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## *International Journal for Computational Civil and Structural Engineering*

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В ноябре 2020 года журнал начал индексироваться в международной базе Scopus.

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## СОДЕРЖАНИЕ

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# IMPLEMENTATION OF GEOTECHNICAL MONITORING ON OPEN-TYPE CARPARKS

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**Abstract**: Special aspects of the two-year (2018–2019) geotechnical monitoring for the parking buildings in use and their deformational behavior are presented in the article. In 2005, open-type car parks No. 1 and No. 2, fell into the influence zone of a two construction projects, a railway line construction, which connects Vnukovo-1 airport and Kievsky railway station in Moscow, and construction of the new Vnukovo-1 airport building.The article discusses tasks to ensure serviceability of existing buildings. Geotechnical monitoring during building operation allows for a timely documentationof negative processes occurring in building structures and to make prompt decisions to prevent emergencies. Geotechnical monitoring is carried out according to a specially developed program that includes a set of works using various methods. This allows us to assess the state of the supporting structures of the monitored buildings to ensure their strength, reliability, and safe operation.

**Keywords**: geotechnical monitoring building structures, safety, reliability, parking.

# ОПЫТ ПРИМЕНЕНИЯ ГЕОТЕХНИЧЕСКОГО МОНИТОРИНГА НА ПАРКИНГАХ ОТКРЫТОГО ТИПА

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**Аннотация**: В данной статье рассмотрены особенности проведения двухлетнего (2018–2019 годы) геотехнического мониторинга за деформационным поведением эксплуатируемых зданий паркингов, представляющих собой автостоянки открытого типа №1 и №2, попавших в 2005 году в зону влияния строительства железнодорожной ветки, соединяющей аэропорт Внуково-1 и Киевский вокзал в Москве, и в зону влияния строительства нового здания аэропорта Внуково-1. В статье рассмотрены вопросы обеспечения эксплуатационной пригодности существующих зданий. Проведение геотехнического мониторинга в процессе эксплуатации позволяет вовремя зафиксировать негативные процессы, происходящие в строительных конструкциях и принять оперативные решения для предотвращения аварийных ситуаций. Геотехнический мониторинг проводится по специально разработанной программе включающей проведения комплекса работ с использованием различных методов. Это позволяет получить оценку состояния несущих конструкций наблюдаемых зданий для обеспечения их прочности, надежности и безопасной эксплуатации.

**Ключевые слова**: геотехнический мониторинг, безопасность, надежность, комплексные наблюдения

## INTRODUCTION

Geotechnical monitoring is widely used to assess reliability and of operation of structural elements in a building, both during construction and during its service [1–11]. With its help it is possible to document negative processes occurring in building structures.This will allow us to make prompt decisions to prevent emergencies, and thereby to save significant material resources. Acomprehensive assessment of the structure state is done according to the monitoring results.Behavior of foundations, soil bases, and aboveground structures is predicted.In the event of negative processes development in structures, suggestions to repair and strengthen building structuresare developed.

In this article, we will focus on the results of geotechnical monitoring carried out by MGSU from 2018 to 2020. Geotechnical monitoring was done for the technical condition of building structures and base soils of the open-type parking lots.

During this period, operated parking buildings No. 1 and No. 2, located at the forecourt of the Vnukovo-1 airport, were under comprehensive observations of their deformation behavior.

## MAIN PART

During construction period (2005) of the railway line connecting the Kievsky railway station in Moscow and the new building of Vnukovo-1 airport, as well as during the construction of the newest building of Vnukovo-1 airport, these car-parks fell into the influence zone of these objects under construction. To ensure the airport operation at that time, it was decided to preserve car-parks No. 1 and No. 2. (Fig.1,2).

Three-tier open-type parking buildings were built in 2004. The construction volume of each car-park is about 44.3 thousand $m^3$. The level of responsibility is normalin accordance with GOST 31937-2011 [19]. Car-parks have a framed structural diagram. Frame of the buildings is made of reinforced concrete columns with a section of 40x40cm. Column grid is 6 ÷ 7.2m x 5.1 ÷ 7.2m, with monolithic beam floor structures in the form of 20 cm thick slabs on beams with a rib section of 40 ÷ 30 cm. The structure of the covering is similar tothat of the floor structures [12–18]. The foundations under the columns are pedestal footing, made of monolithic reinforced concrete. The depth of the foundations is 2.9 m.

Geotechnical monitoring of the technical condition of car-parks' structures was done by MGSU from 2005 to 2012.

Inspections of the car-parks No. 1 and No. 2 buildings carried out in 2016 by OOO "RSP" and OOO "Tekhmeh", allowed to generally assess the state of building structures as "Operational to a limited extent".

In this regard, owner decided to conduct geotechnical monitoring to assess the changes in technical state of car-parks' structures over time,

*Figure 1. Car-park No. 1. General view of the façade*

*Figure 2. Facade fragment of the Car-park No. 2.*

to timely take appropriate measures to prevent negative consequences during their operation.
In April 2018, a corresponding agreement was signed with MGSU. Starting from May 2018 and to the present time, MGSU conducts geotechnical monitoring of car-parks No. 1 and No. 2, in accordance with the approved program.
Geotechnical monitoring program of the observed buildings includes assessment of load-bearing structures state to ensure their strength, reliability, and safe operation, based on a set of works using various methods. The monitoring program ensures following operations:
– geodetic monitoring with recordings of vertical displacements (settlements);
– object monitoring – a visual and instrumental surveying, which allows to assess the current technical condition of the main load-bearing structures of buildings;
– electrical contact dynamic sounding of soils. This type of sounding allows us to study the geological section at a point to a depth of 10–11 meters and to determine physical and mechanical properties of soils;
– seismic/acoustic sounding of foundations, to determine the depth of the foundation base, state of the foundation-soil contact, state of the foundation material and to detect large defects and cracks in the body of the foundation.
For the period from May 2018 to September 2019, four cycles of observations were carried out, excluding the initial one. The next fifth cycle was planned to be completed in March 2020, but due to the COVID 19 epidemic and repair works on car-parks, this monitoring cycle, in agreement with the owner, was postponed to October 2020.
The 1st and 2nd cycle of geodetic monitoring carried out at Car-park No. 1 showed that the total settlement of the building of car-park No. 1 does not exceed the maximum permissible value. The maximum sag on the marks installed in the middle of parking flooring structure span also does not exceed the maximum permissible value;
According to the results of the 1st and 2nd cycle of visual and instrumental surveying of the Car-park No. 1 bearing structures, an increase in the area of damage to floor slabs was noted.
Damage to floor slabs manifested itself in the form of wetting, efflorescence, swelling of the paint layer and traces of rebar corrosion on the lower surface of the slabs. There was also an increase in the area and the appearance of new areas of concrete protection layer destruction with exposure of reinforcement rebars. There was an increase in slab sag between dilatation joints, loss of the bearing capacity of the beam and flooring structure due to systematic soaking (Fig. 3,4).
Category of the technical condition of structures was assessed in accordance with GOST 31937-2011 [19], based on the results of 2nd cycle of geotechnical monitoring of car-park No. 1. In accordance with [19], for the period of the

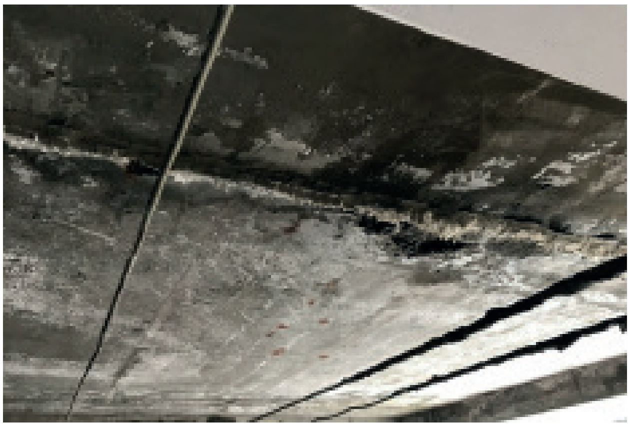

*Figure 3. Car-park No. 1. Destruction of the concrete protective layer.*

*Figure 4. Car-park No. 1. Efflorescences on the flooring surface at the dilatation joints.*

survey, flooring structures (slabs and crossbars), coverings, floors and roofing were in "operational to a limited extent" state. Technical state of the rest of the structures was confirmed as operational.

An analysis of all results of the car-park No. 1 surveys for the period of the 2nd monitoring cycle allowed us to draw conclusions about stability, at that time, of the technical state of foundations and base soils, however, trend to an increase in damage of the above-foundation building structures with a deterioration in their technical condition ntinued. The results of the survey showed that further safe operation of Car-park No. 1 is possible for the next 2 years (until February 2021). To extend the life of car-park No. 1 until 2024 it is necessary to carry out repair and restoration works: replacement of asphalt-concrete floors with waterproofing of the flooring structure; creation of dilatation joints in the new floor; restoration of a protective layer of concrete in places of destruction; injection of cracks with an opening width of more than 0.3-0.4 mm.

*Figure 5. Car-park №1. Dilatation joints and flooring structure after repair.*

*Figure 6. Car-park №1. Reinforced flooring structure beam.*

According to the assessment of damageability, performed according to the method [20], the residual operation time of the Car-park No. 1 structures was 2 years.

During the 3rd cycle of visual and instrumental survey for the period from February to June 2019 at Parking No. 1, as predicted in the 2nd cycle, there was an increase in the area of damage in the form of wetting, efflorescence, swelling of the paint layer and traces of corrosion of rebars on the bottom surface of the slabs. Area of concrete protective layer destruction with exposure of reinforcement rebars also increased, and appearance of the new areas with such damage was noted. There was an increase in slab sag between dilatation joints. The bearing capacity of the beams and floor slabs in some areas was completely exhausted due to systematic soaking. Noted damage was caused by the accumulation of melt and rainwater in the lowered area of the covering, which developed especially intensively in the winter-spring period.

In accordance with [19], at the time of the survey, category of the technical condition of the flooring structure area of the 2nd and 3rd floors was assessed to be in emergency state, and category of roofing, covering, flooring structures and floors was assessed as operational to a limited extent, technical state of the rest of the structures was assessed as operational.

Analysis of all survey results carried out within the 3rd cycle of geotechnical monitoring allowed to draw conclusions about stable technical condition of the foundations and soil bases. But, at the same time, the operated to a limited extent and emergency state of the flooring structures in certain areas and likelihood of further development of damage, and, as a consequence, the deterioration of the technical condition of the above-foundation building structures in the winter-spring period were confirmed.

It should be noted that in connection with the appearance of emergency areas, service life of the car-park No. 1 before the major repairs was exhausted.
Thus, as of June 2019, further safe operation of Car-park No. 1 without restrictions was possible not only after a complex of repair and restoration works indicated earlier, but also after taking measures to strengthen individual structures. At the end of June, car-park No. 1 was closed for repairs.
Due to the need to strengthen individual sections of the flooring structures, MGSU developed a project, according to which reinforcement was carried out.
At the time of the 4th cycle of geotechnical monitoring in September 2019, repair and restoration work on the building of Car-park No. 1 was almost completed.
During repair and restoration works, following repairs of following elements were carried out: protective layer of concrete in damaged areas; dilatation joints; metal mesh fencing; individual sections of flooring structures (Fig. 5, 6)
As a result of the repair and restoration work carried out at Car-park No. 1, previously identified defects and damages affecting the bearing capacity of building structures were eliminated. Technical condition of building structures and the building as a whole as of September 2019, according to GOST 31937-2011 [19], was assessed as corresponding to the norm.
2nd cycle of geotechnical monitoring at Car-park No. 2 and analysis of the obtained results testified to the stability, at that time, of the technical state of the foundations and soil bases, however, there was a trend to an increase in the damageability of the above-foundation building structures with a further deterioration in their technical condition. (Fig. 7,8 )
Assessment of the technical state of Car-park No. 2 by the results of the geotechnical monitoring cycle showed that, in accordance with [19], columns, flooring structures, coverings, floors and roofings are in a operational to a limited extent condition. The rest of the structures are operational.
Damageability assessment carried out according to the method [20], showed that the residual operation time Car-park No. 2 structures is absent. In this regard, MGSU recommended to close the parking lot and carry out a complex of repair and restoration works with strengthening of individual structural units.
Based on the results of the 2nd cycle of geotechnical monitoring, Car-park No. 2 was closed by the ownerfor repair and restoration works.
By the time of the 3rd cycle (June 2019), repair works at Car-park No. 2 were completed. Based on the results of the monitoring, bearing and enclosing structures were brought into a

*Figure 7. Car-park No. 2. Destruction of the plaster layer of the columns.*

*Figure 8. Car-park №2. Reinforcement rebar corrosion marks on the concrete surface*

normative state. According to the assessment of damageability [6], it was found that further safe operation of Car-park No. 2 is possible until 2024. Residual operation time before major repair works as of June 2019 is 31 years.

