p}^{-1})^T ST_{N_p}^{-1}\overline{u}_2^{ie}, \overline{z}_1^{ie}) = \lambda(A_{01}\overline{u}_1^{ie}, \overline{z}_2^{ie}) +$$
$$+ \mu(A_{01}\overline{u}_2^{ie}, \overline{z}_1^{ie}) =$$
$$= \left(\left[\begin{array}{c|c} 0 & \mu A_{01} \\ \hline \lambda A_{01} & 0 \end{array}\right]\begin{bmatrix}\overline{u}_1^{ie}\\ \overline{u}_2^{ie}\end{bmatrix}, \begin{bmatrix}\overline{z}_1^{ie}\\ \overline{z}_2^{ie}\end{bmatrix}\right) =$$
$$= \left(P\left[\begin{array}{c|c} 0 & \mu A_{01} \\ \hline \lambda A_{01} & 0 \end{array}\right]P^T\overline{u}^{ie}, \overline{z}^{ie}\right) =$$
$$= (K_{2,uv}^{ie}\overline{u}^{ie}, \overline{z}^{ie})\,. \quad (4.45)$$

Thus, an expression is obtained for the local stiffness matrix corresponding to the operator L_{uv} within the element number i_e:

$$K_{uv}^{ie} = K_{1,uv}^{ie} - K_{2,uv}^{ie}\,, \quad (4.46)$$

where we have

$$K_{1,uv}^{ie} = P\left[\begin{array}{c|c} 0 & \lambda A_{10} \\ \hline \mu A_{10} & 0 \end{array}\right]P^T; \quad A_{10} = (T_{N_p}^{-1})^T RT_{N_p}^{-1}; \quad (4.47)$$

$$K_{2,uv}^{ie} = P\left[\begin{array}{c|c} 0 & \mu A_{01} \\ \hline \lambda A_{01} & 0 \end{array}\right]P^T; \quad A_{01} = (T_{N_p}^{-1})^T ST_{N_p}^{-1}. \quad (4.48)$$

Let us note that

$$K_{2,uv}^{ie} = (K_{1,uv}^{ie})^T\,. \quad (4.49)$$

Let us further consider (4.32) in a similar way:

$$\int_{x_{1(ie)}}^{x_{Np(ie)}} (\begin{bmatrix}\mu & \\ & 2\mu+\lambda\end{bmatrix}\overline{u}, \overline{z})dx =$$
$$= h_e\int_0^1 (\begin{bmatrix}\mu & \\ & 2\mu+\lambda\end{bmatrix}\sum_{j=1}^{N_p}\varphi_j(t)\overline{\alpha}^j, \sum_{j=1}^{N_p}\varphi_j(t)\overline{\beta}^j)dt =$$
$$= \sum_{i=1}^{N_p}\sum_{j=1}^{N_p}(h_e\int_0^1\varphi_i(t)\varphi_j(t)dt\begin{bmatrix}\mu & \\ & 2\mu+\lambda\end{bmatrix}\overline{\alpha}^j, \overline{\beta}^i) =$$
$$= \sum_{i=1}^{N_p}\sum_{j=1}^{N_p} m_{ij}(\begin{bmatrix}\mu & \\ & 2\mu+\lambda\end{bmatrix}\overline{\alpha}^j, \overline{\beta}^i)\,, \quad (4.50)$$

where we have

$$m_{ij} = h_e\int_0^1\varphi_i(t)\varphi_j(t)dt\,. \quad (4.51)$$

We should note that, in particular $m_{ji} = m_{ij}$, i.e. if $M = \{m_{ij}\}_{i,j=1,..,N_p}$ we get

$$M^T = M\,.$$

For further transformations, we use the representation

$$\left(\begin{bmatrix} \mu & \\ & 2\mu+\lambda \end{bmatrix}\overline{\alpha}^j, \overline{\beta}^i\right) = \mu\alpha_1^j\beta_1^i + (2\mu+\lambda)\alpha_2^j\beta_2^i . \tag{4.52}$$

We can substitute (4.52) in (4.50) and taking into account (4.22)-(4.29) and the adopted notation we get:

$$\begin{aligned}
&\sum_{i=1}^{N_p}\sum_{j=1}^{N_p} m_{ij}(\mu\alpha_1^j\beta_1^i + (2\mu+\lambda)\alpha_2^j\beta_2^i) = \\
&= \mu\sum_{i=1}^{N_p}\sum_{j=1}^{N_p} m_{ij}\alpha_1^j\beta_1^i + (2\mu+\lambda)\sum_{i=1}^{N_p}\sum_{j=1}^{N_p} m_{ij}\alpha_2^j\beta_2^i = \\
&= \mu(M\overline{\alpha}_1, \overline{\beta}_1) + (2\mu+\lambda)(M\overline{\alpha}_2, \overline{\beta}_2) = \\
&= \mu((T_{N_p}^{-1})^T M T_{N_p}^{-1}\overline{u}_1^{ie}, \overline{z}_1^{ie}) + \\
&+ (2\mu+\lambda)((T_{N_p}^{-1})^T B T_{N_p}^{-1}\overline{u}_2^{ie}, \overline{z}_2^{ie}) = \\
&= \mu(A_{00}\overline{u}_1^{ie}, \overline{z}_1^{ie}) + (2\mu+\lambda)(A_{00}\overline{u}_2^{ie}, \overline{z}_2^{ie}) = \\
&= \left(P\left[\begin{array}{c|c} \mu A_{00} & 0 \\ \hline 0 & (2\mu+\lambda)A_{00} \end{array}\right]P^T\overline{u}^{ie}, \overline{z}^{ie}\right) = \\
&= (K_{vv}^{ie}\overline{u}^{ie}, \overline{z}^{ie}) .
\end{aligned} \tag{4.53}$$

Thus, an expression is obtained for the local stiffness matrix corresponding to the operator L_{vv} within the element number i_e:

$$K_{uu}^{ie} = P\left[\begin{array}{c|c} \mu A_{00} & 0 \\ \hline 0 & (2\mu+\lambda)A_{00} \end{array}\right]P^T , \tag{4.54}$$

where we have

$$A_{00} = (T_{N_p}^{-1})^T M T_{N_p}^{-1} . \tag{4.55}$$

5. SEVERAL ASPECTS OF NUMERICAL IMPLEMENTATION

The presented algorithm can be implemented using MATLAB tools. The MATLAB system has convenient functions for working with polynomials. Moreover, the main parameter of these functions is the vector of coefficients of the polynomial. To determine the coefficients of basic polynomials φ_k on an interval $[0\ \ 1]$, we can firstly determine their values at N_p points of the interval $t = [t_1, t_2, ..., t_{N_p}]$, $t_i \in [0\ \ 1]$, $i = 1, 2, ..., N_p$:

$$F_k(i) = \varphi_k(t_i) , \quad i = 1, 2, ..., N_p , \quad k = 1, 2, ..., N_p .$$

Then, using the `polyfit` function, we define their coefficient vector p_k:

```
pk=polyfit(t,Fk,Nk)
```

This function is used to determine the coefficients of the optimal polynomial using the least squares method. In the considering case, we construct polynomial of the $(N_p - 1)$ th degree (i.e. we have to define N_p coefficients of polynomial, according to its N_p values), therefore, we get a polynomial passing through the given values.

In order to calculate the derivatives we can sequentially use the `polyder` function:

```
dpk=polyder(pk)
```
is the vector of coefficients φ_k'.

In order to calculate the product of polynomials we can use the `conv` function:

```
pij=conv(pi,pj)
```
is the vector of coefficients $\varphi_i\varphi_j$;
```
d10pij=conv(dpi,pj)
```
is the vector of coefficients $\varphi_i'\varphi_j$;
```
d01pij=conv(pi,dpj)
```
is the vector of coefficients $\varphi_i\varphi_j'$;
```
dpij=conv(dpi, dpj)
```
is the vector of coefficients $\varphi_i'\varphi_j'$.

In order to calculate the antiderivative of a polynomial we can use the `polyint` function:

```
Pi=polyint(pi)
```
is the vector of coefficients $\int\varphi_i dt$;