During the 4th cycle of visual and instrumental survey in September 2019 no increment of existing and formation of new damages was found on the building of Car-park No. 2. It should be noted that as of September 2019, settlement values of the foundations of the Car-park No. 1 and No. 2 are within the measurement accuracy and do not cause concern.

## CONCLUSIONS

Realized comprehensive geotechnical monitoringprogramon the state ofopen-type car parks'structural elementsover a two-year period allows us to draw the following conclusions:

– as part of geotechnical monitoring, it is necessary conduct complex, mutually complementary surveys. The combination of geodetic, surveying, geophysical and computational-analytical methods allow for timely detection and prevention of the possible emergencies in building structures at an early stage of their development;

– geotechnical monitoring during operational cycle for the buildings affected by the new construction several years ago, ensures safety and reliability of building structures

– as part of the monitoring, engineering proposals were developed to prevent and eliminate deviations exceeding permissible values.Implementation of thoseproposals was monitored;

– it was found that the highest intensity of damage to car parks' structural elementswas observed during the winter-spring period due to wetting, efflorescence, destruction of concrete protective layer, exposure and corrosion of rebars;

– assessed impact of damage to building structures on their reliability,discovered during geotechnical monitoring allowed to determine remaining lifetime for safe operation of car parks.

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# NUMERICAL ANALYSIS OF LONGITUDINAL REINFORCEMENT EFFECT ON RC SLAB PUNCHING SHEAR RESISTANCE BY STRENGTH AND CRACK PROPAGATION CRITERIA

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**Abstract**: The article deals with the influence of longitudinal reinforcement of the support zone of reinforced concrete slabs on the strength and crack resistance under the criterion for punching failure. The evaluation of impact was carried out by the method of numerical studies based on finite-element computational technologies. The results of physical experiments published in the scientific literature are taken as the basis for the conducted research. The existing provisions of the existing domestic and foreign standards for the calculation of slab reinforced concrete structures according to the criterion for punching failure are considered. The main provisions of the applied finite element approach are presented, verification is performed and the correctness of the applied technique is justified. In the numerical studies, the forecast of strength and crack resistance was done for considered reinforced concrete slab structures; the results of numerical studies were compared with the data from physical experiments and the evaluation results based on the relevant domestic and foreign regulations. According to numerical studies results it was stated that longitudinal reinforcement of the tensile zone of slab structure has a significant impact on both the level of load-bearing capacity and the scheme of crack formation and propagation. The results of the implemented studies justify the necessity to revise the national standards of structural analysis for reinforcement concrete slab structures under the criterion for punching failure.

**Keywords**: modeling, numerical methods, design model, stress-strain state, reinforced concrete structures, punching failure.

# ОЦЕНКА ВЛИЯНИЯ ПРОДОЛЬНОГО АРМИРОВАНИЯ НА ПРОЧНОСТЬ И ТРЕЩИНОСТОЙКОСТЬ ЖЕЛЕЗОБЕТОННОЙ ПЛИТЫ ПО КРИТЕРИЮ ПРОДАВЛИВАНИЯ МЕТОДОМ ЧИСЛЕННЫХ ИССЛЕДОВАНИЙ

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**Аннотация**: В статье рассматривается вопрос влияния продольного армирования приопорной зоны железобетонных плит на прочность и трещиностойкость по критерию продавливания. Оценка влияния выполнена методом численных исследований на основе конечноэлемпентных расчетных технологий. В качестве основы проведенных исследований приняты результаты физических экспериментов, опубликованных в научной литературе. Рассмотрены существующие положения действующих отечественных и зарубежных норм по расчету плитных железобетонных конструкций по критерию продавливания. Представлены основные положения используемого конечноэлементного подхода, выполнена верификация и обоснована корректность применяемой методики. В выполненных численных исследованиях получен прогноз прочности и трещиностойкости рассмотренных вариантов железобетонных плитных конструкций, представлено сравнение полученных результатов численных исследований с данными физических экспериментов и результатами оценки на основе действующих отечественных и зарубежных норм. По результатам численных исследований установлено, что продольное армирование растянутой зоны плитных конструкций оказывает существенное влияние как на уровень несущей способности, так и на схему формирования и развития трещин. Результатами выполненных исследований обосновывается необходимость совершенствования отечественных норм по расчету железобетонных плитных конструкций по критерию продавливания.

**Ключевые слова**: моделирование, численные методы, расчетная модель, напряженно-деформированное состояние, железобетонные конструкции, продавливание.

## INTRODUCTION

The studies of the punching failure phenomenon for reinforced concrete structures has a centennial timeline. Thus, one of the first domestic books dedicated to structural analysis and design of reinforced concrete structures [1] contains guidelines for design methods of supporting joints of slab structures as well as the number of structural requirements for the dimensions of zone adjacent to core support in reinforced concrete slabs (pp. 19–22). In the early XXth century, the book [2] containing guidelines for structural analysis and design of supporting joint of floor slabs, with probable manifestation of punching failure, was widely used in the engineering practice for reinforced concrete structures (pp. 524–525).

The studies of reinforced concrete structures carried out in the second half and in the late XX century have taken a broadside approach to the issue of punching failure. Thus, in the works of Zalesov A.S. [3, 4, 5], Karpenko N.I. [6, 7] and others [8–10], as well as of some foreign researchers (see, for example [11–19]), the various aspects of punching failure for reinforced concrete slabs are factored in. The studies implemented by Klovanich S.Ph. and Shekhovtsov V.I. constitute those few works investigating cruciform and angle shape (the research results are outlined in the monograph [20]). It is shown that punching failure is characterized by rather complicated mechanisms defined not only (and not so much) by the performance of the concrete body of a structure but by the impact of both longitudinal and transverse reinforcement in the support zones.

A number of models describing the performance of slabs under the punching failure have been proposed (the detailed analysis of models is presented in [5]).

Along with that, there is commonly held opinion that structural failure under the punching failure mechanism is nothing other than a particular case of sloping section failure of reinforced concrete structure. However, the works [21, 22] prove that such an approach is not the correct one. Thus, punching failure phenomenon should be thoroughly studied, with special consideration given to its complicated mechanism.

When doing numerical prediction of the bearing capacity of reinforced concrete slabs under the criterion for punching failure, the most significant aspect is the factor of longitudinal reinforcement of tensile zone of slab structures. However, the present regulations and codes of practice do not examine the factor of longitudinal reinforcement. So, it is worth considering the fundamental statements of domestic and foreign regulations related to strength prediction of a reinforced concrete structure without transverse reinforcement of support zones.

*Analysis of the current normative approach to the structural design of reinforced concrete slabs under the criterion for punching failure.*

The following regulations of codes of practice have been considered: Construction Rules CR 63.13330.2018 (hereinafter CR63) [23], EN 1992-1-1 Eurocode 2 (hereinafter EC2) [24] and Model Code 2010 (hereinafter MC 2010) [25].

In the ***EC2*** [24], impact puncture strength, regardless of transverse reinforcement (VRd,c) is estimated under the formula (1).

$$V_{Rd,c} = \frac{0.18}{\gamma_c} \cdot b_{0,EC} \cdot d \cdot k \cdot (100 \cdot \rho \cdot f_c)^{1/3} \geq v_{min} \cdot b_{0,EC} \cdot d \quad (1)$$

where: $\gamma_c$ – reliability factor of concrete under compression; $b_{0,EC}$ – perimeter of the effective cross section circuit with rounded corners at the 2$d$ distance from the loaded area; $d$ – effective operating height of cross section; $f_c$ – compressive strength of concrete; $\rho$ – coefficient of longitudinal reinforcement (the maximum value of 2%); $k$ – coefficient factoring in scale effect (relative reduction of impact puncture strength at the increase in its effective operating height of cross section), is calculated as: $k = 1 - \sqrt{200/d}$; $V_{min}$ – minimum impact puncture strength that makes allowance only for tensile strength of concrete and scale effect, is calculated under the formula (2).

$$v_{min} = 0.035 \cdot k^{3/2} \cdot f_c^{1/2} \quad (2)$$

It should be noted that when evaluating bearing capacity of reinforced concrete slabs under the criterion for punching failure, EC2 factor in normal stresses in concrete along the slab orthogonal axes (Y и Z) in critical cross sections. The mentioned normal stresses may appear due to, for example, prestressed longitudinal reinforcement or due to the forces formed in the bearing structures exposed to the loads. The normal stresses factor is taken into account by insertion of additive component $\sigma_{cp}$ into the right side of the formula (1).

$$\sigma_{cp} = \frac{\sigma_{cy} + \sigma_{cz}}{2} \quad (3)$$

where: $\sigma_{cy}$ and $\sigma_{cz}$ – axis stresses faired along the span width for the intermediate columns and along the width of calculated perimeter for edge columns. Caused by external actions or stretching, stresses are taken into consideration.

Factoring in normal stresses in the concrete of support zone, the formula (1) takes the following form:

$$V_{Rd,c} = \frac{0.18}{\gamma_c} \cdot b_{0,EC} \cdot d \cdot k \cdot (100 \cdot \rho \cdot f_c)^{1/3} \geq v_{min} \cdot b_{0,EC} \cdot d + 0.1 \cdot \sigma_{cp} \quad (4)$$

In the ***MC2010*** [25], impact puncture strength regardless of transverse reinforcement ($V_{Rd,c}$) makes allowance for angular rotation of the slab support zone $\psi$ and is estimated under the formula (5).

$$V_{Rd,c} = k_\psi \cdot \frac{\sqrt{f_c}}{\gamma_c} \cdot b_0 \cdot d_v \quad (5)$$

where: $\gamma_c$ – reliability factor of concrete under compression; $b_0$ – perimeter of the effective cross section circuit with rounded corners at the 0.5$d_v$ distance from the loaded area; $d_v$ – effective operating height of cross section; $f_c$ – compressive strength of concrete; $k_\psi$ – coefficient factoring in the angle of slab rotation $\psi$, is calculated under the formula (6).

$$k_\psi = \frac{1}{1.5 + 0.9 \cdot k_{dg} \cdot \psi \cdot d_v} \quad (6)$$

where: $k_{dg}$ – coefficient factoring in the grain size of coarse aggregate; $\psi$ – the angle of slab rotation, estimated by the formula (7) for the recommended MC2010, the approximation level II.

$$\psi = 1.5 \cdot \frac{r_c}{d_v} \cdot \frac{f_y}{E_c} \cdot \left(\frac{m_s}{m_R}\right)^{1.5} \quad (7)$$

where: $r_s$ – the distance from the point where the radial moment of flection is equal to 0 (for the tested samples, the distance from the sample center to the fixing point); $E_s$ – reinforcement modulus of elasticity; $f_y$ – yield point of reinforcement; $m_s$ – moment of flection in the slab exposed to the load, averaged at the width $b_s = 1.5r_s$; $m_R$ – bending strength of the slab, estimated by the formula (8).

$$m_R = \rho \cdot d_v^2 \cdot f_y \cdot \left(1 - \frac{\rho \cdot f_y}{2 \cdot f_c}\right) \quad (8)$$

It bears mentioning a significant peculiarity of the ***MC2010*** [25], i.e. the application of so-called approach in the form of the Levels of Approximation (Level of Approximation, hereinafter LoA) – see figure 1.

LoA I represents preliminary evaluation of bearing capacity of a floor slab support zone at punching failure on the hypothesis that longitudinal reinforcement has reached yield point at the moment of failure.

LoA II is distinguished from LoA I by refinement of the reinforcement use factor for longitudinal reinforcement at the moment of punching failure when the approximate data could be obtained analytically on the base of punching force and the parameters of longitudinal reinforcement. LoA III presupposes specifying strain capacity of reinforcement by dimensional linear calculation, and the level IV provides for determining the angle of rotation directly from the non-linear slab design (including the slab modelling by shell structural elements).