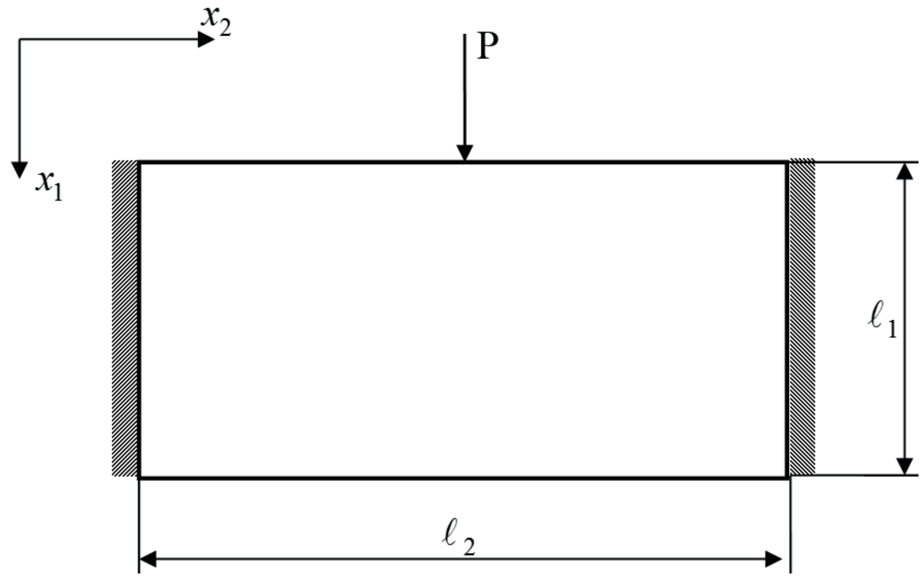


Figure 5.1. Formulation of the problem (Sample).

`Pij=polyint(pij)`

is the vector of coefficients $\int \varphi_i \varphi_j dt$;

`d10Pij=polyint(d10pij)`

is the vector of coefficients $\int \varphi_i' \varphi_j dt$;

`d01Pij=polyint(d01pij)`

is the vector of coefficients $\int \varphi_i \varphi_j' dt$;

`dPij=polyint(dpij)`

is the vector of coefficients $\int \varphi_i' \varphi_j' dt$;

Then the calculation of

$$B(i,j),\quad R(i,j),\quad S(i,j),\quad M(i,j)$$

can be done in accordance with formulas

$M(i,j) =$ `he[polyval(Pij,1)-polyval(Pij,0)];`

$R(i,j) =$ `polyval(d10Pij,1)-polyval(d10Pij,0);`

$S(i,j) =$ `polyval(d01Pij,1)-polyval(d01Pij,0);`

$B(i,j) =$ `[polyval(dPij,1)-polyval(dPij,0)]/he,`

where the function `polyval (p, t)` allows researcher to calculate the values of a polynomial with a vector of coefficients p at a given point t.

5. EXAMPLE OF ANALYSIS

5.1. Formulation of the problem.

As a model example, let us consider the determination of the displacements of a beam-wall, fixed along the side faces in both directions, under the influence of a load concentrated in the center (Figure 5.1).

Let us consider the following geometric parameters: $\ell_1 = 6$ m, $\ell_2 = 12$ m.

Let us consider the following design parameters of material of plate: coefficient of elasticity $E = 26500 \cdot 10^4$ kN/m^2, Poisson's ratio $\nu = 0.15$. Let external load parameter be equal to $P = 1\,00$kN.

5.2. Structural analysis with allowance for localization.

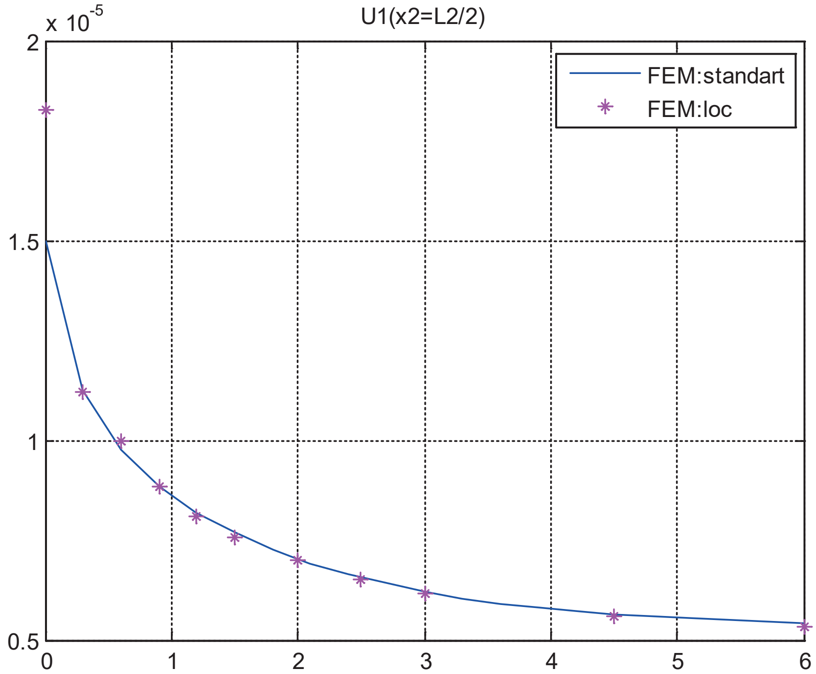
Let the number of elements be equal to $N_e = 6$.