The authors of the MC 2010 [25] recommend the application of the level I (LoA I) for preliminary calculations. LoA II is recommended for designing the most part of new structures with the regular column grid, LoA III is recommended for the analysis of existing structures and the structures with irregular geometry; whereas LoA IV is a good practice for particular cases or for more specified evaluation of the longitudinal floor slab to column joint.

In the CP 63.13330.2018 [23], bearing capacity of reinforced concrete slab without transverse reinforcement is calculated by the formula (9).

$$F_{b,ult} = R_{bt} \cdot A_b \qquad (9)$$

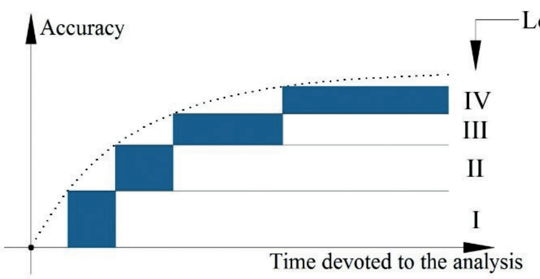


*Figure 1. Accuracy of normative evaluation as a function of time spent on the calculation of different approximation levels.*

where: $F_{b,ult}$ – the ultimate force taken by concrete; $R_{bt}$ – tensile strength of concrete; $A_b$ – area of the effective cross section located at the distance of $0.5h_0$ form the border of the concentrated force application area $F$ with the operating height of the section $h_0$. The area $A_b$ is determined by the formula (10).

$$F_{b,ult} = R_{bt} \cdot A_b \qquad (10)$$

where: $u$ – perimeter of the effective cross section circuit.

Outlined in domestic and foreign codes and regulations, the comparative analysis of bearing capacity rating for reinforced concrete slabs under the criterion for punching failure has shown that allowance had been made for the factor *fib* regulations [25] and in the *EC* codes [24]. The given approach seems rather logical as longitudinal reinforcement of tensile zone of reinforced concrete slab at the support zone has an immediate impact on crack formation and propagation in the zone of superior limits of stress for slab support zone. In its turn, bearing capacity of support joint depends on the crack formation processes under the criterion for punching failure. Thus, the detailed study of the impact of the tensile zone longitudinal reinforcement on the value of bearing capacity of reinforced concrete slabs under the criterion for punching failure is regarded as the relevant task. The solution of this task will enable to refine current domestic codes and regulations.

## OBJECTIVE AND METHOD OF RESEARCH

***Description of finite element computer system designated for numerical methods.***

The study of stress-strain state, bearing capacity and crack formation has been carried out by means of computer system (CS) ATENA [26].

Structural modeling by ATENA, generally speaking, is done based on specific properties of materials: concrete is modelled by the volume finite elements; whereas reinforcing members are usually modelled as rods. However, for some cases, it is quite conceivable that reinforcement modelling is done by means of distributed reinforcement, when steel ration of concrete is set. The user models concrete-to-steel bond by means of introduction of special bonds performing under the desired law. In addition, the software complex allows modelling of other materials, such as steel, soils, mason work, fiber-reinforced concretes, ultrahigh-strength concretes, carbon fiber and others.

Computer software ATENA comprises special constitutive models for finite element analysis of concrete and reinforced concrete structures. In compliance with the official reference book describing the software theoretical basis [26], a model of concrete combines theory of plastic behavior equation (under compression) and fracture mechanics (under tension). The model applies the criterion of superior stress limits for evaluation of strength, exponential softening law, when a crack could be specified as turning or restrained one.

Under tension, concrete behavior is simulated by non-linear methods of fracture mechanics combined with the smeared crack model. The main parameters of the given approach are the following: concrete tensile strength, pattern and shape of crack formation, and fracture energy. The phenomenon of crack formation is described by the smeared crack model in the form of crack band [27]. In its general form, the law of crack formation is presented at the figure 2.

Crack opening width is calculated as full movement within the cracked element [26]. The width w is determined by the formula (11).

$$w = \varepsilon \cdot L_t \tag{11}$$

where: $\varepsilon$ – average relative deflorations of cracked finite element at the lack of strains; $L_t$ – the size of finite element.

As demonstrated at the figure 3, the process of crack formation could be divided into three stages. So-called uncracked stage corresponds to the performance of the material before reaching its ultimate tensile strength. Crack formation occurs in the zone of a potential crack as tension stresses are being reduced in this area. Then crack opening continues at zero tension stresses occurring at the crack tip.

CS ATENA gives expansive opportunities to set the loads and effects on the researched finite element model. Thus, there exist conventional static and dynamic loading as well as the effect forming due to creep, contraction, materials degradation and corrosion.

For plasticity model describing concrete compression, the Menetrèy-Willam failure criterion is applied. Separately, the algorithm combining the crack formation model and plasticity model has been developed. The principal peculiarity of CS ATENA is as follows: though two mentioned models are simulated independently, they are jointly applied in computation.

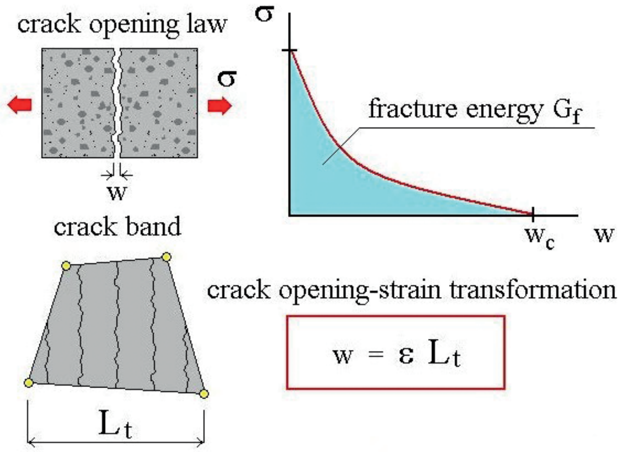


*Figure 2. The law of crack formation in a general form CS ATENA according to the [27].*

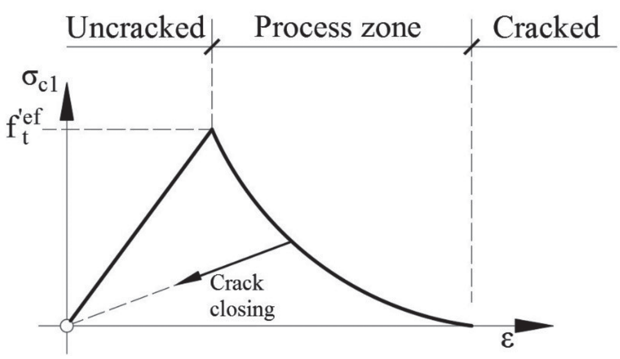


*Figure 3. The stages of crack formation and propagation CS ATENA according to [27].*

The other specific feature of concrete model is the use of so-called restrictions of failure strain isolation. The given notion is applied for identifying discrete failure planes, independent of finite element grid.

For the case of tension, these planes are presented by cracks, for the case of compression – by the areas of grinding respectively. In the computed model, these discrete areas of failure have the dimensions independent of the element dimensions. For this particular reason, failure planes are presented in the model as the planes independent of the finite element grid dimensions. For the case of tensile rupture, the present approach is known as the crack band model mentioned above. In the CS ATENA, the similar approach is also applied for compression failure. Thus, restrictors of failure strain isolation enable to eliminate two principal shortcomings of traditional finite element concrete model, i.e. the effect of size and orientation of finite element grid on the result.

The applied concrete model enables to consider such phenomena as:

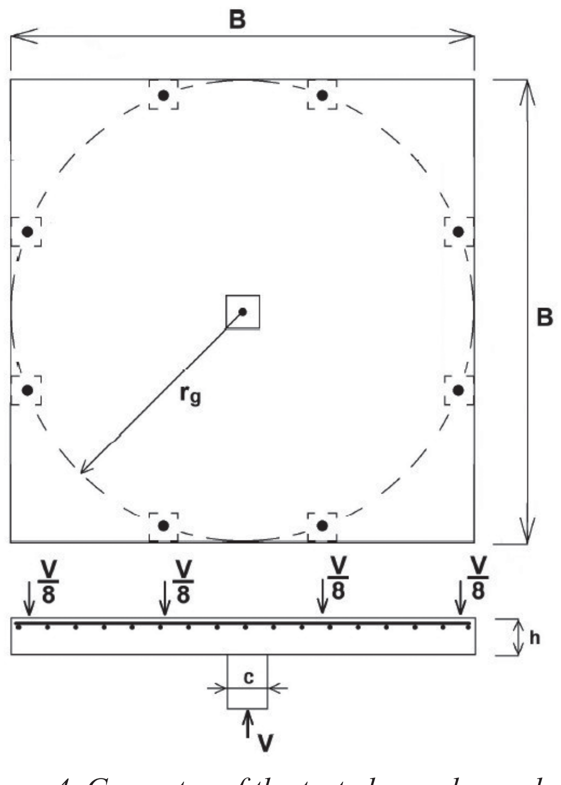


*Figure 4. Geometry of the tested samples under the [29].*

– non-linear behavior of a material exposed to tension and compression;
– crack formation and crack opening/closing;
– lowering of compressive and shear strength occurred after cracking in either direction;
– enhancement of concrete strength properties due to two-three-side squeeze reduction;
– mesh of crack edges in shear;
– grinding at the high degree of squeeze reduction;
– crack closing occurred due to material crushing in other directions.

Applying smeared crack model as an integral part of the above-mentioned constitutive model of concrete behavior allows accurate computation and visualization of discrete cracks propagation. Moreover, as claimed by the software developers [28], the present model compares favorably in accuracy with the models realizing discrete cracks.

***Verification of the adopted tool of numerical study (CS ATENA).***

Verification of the research tool (CS ATENA) has been implemented by the method of comparative analysis of the results obtained by finite element calculation of CS ATENA models. The models correspond to the published results of the physical experiments carried out in the University of Lausanne [29]. The tests were done in the framework of the study of reinforced concrete slab-to column joint exposed to punching failure at the low factor of longitudinal reinforcement in slabs (Figure 2). For all patterns, the samples were reinforced only by longitudinal reinforcement. The samples' characteristics are demonstrated in the table 1.

Additional parameters for concrete, that are required for numerical calculations, have been obtained on the basis of experimental compression strength by means of the equation from MC 2010 and factoring in given experimental parameters.

By analogy with [18], there have been done test calculations of the models corresponding to [29]. The correlation results of verification computation ($V_{sim,\,\mathrm{ATENA}}$) and the data of physical experiments ($V_{ex}$) are given in the table 2. In addition, for reference only, the table 2 gives the computation results under the codes and regulations [23, 24, 25].

The analysis of numerical studies of bearing capacity of concrete slab with the reinforced tensile zone under the criterion for punching failure implemented by CS ATENA exhibits good correlation with the results of physical tests, i.e. average deviation from the physical test data figures up to 7%. Thus, verification of CS ATENA for the purpose of further investigation is fulfilled.

## RESEARCH RESULTS

***Description of the model of reinforced concrete slab for numerical studies.***

Numerical studies have been implemented with the purpose to assess to impact of longitudinal reinforcement of the tensile zone for reinforced concrete slab on the value of bearing capacity under the criterion for punching failure. In order to carry out investigation for the punching failure scenario, design diagram of slab-to-column joint has been developed.

*Table 1*

| Sample | *B, m* | *c, m* | *h, m* | $h_0$, *m* | $r_g$, *m* | Parameters of longitudinal reinforcement of tensile zone: diameter (mm)/spacing (mm) | Model's parameters | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | ρ, % | $f_c$, *MPa* | $R_{bt}$, *MPa* | $d_g$, *mm* | $f_y$, *MPa* |
| PG1 | 3 | 0.26 | 0.25 | 0.21 | 1.5 | Ø20 / 100 | 1.5 | 27.6 | 2.4 | 16 | 573 |
| PG2 | 3 | 0.26 | 0.25 | 0.21 | 1.5 | Ø10 / 150 | 0.25 | 40.5 | 3.2 | 16 | 552 |
| PG3 | 6 | 0.52 | 0.5 | 0.456 | 2.85 | Ø16 / 135 | 0.33 | 32.4 | 2.7 | 16 | 520 |
| PG4 | 3 | 0.26 | 0.25 | 0.21 | 1.5 | Ø10 / 150 | 0.25 | 32.2 | 2.7 | 4 | 541 |
| PG5 | 3 | 0.26 | 0.25 | 0.21 | 1.5 | Ø10 / 115 | 0.33 | 29.3 | 2.5 | 4 | 555 |
| PG7 | 1.5 | 0.13 | 0.125 | 0.1 | 0.75 | Ø10 / 105 | 0.75 | 34.7 | 2.8 | 16 | 550 |
| PG8 | 1.5 | 0.13 | 0.125 | 0.117 | 0.75 | Ø8 / 155 | 0.28 | 34.7 | 2.8 | 16 | 525 |
| PG9 | 1.5 | 0.13 | 0.13 | 0.117 | 0.75 | Ø8 / 196 | 0.22 | 34.7 | 2.8 | 16 | 525 |
| PG10 | 3 | 0.26 | 0.25 | 0.21 | 1.5 | Ø10 / 115 | 0.33 | 28.5 | 2.5 | 16 | 577 |
| PG11 | 3 | 0.26 | 0.25 | 0.21 | 1.5 | Ø16/18 / 145 | 0.75 | 31.5 | 2.7 | 16 | 570 |

Notes: designators in the table 1 are accepted in compliance with [24, 25] – see the section 1.

*Table 2*

| N | Sample | $V_{ex}$, kN | $V_{sim,}$ ATENA, kN | $\frac{V_{ex} - V_{sim,ATENA}}{V_{ex}} \cdot 100\%$ | $V_{Rd,c}$ [25], kN | $V_{Rd,c}$ [24], kN | $F_{b,ult}$ [23], kN |
|---|---|---|---|---|---|---|---|
| 1 | PG1 | 1023 | 919.5 | 10.12 | 841 | 950 | 947.5 |
| 2 | PG2 | 440 | 434.2 | 1.32 | 420 | 594 | 1263.36 |
| 3 | PG3 | 2153 | 2209 | -2.60 | 1730 | 2340 | 4806.6 |
| 4 | PG4 | 408 | 419.9 | -2.92 | 344 | 550 | 1066 |
| 5 | PG5 | 550 | 551.7 | -0.31 | 455 | 583 | 987 |
| 6 | PG7 | 241 | 290.4 | -20.50 | 197 | 189 | 257.6 |
| 7 | PG8 | 140 | 146.2 | -4.43 | 137 | 178 | 323.7 |
| 8 | PG9 | 115 | 117 | -1.74 | 109 | 165 | 323.7 |
| 9 | PG10 | 540 | 571.1 | -5.76 | 454 | 577 | 987 |
| 10 | PG11 | 763 | 922.5 | -20.90 | 682 | 788 | 1066 |

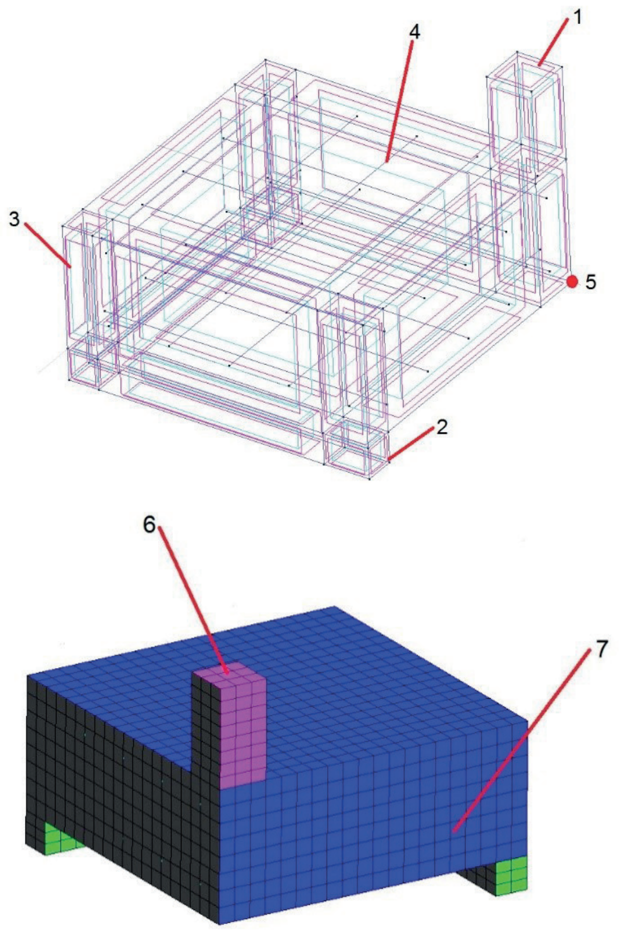


*Figure 5.* *3D (to the left) and finite model (to the right) model for numerical studies done by CS ATENA.*
*where: 1 – column; 2 – bearing support;*
*3 – floor slab; 4 – reinforcement rods;*
*5 – monitoring point for displacement along the vertical axis; 6 – point of load application;*
*7 – sample fixture along the symmetry axis.*

Design diagram represents ¼ of support zone of reinforced concrete slab (fig. 5). Longitudinal reinforcement of tensile and compressed zones of the slab has been simulated by means of rod finite elements. The diameter and spacing of rods have been adopted in a such way that sample fracture occurred due to punching load and not on the account of slab bending; or combined fracture took place. Slab load has been transferred via the column in geometrical center of the slab. There has been simulated sample supporting that occurred along the circuit on the rectangular distribution frame. The column material has been taken as elastic one, with concrete modulus of elasticity, to simplify design diagram. All the parameters that are required for describing concrete performance under MC 2010 [25] have been calculated on the basis of cube strength of concrete. Slab, column and support structure have been simulated by volume finite elements. For the design diagram, volume finite elements have taken shape of rectangular prisms with the dimensions 25x25x25 mm.

The main parameters of numerical studies are given in the table 4.

The results of numerical studies of bearing capacity for reinforced concrete slab factoring in longitudinal reinforcement in comparison with the results of calculation under the current codes and regulations [23, 24, 25] are outlined in the table 5 and at the figures 6–8.

The analysis of the results obtained by means of numerical studies enables to state the number of

*Table 4*

| Sample | $b$, m | $h$, m | $h_0$, m | Reinforcement, diameter (mm)/spacing (mm) | ρ, % | $f_c$, MPa | $R_{bt}$, MPa | $d_g$, m | $f_y$, MPa |
|---|---|---|---|---|---|---|---|---|---|
| P1 | 0.15 | 0.2 | 0.163 | Ø10 / 100 | 0.48 | 22.4 | 1.78 | 20 | 500 |
| P2 | 0.15 | 0.2 | 0.163 | Ø14 / 100 | 0.94 | 22.4 | 1.78 | 20 | 500 |
| P3 | 0.15 | 0.2 | 0.163 | Ø16 / 100 | 1.23 | 22.4 | 1.78 | 20 | 500 |
| P4 | 0.15 | 0.2 | 0.163 | Ø20 / 100 | 1.93 | 22.4 | 1.78 | 20 | 500 |

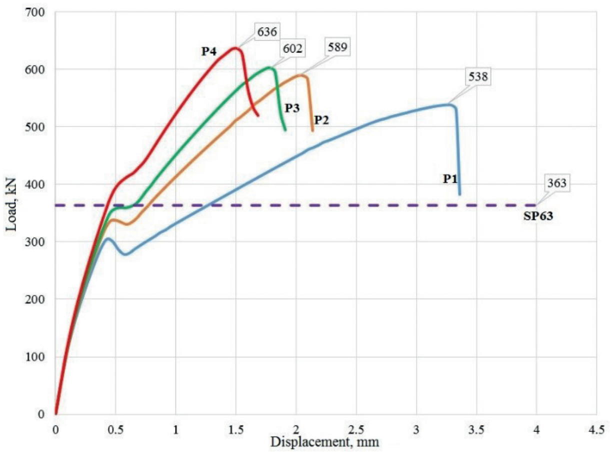


*Figure 6. Deformation curve for the calculation models and the values of ultimate loading (kN).*

new aspects in the pattern of formation of stress-strain state for concrete slab with longitudinal reinforcement of the tensile zone:

1. Longitudinal reinforcement of concrete tensile zone blocks formation, propagation and opening of cracks in the given zone. The main crack volume is formed in the tensile zone of concrete that gives grounds to determine a pattern on joint fracture under the punching failure mechanism predominately.
2. Stresses in the reinforcing rods of concrete tensile zone is lowered with the increase in reinforcing rod diameter. At the same time, the level of bearing capacity of the joint under the criterion for punching failure is augmenting. Thus, increase in the ratio of longitudinal reinforcement of concrete tensile zone ensures significant enhancement of reinforced concrete slab under the criterion for punching failure.

*Table 5*

| Sample | $V_{sim,}$ ATENA, kN | $F_{b,ult}$ [23], kN | $V_{Rd,c}$ [24], kN | $\frac{F_{b,ult} - V_{sim,ATENA}}{F_{b,ult}} \cdot 100\%$ | Stresses arising in reinforcement at fracture point under CS ATENA, MPa |
|---|---|---|---|---|---|
| P1 | 538 | 363 | 343 | -48.21 | 493 |
| P2 | 589 | 363 | 430 | -62.26 | 346 |
| P3 | 602 | 363 | 467 | -65.84 | 277 |
| P4 | 636 | 363 | 467 | -75.21 | 200 |

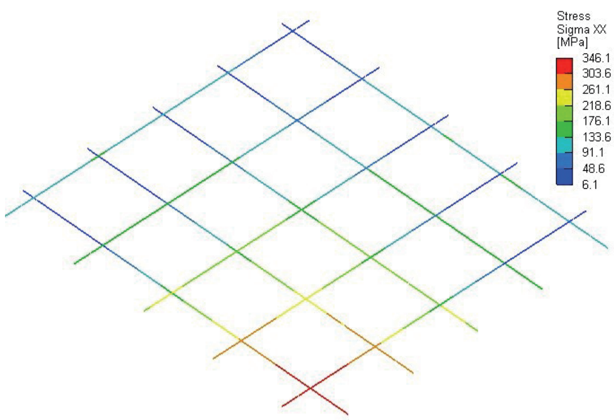


*Figure 7. Stresses in the reinforcement of concrete tensile zone at fracture point for the sample P2.*

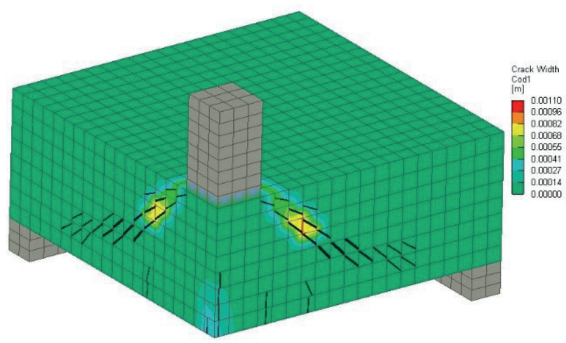


*Figure 8. Representative pattern of crack formation in the sample P2 at the fracture point.*

3. In domestic codes and regulations, the lack of clauses allowing for the factor of longitudinal reinforcement when determining bearing capacity of reinforced concrete slabs under the criterion for punching failure should be regarded as the main shortcoming. Thus, codes regulating analysis and design of reinforced concrete structures require further improvement.

## CONCLUSION

The implemented research demonstrates significant impact of longitudinal reinforcement of concrete tensile zone on the value of bearing capacity of the support joint under the criterion for punching failure that until the present time had not been evidenced in the current national codes. The results of investigation proved plausible proof that the issue of the pattern of stress-strain state formation in the support zone of reinforced concrete slabs had not been studied thoroughly. It is quite evident that the mechanisms defining the level of bearing capacity of the support zone have been studied at a lesser extent than the conditions affecting bearing capacity of reinforced concrete slabs under standard cross-section. The patterns of formation and developments of local damages of reinforced concrete slabs in the support zones should be thoroughly studied under two mechanisms – bending and punching failure. Setting standards for bearing capacity of the support zone of reinforced concrete structures requires further improvement.