Then we have the following element length:

$$h_e = \ell_1 / N_e = 6/4 = 1.5\,.$$

Let's define localization in the load area.

For the first element we have $N_k = 5$ and fifth-order spline; distance between the coordinates of the nodes of the first element is equal to $h_1 = 1.5/5 = 0.3$.



Figures 5.2. Comparison of the results of analysis in the middle sections along x_1 direction (discrete direction).

For the second element we have $N_k = 3$ and third-order spline; distance between the coordinates of the nodes of the second element is equal to $h_2 = 1.5/3 = 0.5$.

For the third element and for the fourth element we have $N_k = 1$ and first-order spline; distance between the coordinates of the nodes of the third element and of the fourth element is equal to $h_3 = h_4 = 1.5/5 = 0.3$.

With such approximation the total number of nodes for all elements is equal to

$$N_x = 5 + 3 + 2 \cdot 1 + 1 = 11.$$

Then the total number of unknown nodal values for vectors $\bar{u}$ and $\bar{v} = \bar{u}'$ is equal to

$$N_g = 4 \cdot N_x = 4 \cdot 11 = 44.$$

5.3. Structural analysis without localization.
In this case, we will consider only the standard linear fulfilment. In this case, the length of the element is taken equal to the minimum distance between the nodes, i.e. $h_e = 0.3$. Then the number of elements is equal to

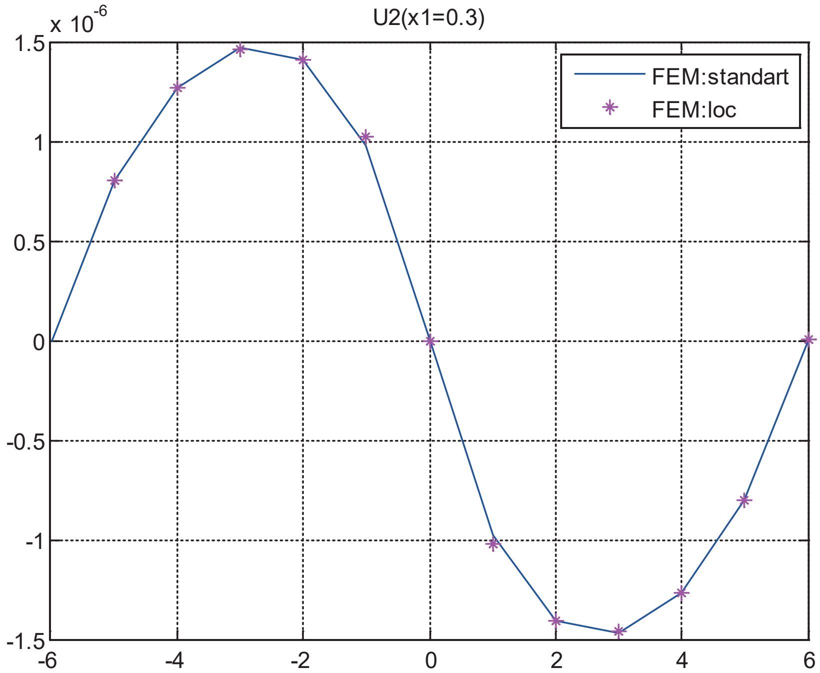
$$N_e = 6/0.3 = 20$$

and the total number of nodes is equal to $N_x = 21$. In this case, the number of nodal unknowns for each component of the vectors $\bar{u}$ and $\bar{v} = \bar{u}'$ is equal to

$$N_g = 4 \cdot N_x = 4 \cdot 21 = 84.$$

Graphical comparison of corresponding results of analysis is presented at Figure 5.2 and Figure 5.3 (`FEM:loc-spline` are nodal values computed with allowance for localization; `FEM:standart` are nodal values computed without localization).

As researcher can see, the results obtained are almost completely identical. Besides, the use of localization based on application of B-splines of various degrees leads to a significant decrease in the number of unknowns. The difference for this example is equal to $\Delta = 84 - 44 = 40$.



Figures 5.3. Comparison of the results of analysis in the middle sections along x_2 direction (continual direction).

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