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# CALCULATION MODEL OF A COMPLEX-STRESSED REINFORCED CONCRETE ELEMENT UNDER TORSION WITH BENDING

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**Abstract**. The article presents the methodology and principles of creating calculation models for reinforced concrete structures operating in conditions of complex resistance. A block calculation model of reinforced concrete bar structures in torsion with bending is presented. This model consists of a support block formed by a spatial crack and a compressed zone of concrete closed on it and a second block formed by a vertical section running perpendicular to the longitudinal axis of a reinforced concrete element along the edge of the compressed zone closing the spatial spiral. Cases are considered when the torque effect has the greatest influence on the stress-strain state of structures. In this case, the following forces are taken into account as the calculated forces in the spatial section: normal and tangential forces in the concrete of the compressed zone; components of axial and shear forces in the reinforcement crossed by a spatial crack. A feature of the proposed calculation model is that it considers independently of each other the strength of an element in spatial sections passing along a spatial crack, and the strength of an element between spatial cracks. The spatial section is formed by a crack located on three sides of the element and a compressed zone located on the fourth side and closing the ends of the spiral crack. In this case, the compressed zone, depending on the ratio of the bending and torque moments, can be located along the horizontal and vertical (lateral) edges of the element. The governing equations are written in the form of static equations for the adopted calculation cross-sections and a closed-loop system that unites them, written as a function of many variables with Lagrange multipliers $\lambda_i$. On the basis of the constructed function for all the variables included in it, an additional non-decaying system of equations has been compiled, from which follows a dependence that allows finding the projection of a dangerous spatial crack.

**Keywords**: reinforced concrete, calculation model, torsion with bending, spatial crack, complex stress state.

# РАСЧЕТНАЯ МОДЕЛЬ СЛОЖНОНАПРЯЖЕННОГО ЖЕЛЕЗОБЕТОННОГО ЭЛЕМЕНТА ПРИ КРУЧЕНИИ С ИЗГИБОМ

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**Аннотация**. Изложена методология и принципы создания расчетных моделей, железобетонных конструкций, работающих в условиях сложного сопротивления. Приведена блочная расчетная модель железобетонных стержневых конструкций при кручении с изгибом, состоящая из приопорного блока образованного пространственной трещиной и замыкаемой на нее сжатой зоной бетона и второго блока, образуемого вертикальным сечением, проходящим перпендикулярно к продольной оси железобетонного элемента по краю сжатой зоны, замыкающей пространственную спиралеобразную трещину. Рассмотрены случаи, когда наибольшее влияние на напряженно-деформированное состояние конструкций оказывает действие крутящего момента. При этом в качестве расчетных усилий в пространственном сечении учитываются: нормальные и касательные усилия в бетоне сжатой зоны; составляющие осевых и нагельных усилий в рабочей арматуре, пересекаемой пространственной трещиной. Особенностью предлагаемой расчетной модели является то, что в ней рассматриваются независимо друг от друга прочность элемента по пространственным сечениям, проходящим по пространственной трещине, и прочность элемента между пространственными трещинами. Пространственное сечение образуется трещиной, располагающейся по трем сторонам элемента, и сжатой зоной, располагающейся по четвертой стороне и замыкающей концы спиральной трещины. При этом сжатая зона, в зависимости от соотношения изгибающего и крутящего моментов, может располагаться по горизонтальным и вертикальным (боковым) граням элемента. Разрешающие уравнения составлены в виде уравнений статики для принятых расчетных сечений и объединяющей их замкнутой системы, записанной в виде функции многих переменных с множителями Лагранжа $\lambda_i$. На основе построенной функции по всем входящим в нее переменным, составлена дополнительная не распадающихся система уравнений, из которой следует зависимость, позволяющая находить проекции опасной пространственной трещины.

**Ключевые слова**: железобетон, расчетная модель, кручение с изгибом, пространственная трещина, сложное напряженное состояние.

## INTRODUCTION

There is a whole class of reinforced concrete structures (central load-bearing core of high-rise buildings, beams for various purposes with side consoles, inclined arches, L-shaped frames of various types, etc.), in which the lack of consideration of torsion in the calculations can lead to a decrease in their reliability, and in extreme cases, and to the collapse of individual elements of structures and even the entire object. Over the past few years, a number of domestic and foreign studies have been carried out on this topic devoted to the development and experimental verification of new generation calculation models for certain types of complex resistance, which most fully reflect the features of their deformation, cracking and failure with complex resistance, including torsion with bending [1–10, 11–15]. However, the solution of the problem of complex resistance of reinforced concrete, despite these and other works of the considered direction, is still fragmentary, and most importantly, it does not have a reliable experimental justification.

Therefore, a critical analysis, development and substantiation of general calculation models of complex resistance of reinforced concrete, as close as possible to the real work of reinforced concrete structures under a complex stress state, is an urgent task. This is especially true of structures made of high-strength reinforced concrete and fiber-reinforced concrete, the specificity of deformation of which under the considered stress state has significant features, and their calculation using traditional models of reinforced concrete does not reflect the stress-strain state of structures made of such materials.

Analysis of domestic and foreign models of deformation of reinforced concrete with cracks most often used in modern software systems shows that the calculation of reinforced concrete structures for the limit states of the second group is performed most often using the criteria for achieving the principal stresses or principal deformations of concrete of their limit values [3–9]. However, an analysis of experimental studies of the last two to three decades [17–21] shows that such criteria reflect the appearance of only dispersed (regular) cracks in reinforced concrete structures. Completely different criteria are needed when modeling the appearance and development of single discrete cracks in reinforced concrete. Here, as shown by experimental studies [1–4, 17, 18, 20], the main role is played by the concentration of deformations in places of change in geometric dimensions, in zones of concentration of force and deformational actions, zones of cross-media concentration [16, 22], cracking in structures made of high-strength reinforced concrete and fiber-reinforced concrete, etc.

An important issue in the practice of research and calculation of reinforced concrete structures in modeling the stress state remains the issue of the correctness of the accepted calculation models as close as possible to the real work of reinforced concrete structures [23]. Problems related to the fundamental physical foundations of reinforced concrete deformation and their experimental substantiation, deformational diagrams of concrete and reinforcement in the zones of single cracks and, accordingly, the quantitative values of the width of the opening of such cracks cannot be determined on the basis of traditional deformational models of reinforced concrete built on imperfect hypotheses of the theory of reinforced concrete , in particular, hypotheses about the work of tensile concrete between cracks and above a crack are very conditional.

In connection with the above, some possible directions of development of the modern theory of reinforced concrete are formulated.

– methodology and basic principles for creating of calculation models, reinforced concrete structures operating in conditions of complex resistance and a deformational model of a complex stressed state of reinforced concrete in torsion with bending, as well as its experimental justification. At the same time, the following directions in solving the formulated research goal are considered:

– peculiarities of solving problems of the theory strength, crack resistance and stiffness of reinforced concrete under complex stress state;

– new patterns of crack resistance and deformability of reinforced concrete structures;

– proposals for the construction of physical and calculation models of the resistance of reinforced concrete of a new generation and the modernization of existing hypotheses of the theory of reinforced concrete;

– construction of a calculation model of reinforced concrete deformation with complex resistance – bending with torsion;

– experimental substantiation of the deformational model of reinforced concrete with complex resistance.

– features of solving problems of the theory of strength, crack resistance and stiffness of reinforced concrete in a complex stressed state.

Conducted in the last decade, experimental and theoretical studies of complexly stressed reinforced concrete structures, for example [7, 11–13, 21], including the authors of this publication [1–4, 8–10], show that the criteria used in the calculation models of the criteria for achieving the principal stresses or principal deformations of concrete of their limiting values reflect the appearance and development of only a network of regular cracks in reinforced concrete, for which they were actually formulated.

When solving such problems, as again the accumulated experience of experimental studies on a new basis [1–4, 13, 17–21] shows, the concentration of deformations at the points of change in geometric dimensions in the zones of concentration of force and deformational actions, in the zones of the so-called cross-media concentration in two-layer composite and monolithic structures, in structures made of high-strength concrete and in other cases where the formation of discrete cracks is possible. It is especially important to take into account such a concentration for high-strength reinforced concrete and fiber-reinforced concrete, when the formation of single cracks and the deformational effect most significantly affect the relative mutual displacements of concrete and reinforcement, the assessment of the crack opening width and the stiffness of the structure. Methods for modeling this type of cracks have not yet been developed.

The general methodology for solving the problems of crack resistance, stiffness and strength of complexly stressed reinforced concrete for the limiting states of the first and second groups in this formulation can be presented in the form of a level diagram (Figure 1).

In accordance with this scheme, the solution to the problem of complex resistance of reinforced concrete is built on a unified methodological basis for the limiting states of the first and second groups. Contradictory in the theory of reinforced concrete is the statement about the work of tensile concrete between the crack and above the crack. Many effects associated with this phenomenon require clarification of their physical nature. There is a need to harmonize the experimental data obtained at one time using a microscope, and in recent years, in connection with the rapid development of measurement technologies and instrumental base, digital video cameras, which provide fixation of development and opening in parts of a millisecond with an accuracy of 0.001 mm, up to 0.001 mm, and their theoretical values calculated according to the theory of the existing calculation apparatus for reinforced concrete.

Moreover, when solving the problems of stiffness of reinforced concrete elements with dispersed cracks, the traditional model of V.I. Murashev in one or another of its modifications. Meanwhile, as shown in [8, 17, 18, 22], the distance between the cracks with increasing load remains constant only

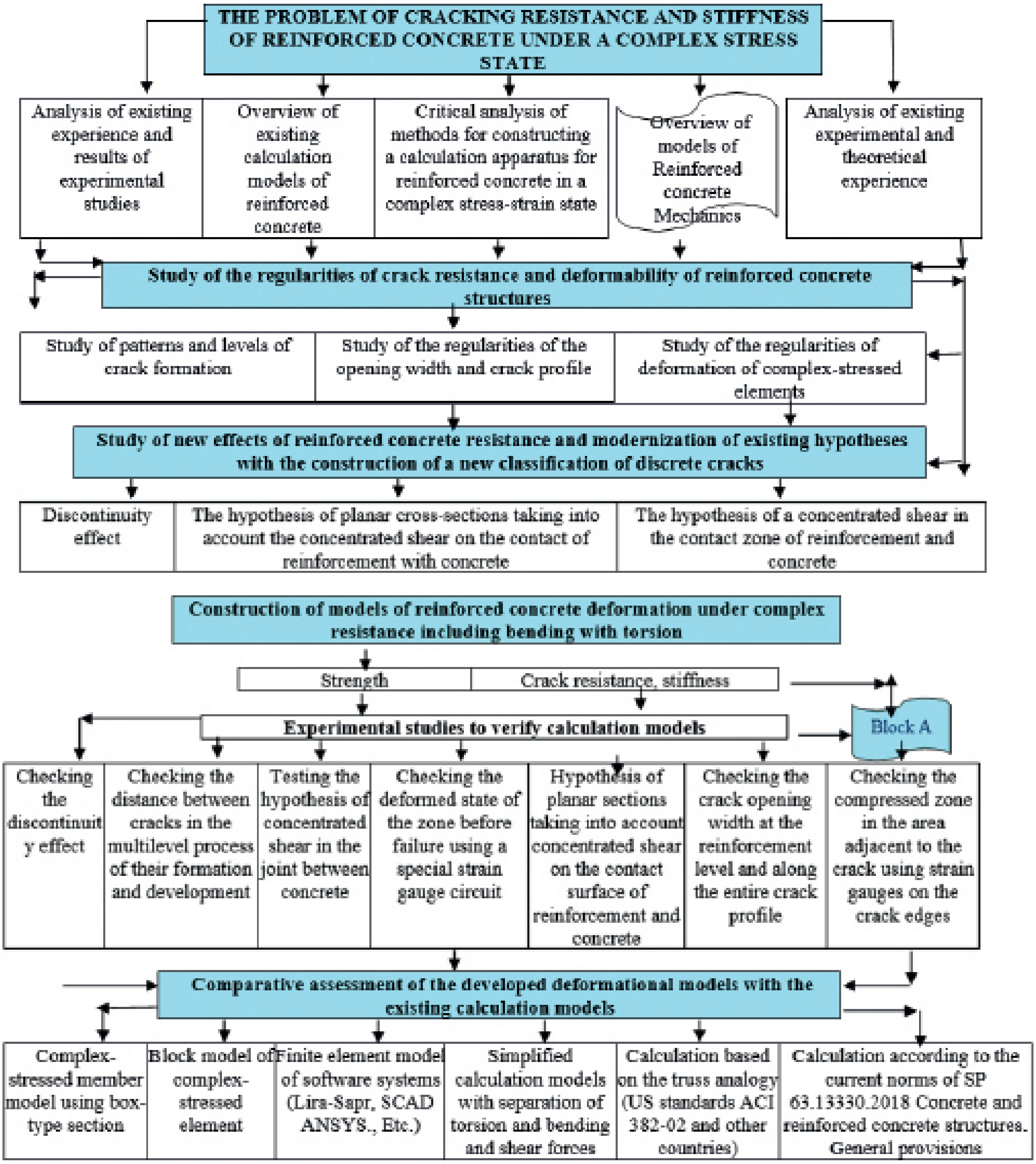



*Figure 1. General level scheme for solving the problem of stiffness, crack resistance and strength of reinforced concrete for limiting states*

until the appearance of a new crack. Analysis of the deformation diagram of tensile concrete, together with the nature of the surface warping, confirmed by extensive experimental material, shows that the nature of the deformation diagram of reinforcement changes qualitatively with increasing load (Figure 2.).

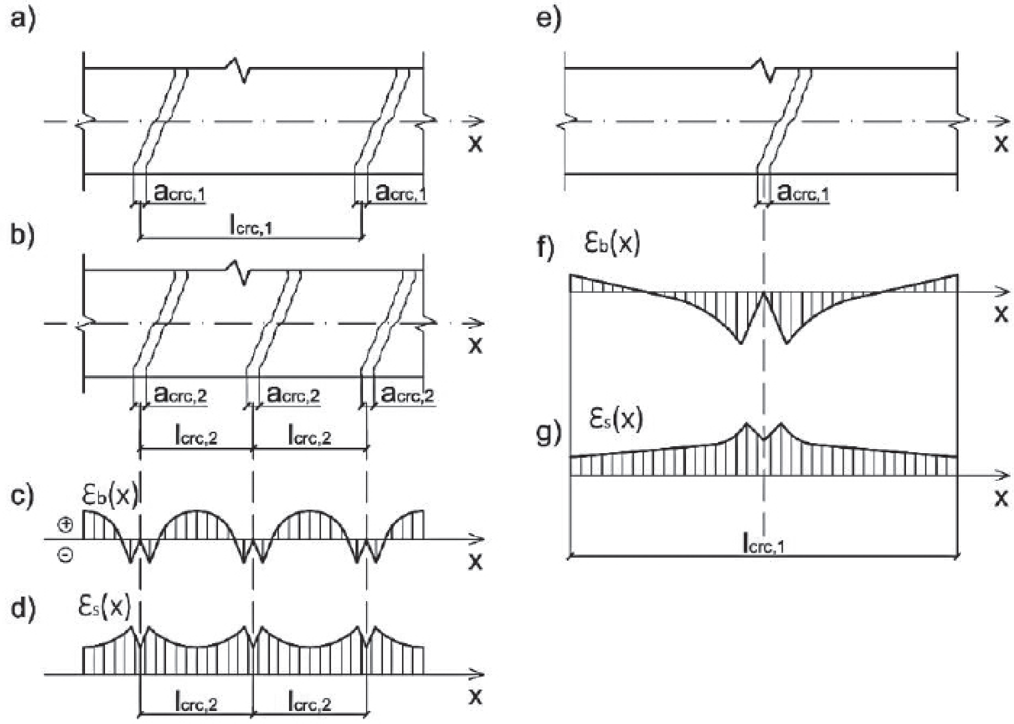



*Figure 2: Level picture of cracking in a complexly stressed reinforced concrete element made of usual (a-d) and high-strength (e-g) reinforced concrete*

In areas adjacent to cracks, reinforcement deformations begin to decrease and even change sign. Deformations in the middle of the section increase. It is not always correct to use software systems in studies of structures made of high-strength reinforced concrete, in which traditional physical models built for usual reinforced concrete are used to assess the limiting states of the second group, with averaging deformations for concrete and reinforcement. Meanwhile, the experimental studies of structures made of high-strength reinforced concrete and cracks carried out in recent years show the need to take into account the deformation effect in the crack [6, 16, 22]. It is especially important to take into account such an effect for single cracks, when the deformational effect most significantly affects the relative mutual displacements of concrete and reinforcement, the assessment of the crack opening width and the stiffness of the structure. Experimental studies of structures made of fiber-reinforced concrete carried out in recent years, for example [1–4], have shown that in structures made of such materials, a qualitatively different nature of deformation and cracking is observed. In these tests, it was found that the pattern of cracks in a reinforced concrete beam made of usual and high-strength reinforced concrete is qualitatively different. In a usual concrete structure, a whole network of cracks forms in the tension zone. Moreover, as the load increases, new cracks ($a_{crc,2}$) are added to the already formed cracks at the first stage of cracking (cracks $a_{crc,1}$) at higher stages of loading and, accordingly, the level distances between the cracks $l_{crc,1}$, $l_{crc,2}$ change (see Figure 2.).

In structures made of high-strength concrete, as a rule, single cracks are formed. The nature of the deformation diagrams of concrete and reinforcement in the zone of such cracks and, accordingly, the quantitative values of the width of the opening of such cracks cannot be determined on the basis of the traditional deformation model of reinforced concrete deformation.

The solution of the indicated problems of crack resistance and stiffness of reinforced concrete under a complex stress state at the first level is undoubtedly associated with a deep critical analysis of the existing calculation models of the theory of reinforced concrete, experience and results of experimental studies, including those carried out using a laboratory base of a new generation (see Figure 2 ).

At the second and third levels, it is required to study the physical laws of the deformability and crack resistance of reinforced concrete elements under various types of stress state and to study new effects of reinforced concrete resistance, including the formation of discrete cracks in high-strength reinforced concrete.

The next level in the hierarchy of solving the considered problem is the construction of calculation models of a new generation that most fully reflect physical phenomena under the force and medium resistance of reinforced concrete, including under special actions, and their experimental verification using more advanced equipment and measuring instruments for the investigated parameters.

The most important final level for assessment the effectiveness of solving the considered problems and the reliability of the obtained solutions is a comparative assessment of the developed deformation models of the theory of complex resistance of reinforced concrete with the existing calculation models for the limiting states of the first and second groups.

## IMPLEMENTATION OF THE GENERAL SCHEME FOR BENDING WITH TORSION

(experiment comparison with norms).

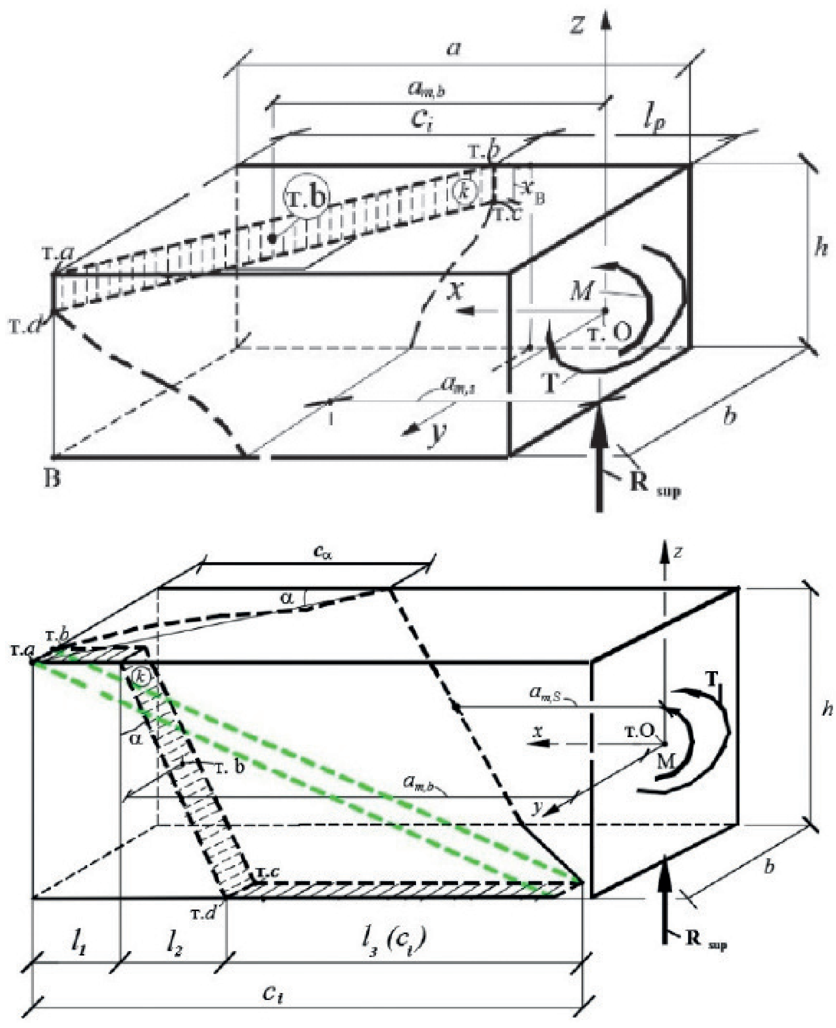
Using the methodology of the presented general level scheme for solving the problem of stiffness, crack resistance and strength of reinforced concrete, the construction of a calculation model of a complexly stressed reinforced concrete element undergoing torsion with bending is considered.

The proposed calculation model for analyzing the stress-strain state of reinforced concrete elements with complex resistance – torsion with bending is based on the accepted in domestic practice in the works of A.A. Gvozdev [24], N.I. Karpenko [5,6], Vl.I. Kolchunov [8–10], Fedorov V.S. [25], V.I. Morozov [7] and other scientists approach, which is that the strength of an element in spatial sections passing through a spatial crack and the strength of an element between spatial cracks are considered independently of each other. The spatial section is formed by a crack located on three sides of the element and a compressed zone located on the fourth side and closing the ends of the spiral crack. In this case, the compressed zone, depending on the ratio of the bending and torques, can be located along the horizontal and vertical (lateral) edges of the element. The calculation is based on the equilibrium of the moments of external and internal forces in a spatial section. In this case, in the spatial section, the forces in the longitudinal and transverse reinforcement are taken into account, crossing the spatial section at the face opposite to the compressed zone. The forces in the longitudinal and transverse reinforcement are entered into the calculation with their calculation resistances at a certain ratio established from the analysis of experimental data.

The compilation of equations requires some explanation. The upper, lower and lateral longitudinal reinforcement (in the presence of multi-tiered reinforcement), in Figure 3, a (Scheme A) and Figure 3, a (Scheme B) are conventionally not shown in order to eliminate the bulkiness of the image. In conditions of equilibrium, the stresses arising in the noted reinforcement are taken into account. The only exception is the equation of equilibrium of the moments of internal and external forces acting in section I–I relative to the axis perpendicular to this section and passing through the point of application of the resultant forces in the compressed zone ($T_{b,I} = 0$).

When considering the normal section I – I (III – III) and the spatial section k, the following are taken into account: the limiting support reaction $R_{sup}$; the height of the compressed zone of concrete in a normal section $x_{B,1}$; coefficient for determining the lateral force $\gamma_{Q,t}$; stress of longitudinal reinforcement in normal section $\sigma_{s,I}$; stress in the reinforcement at the side faces of the section of the structure in the spatial section $\sigma_{s,sid,k}$; the height of the compressed zone of the spatial section $x_{B,k}$; linear force in transverse reinforcement located at the side, top and bottom edges $q_{sw,\sigma,up}$, $q_{sw,\sigma}$, $q_{sw,lef}$; normal stresses in concrete $\sigma_{bu,x,I}$; components of axial stresses in the working reinforcement crossed by a spatial crack $\sigma_{s,up}, \sigma_{sc,up}, \sigma_{s,d}, \sigma_{sc,d}$; shear forces in concrete with the completeness coefficients of their diagrams $\omega_1\tau_{yz}$, $\omega_2\tau_z$; components of the thrust forces in the working reinforcement crossed by the spatial crack $k$, as well as the lengths of the projections of the spatial crack sections onto the horizontal axis $l_1$, $l_2$, $l_3$ ($c_i$).

In the spatial section k-k for block 2, cut off by a complex section, passing along a spiral spatial crack and along a broken section of the compressed zone, all reinforcement falling into this section is taken into account. In this case, in the compressed upper longitudinal reinforcement, cut off by sections I – I and III – III, the nagel effect is not taken into account, and in the rest of the longitudinal and transverse reinforcement, the components of the nagel effect are taken into account.

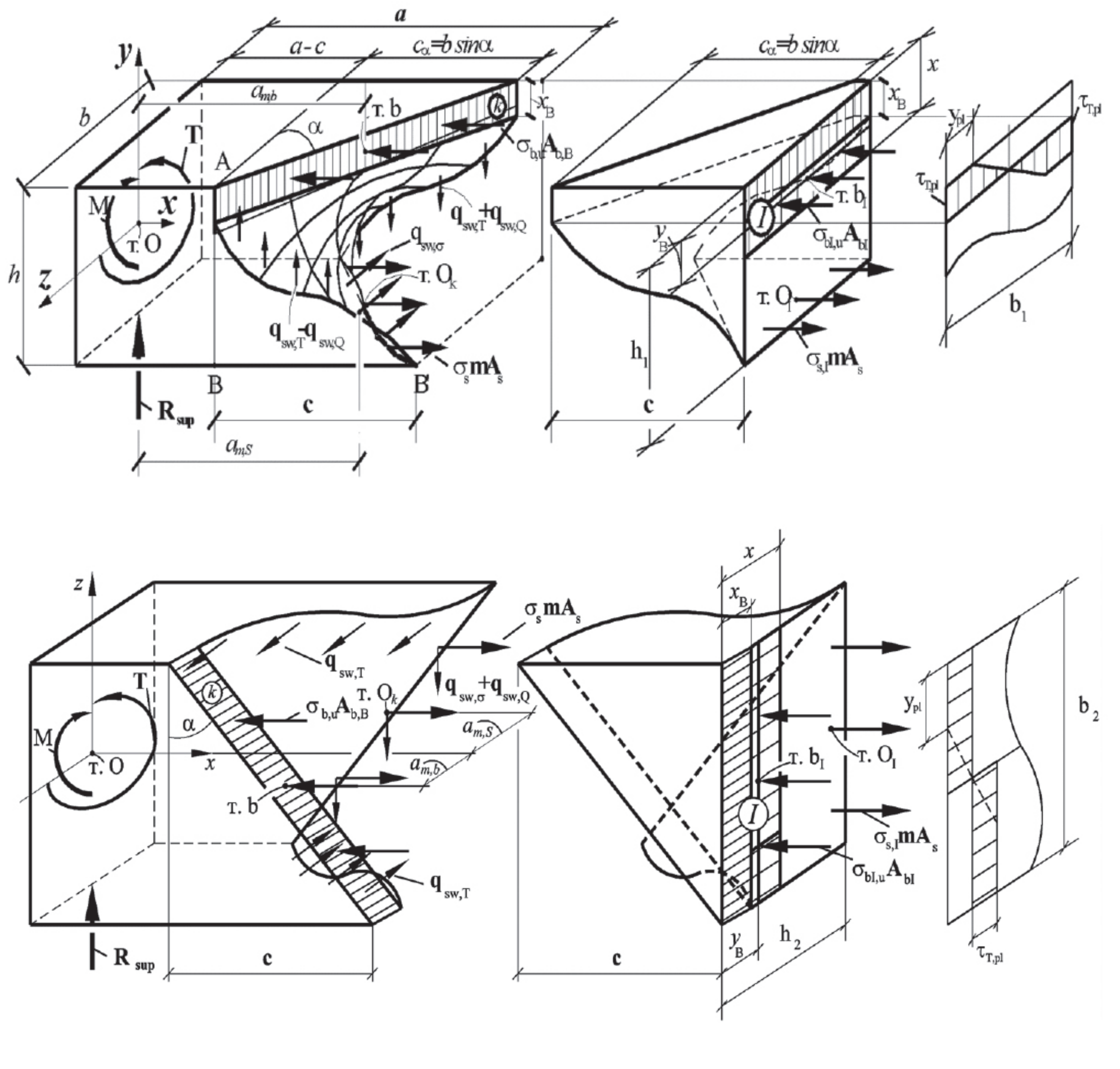



The need to use a complex broken section of the compressed zone of concrete is due to the fact that, according to experimental data, its destruction occurs in a certain volume, located not along the entire length between points $a$ and $b$ (see Figure 3.1), but only in a certain volume located in the middle part. In this case, the failure occurs in the middle part not along the line $ab$, but at an angle close to 45° to the upper edge of the reinforced concrete structure, which predetermined the direction of the middle part of the broken section, where the ultimate stress-strain state is reached.

In the areas of the compressed zone located at the edges of the broken section, the stress-strain state varies from sections I – I and III – III to the middle zone according to linear dependences, respectively. In this case, it is assumed that the height of the compressed zone decreases with an increase in the bending moment.

The lateral surfaces of the broken section in compressed concrete (see Figure 3 b) coincide with the planes of the axis or "smeared" plane of the working longitudinal reinforcement.

In this case, angular reinforcement when crossing by a broken section is considered to be located on the left for section I – I and on the right for section III – III. Thus, it is intersected by planes I – I, III – III, respectively, at the end sections of a complex broken section (see Figure 3 - scheme A and Figure 4 - scheme B). Diagrams of the distribution of forces from torques in the compressed and extended zones in the middle section I – I using the example of case B are shown in Figure 4.



The general solution of the considered problem is constructed as follows. The statics equations are written for the calculation sections I – I and the spatial section k-k adopted in the block model. In this case, when projecting the stress components in the k plane onto the I – I plane, a parameter $\varphi_{10,*}$ is introduced that takes into account such a projection of the stress components.

Also introduced: a parameter $K_M$ that takes into account the static loading scheme from the standpoint of additional bending moments along the length of the bar; parameter $K_{pr,M}$ that takes into account the relationship between the generalized support reaction $R_{sup}$ and the bending moment $M$ and a static-geometric parameter $\varphi_y(x_b, x)$ = const that takes into account the location of the center of gravity of the compressed zone of concrete in section I–I, where the compressive stress diagram is rectangular in the area $x_B$ and triangular in the area $x - x_B$).

As a result, from the equilibrium equation of the projections of all forces acting in section I–I on the $x$ axis, the height of the compressed concrete zone $x$ in this section ($\sum X = 0$) is determined, from the equation of the sum of torques relative to the point of application of the resultant forces in compressed concrete $b_I$ ($\sum T_{b,I} = 0$) we obtain a value $x \cdot \lambda_{II}$ characterizing the height of the compressed zone, where the shear stresses from the torque reach the yield point, and from the hypothesis of planar cross-sections, the average value of stresses in the reinforcement $\sigma_{s,m,I}$ is found.

Equations are written in a similar way for the spatial section *k-k* formed by a spiral-shaped crack and a vertical section passing through the compressed zone of concrete through the end of the front of the spatial crack (see Figure 4).

The missing unknowns of the considered calculation model are determined by compiling a function of many variables with Lagrange multipliers $\lambda_i$ for mechanical systems of the form $F(R_{sup}, x, x_B, \gamma_{Q,t}, \gamma_{T,k}, \sigma_{s,I}, \sigma_{S,k}, q_{sw,rig}, q_{sw,lef}, q_{sw,\sigma}, c, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8, \lambda_9)$: and equating to zero its partial derivatives with respect to all variables included in it. As a result, an additional system of equations is obtained for determining the unknowns.

## EXPERIMENTAL SUBSTANTIATION OF THE PROPOSED CALCULATION MODEL

of the strength of a complex stressed reinforced concrete element in torsion with bending was carried out by conducting a series of experimental studies of reinforced concrete structures. Thus, in [1], it was found that for reinforced concrete structures of square section made of high-strength concrete of class B100 during torsion with bending, as a rule, there is only one main crack (the fragile structure of high-strength concrete contributes to such concentration), along which failure occurs. For the box-shaped section, several cracks took place, from which the one along which the failure occurs. At the steps preceding failure, this crack begins to prevail over the rest and has a maximum opening width. Such features of cracking are fully consistent with the hypotheses underlying the calculation model for high-strength concrete.

Based on experimental studies of reinforced concrete structures made of high-strength concrete [2,4] and fiber-reinforced concrete [3] of circular cross-section, data on the complex stress-strain state in the studied regions of resistance were obtained, such as: values of the generalized load of cracking, and failure, its level relative to the ultimate load ; the distance between cracks at different levels of cracking (two or three levels are usually formed before the failure occurs); width of crack opening at the level of the axis of the working reinforcement, at a distance of two diameters from the axes of the reinforcement and along the entire profile of the crack at various stages of loading; coordinates of points ($x; y; z$) of formation of spatial cracks; schemes of the formation, development and opening of cracks in reinforced concrete structures during torsion with bending, which confirmed their satisfactory agreement with the results of the calculation according to the considered calculation model.

## CONCLUSIONS

1. A general calculation model of the complex resistance of reinforced concrete structures in

torsion with bending is proposed. The model includes a support block formed by a spatial crack and a compressed zone of concrete closed on it, – a spatial section *k* and a second block formed by a vertical section I–I, passing perpendicular to the longitudinal axis of the reinforced concrete element along the edge of the compressed zone closing the spatial crack. In this case, option A of the calculation model A is used for a spiral-shaped spatial crack, option B – for a spatial X-shaped crack.

2. In the consideration of the general model, the first and second cases are accepted, when of the three external influences during torsion with bending (*Q, M, T*), the torque T exerts the greatest influence on the stress-strain state of the structure.

3. The resolving equations of the proposed block model are composed in the form of static equations for the adopted sections and a closed-loop system that unites them, written in the form of a function of many variables with Lagrange multipliers $\lambda_i$. On the basis of the constructed function for all the variables included in it, an additional system of equations has been compiled, from which a dependence follows, which makes it possible to find the projections of a dangerous spatial crack.

4. The proposed calculation model was experimentally confirmed by tests of a fairly representative group of reinforced concrete structures made of ordinary and high-strength concrete and fiber-reinforced concrete, carried out with the participation of the authors and other researchers.

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# UNSYMMETRIC OSCILLATIONS OF ANISOTROPIC PLATE HAVING AN ADDITIONAL MASS

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**Abstract**: The problem of unsymmetric oscillations of circular plate made from anisotropic material is examined. The plate under consideration has an additional point mass attached off the center or a system of additional masses. Also the oscillations of anisotropic circular plate with a point support placed off the center are studied. The exact analytical approach is used for the decision of the above-mentioned problems; the method of compensating loads is applied. For this aim the basic and the compensating solutions are received. The basic solution satisfies to the resolving differential equation of the problem under study. The compensating solution satisfies to the corresponding homogeneous equation and this solution amounting to the basic one also satisfies to the boundary conditions. The Nielsen's equation is used for the receiving of the exact solutions expressed in terms of Bessel functions. The equation for determination of frequencies of natural vibrations is obtained.

**Keywords**: oscillations, circular anisotropic plate, additional mass, point support, Bessel functions.

# НЕСИММЕТРИЧНЫЕ КОЛЕБАНИЯ АНИЗОТРОПНОЙ ПЛАСТИНЫ С ПРИСОЕДИНЕННОЙ МАССОЙ

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**Аннотация**: В работе изучаются несимметричные колебания круглой пластины, сделанной из анизотропного материала. Рассматриваемая пластина имеет дополнительную внецентренно расположенную массу или систему масс, или внецентренную опору точечного типа. Для получения решения поставленной задачи используется точный аналитический метод; используется метод компенсирующих нагрузок (МКН). Для этого строятся основное и компенсирующее решения. Основное решение удовлетворяет разрешающему дифференциальному уравнению. Компенсирующее решение удовлетворяет соответствующему однородному уравнению и совместно с основным решением удовлетворяет граничным условиям. Для получения решения однородного дифференциального уравнения, выраженного в цилиндрических функциях, применяется новый способ, связанный с использованием уравнения Нильсена. Для определения частот собственных колебаний получены соответствующие уравнения. Решения поставленных задач выражены в функциях Бесселя.

**Ключевые слова**: несимметричные колебания, круглые анизотропные пластины, присоединенные массы, точечные опоры, функции Бесселя.

## INTRODUCTION

The analytical method for the solution of unsymmetric vibrations of anisotropic circular plate having an additional point mass attached off the center or a system of masses, also having a point support placed off the center is proposed. The method of compensating loads (MCL) is applied. The sought solutions are obtained in closed form and expressed in terms of Bessel functions.

The problem of oscillations anisotropic or inhomogeneous plates of various geometry is very urgent. The analytical method application, in particular, connected with the theory of special function for plates and shells analysis, is known in literature. The works [1]–[8] are devoted to the plates of variable thickness and shells examination; the solutions are received in terms of hypergeometric and confluent functions. The works [9]–[13] concern to the problems of statics and vibrations of isotropic plate, many of them are obtained in terms of cylindrical functions. In [14], [15] the analytical methods are applied for computation of plates and shells made from orthotropic material. In the monograph [16] for plates of variable thickness of different forms, subjected to the action of complicated loads, the theory of special functions is widely used; the solutions are expressed in terms of hypergeometric and confluent functions, Legendre functions, in orthogonal polynomials.

The present work concerns to unsymmetric oscillations of anisotropic circular plates with additional point mass attached off the center or a system of masses, also a point support placed off the center. The analytical approach and the method of compensating loads (MCL) are used. The new method, suggested in [17], [18], was also utilized. The mentioned method allows to resolve some statics and vibration problems of anisotropic circular plates and to receive the solutions in terms of Bessel functions. This method uses the Nielsen's equation and makes it possible to resolve some statics and vibration problems of anisotropic elastic solids.

The works [20], [21] are devoted to the investigation of symmetric vibrations of circular plates made from orthotropic material. The work [20] received the solution of the problem of the similar plates forced vibrations, the plates under study are subjected to an action of concentrated force $P\sin pt$. In the work [21] the action of loads distributed along concentric circumferences and over ring surfaces are considered.

In [23] the problem of circular orthotropic plates natural oscillations is considered. The plates under study have a point support in the center or a ring support or they have an additional mass in the center or a system of masses.

## FORCED VIBRATIONS OF ORTHOTROPIC PLATE SUBJECTED TO AN ACTION OF A CONCENTRATED FORCE

First the auxiliary problem of unsymmetric vibrations of orthotropic plate subjected to an action of concentrated force $P\sin pt$, acting at the point $A$ off the center is considered. Let us denote the distance from the center to the point $A$ by $\alpha$. We introduce the notation of the angle between the line $OA$ and the fixed radius by $\theta$. Further the method of compensating loads (MCL) [18], [20], [21] is applied. For this aim the basic solution $W_0$ and the compensating solution $W_k$

$$W = W_0 + W_k \tag{1}$$

are determined.

The basic solution $W_0$ satisfies to the differential equation, describing the problem, and shows the peculiarities of the external loads. The compensating solution $W_k$ satisfies to the corresponding homogeneous equation and amounting to $W_0$ satisfies to the boundary conditions.

For the receiving of the solution of the resolving differential equation the Nielsen's equation is used. This method is described in detail in [20], [21]. Using the mentioned approach we can write the following expression for the basic solution in terms of Bessel functions:

$$W_0 = -\frac{P}{8Dn_2b^2}\left[Y_0(z)+\frac{2}{\pi}K_0(z)\right], \tag{2}$$

here $b=\sqrt[4]{\frac{\gamma h}{gn_2D}p_S^2}$,

where $D$ is the cylindrical rigidity, $\gamma$ – the volume weight, $p_S$ is the circular frequency of natural vibrations, $S$ is the member of nodal circles.

$$E_r=\frac{E}{n_1},\ E_\theta=En_2,\ \sigma_r=\frac{\sigma}{n^2},\ \sigma_\theta=\sigma,$$
$$n^2=n_1n_2,$$

here $z=\sqrt{\alpha^2+x^2-2\alpha x\cos(\theta-\varphi)}$.

Using the formulae of the Bessel functions addition [9], [21] we can write when $x \geq \alpha$:

$$W_0 = -\frac{P}{8Dn_2b^2}\times$$
$$\times 2\sum{}'\left[Y_n(x)J_n(\alpha)+\frac{2}{\pi}K_n(x)I_n(\alpha)\right]\cos n(\theta-\varphi). \tag{3}$$

The symbol $'$ indicates that when summing the series the term with zero index multiplies on $\frac{1}{2}$.

Let us write the compensating solution in the form of series:

$$W_k = \sum\left[A_nJ_n(x)+B_nI_n(x)\right]\cos n(\theta-\varphi), \tag{4}$$

where $J_n$, $I_n$, $Y_n$, $K_n$ are the Bessel functions.

The coefficients $A_n$ and $B_n$ are determined from the boundary conditions.

Let us assume that the contour of the plate under study is clamped; the reduced radius is equal to $\beta$. When $x=\beta$

$$W=0,\ \frac{\partial W}{\partial x}=0, \tag{5}$$

where $W$ is determined from the formula (1).

The sum of the basic and the compensating solutions introduces in the conditions (5). Then the coefficients at the cosines of the identical arguments are equated and we receive the following equations for determination of the coefficients $A_n$ and $B_n$:

$$-\frac{P}{4Dn_2b^2}\left[Y_n(\beta)J_n(\alpha)+\frac{2}{\pi}K_n(\beta)I_n(\alpha)\right]+$$
$$+A_nJ_n(\beta)+B_nI_n(\beta)=0, \tag{6}$$

$$-\frac{P}{4Dn_2b^2}\left[Y_n'(\beta)J_n(\alpha)+\frac{2}{\pi}K_n'(\beta)I_n(\alpha)\right]+$$
$$+A_nJ_n'(\beta)+B_nI_n'(\beta)=0. \tag{7}$$

Solving the system (6), (7) and using the expression for Bessel functions Wronskian [9], [23], we get:

$$A_n = \frac{P}{4Dn_2b^2}\,\frac{J_n(\alpha)\left[-Y_n(\beta)I_n'(\beta)+Y_n'(\beta)I_n(\beta)\right]\frac{2}{\pi\beta}I_n(\alpha)}{I_n(\beta)J_n'(\beta)-J_n(\beta)I_n'(\beta)}, \tag{8}$$

$$B_n = \frac{P}{4Dn_2b^2}\,\frac{\frac{2}{\pi}I_n(\alpha)\left[J_n'(\beta)K_n(\beta)-J_n(\beta)K_n'(\beta)\right]-\frac{2}{\pi\beta}J_n(\alpha)}{I_n(\beta)J_n'(\beta)-J_n(\beta)I_n'(\beta)}. \tag{9}$$

## UNSYMMETRIC VIBRATIONS OF ANISOTROPIC PLATE WITH ADDITIONAL JOINT MASS OR WITH SEVERAL MASSES

Further the problem of oscillations of anisotropic circular plate with additional mass $M$ in the point $(\alpha, 0)$ is considered. The equation for frequencies [22] has the following form:

$$M\omega^2W(\alpha,0;\alpha,0)=1. \tag{10}$$

Substituting instead $W$ the expression received above for the forced vibrations consideration, we can receive the corresponding equation for frequencies. Setting the right part of the received equation equal to zero we obtain the equation for frequencies for the case when the joint mass attached off the center is extremely great. This fact is identical for the case of the plate with single rigid support of the point type placed off the center. We mark that for the case of the zero term of the series representing basic and compensating solutions, we get the frequency equation for the plate having ring intermediate support.

The natural oscillations of the circular orthotropic plate with several additional masses ($n = 1,2,3,...$) applied in the points with the coordinates ($\alpha_n$, $\theta_n$) are examined. For this case we have the following equation for frequencies:

$$\begin{vmatrix} \omega^2 M_1 w(\alpha_1,\theta_1;\alpha_1,\theta_1;\omega)-1 & .... & \omega^2 M_l w(\alpha_1,\theta_1;\alpha_l,\theta_l;\omega) \\ \omega^2 M_1 w(\alpha_2,\theta_2;\alpha_1,\theta_1;\omega) & .... & \omega^2 M_l w(\alpha_2,\theta_2;\alpha_l,\theta_l;\omega) \\ .................................... & .... & .................................... \\ \omega^2 M_1 w(\alpha_l,\theta_l;\alpha_1,\theta_1;\omega) & .... & \omega^2 M_l w(\alpha_l,\theta_l;\alpha_l,\theta_l;\omega)-1 \end{vmatrix} = 0 . \tag{11}$$

For the study of the problem of oscillation of orthotropic circular plate with elastic supports of the point type the frequency equation has the form which is similar to (11). However, we must assume in the frequency equation that $M_n = -\frac{D_n}{\omega^2}$, where $D_n$ is the rigidity of the support.

When solving the vibration problem of the orthotropic circular plate with several additional masses uniformly distributed along a circumference the system under study will have the same number of axes of symmetry as the number of joint masses. In this case the frequency equation becomes more simple.

Let us consider the following computation example: the forced vibration of the circular orthotropic plate which contour is clamped and the radius is equal to $b$. This plate is loaded by the force $P\sin pt$ which applied at the distance $a = 0{,}5b$ from the center.

For computation it is assumed that the plate's radius $b = 3m$, the plate's thickness $h = 0{,}2m$, modulus of elasticity $E = 2 \cdot 10^6 t/m^2$, Poisson's ratio $\sigma = \frac{1}{6}$; we put $n_2 = 1$.

The calculations show that it is sufficient to retain four terms of the series.

The results of evaluation are given in the table 1.

*Table 1. The values of the deflections*

| Point ($x$; $\phi$) | 0; 0 | 0,2; 0 | 0; 0 | 2,0; 0 | 0,2; $\frac{\pi}{2}$ | 1,0; $\frac{\pi}{2}$ | 2,0; $\frac{\pi}{2}$ | 0,2; $\pi$ | 1,2; $\pi$ | 2,0; $\pi$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $w$ | 0,3071 | 0,3475 | 0,3895 | 0,0013 | 0,2916 | 0,1448 | 0,000 | 0,2641 | 0,0981 | 0,0003 |

The first line shows the coordinates of the points where the plate's deflections are determined. In this table the fixed value $\frac{P}{8Dn_2b^2}$ is omitted.

We note that the function $w$ reduces to infinity when

$$I_n(\beta)J_n'(\beta) - I_n'(\beta)J_n(\beta) = 0 . \tag{12}$$

In this case we have the phenomenon of resonance; the corresponding frequency of forced vibrations is

$$P = \sqrt{\frac{Dn_2 g}{\gamma h}} \frac{\beta_n^2}{b^2},$$

where $\beta_n$ are the roots of the equation (12).

## CONCLUSIONS

The exact analytical solutions of the problem of unsymmetric vibrations of the circular plate made from orthotropic material are obtained. An action of a force applied in arbitrary point, an action of a joint additional mass attached off the center or a system of masses, an influence of a point support, placed of the center are considered.

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# LOCALIZATION OF SOLUTION OF THE PROBLEM OF ISOTROPIC PLATE ANALYSIS WITH THE USE OF B-SPLINE DISCRETE-CONTINUAL FINITE ELEMENT METHOD

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**Abstract**: Localization of solution of the problem of isotropic plate analysis with the use of B-spline discrete-continual finite element method (specific version of wavelet-based discrete-continual finite element method) is under consideration in the distinctive paper. The original operational continual and discrete-continual formulations of the problem are given, some actual aspects of construction of normalized basis functions of a B-spline are considered, the corresponding local constructions for an arbitrary discrete-continual finite element are described, some information about the numerical implementation and an example of analysis are presented.

**Keywords**: localization, wavelet-based discrete-continual finite element method, B-spline discrete-continual finite element method, discrete-continual finite element method, finite element method, B-spline, numerical solution, isotropic plate, plate analysis

# ЛОКАЛИЗАЦИЯ РЕШЕНИЯ ЗАДАЧИ О ПОПЕРЕЧНОМ ИЗГИБЕ ИЗОТРОПНОЙ ПЛАСТИНЫ НА ОСНОВЕ ВЕЙВЛЕТ-РЕАЛИЗАЦИИ ДИСКРЕТНО-КОНТИНУАЛЬНОГО МЕТОДА КОНЕЧНЫХ ЭЛЕМЕНТОВ С ИСПОЛЬЗОВАНИЕМ В-СПЛАЙНОВ

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**Аннотация**: В настоящей статье рассматривается локализация решения задачи о поперечном изгибе изотропной пластины на основе вейвлет-реализации дискретно-континуального метода конечных элементов с использованием В-сплайнов. Приведены исходные операторные континуальная и дискретно-континуальная постановки задачи, рассмотрены некоторые актуальные вопросы построения нормализованных базисных функций В-сплайна, описаны соответствующие локальные построения для произвольного дискретно-континуального конечного элемента, представлены некоторые сведения о численной реализации и пример расчета.

**Ключевые слова**: локализация, вейвлет-реализация метода конечных элементов, дискретно-континуальный метод конечных элементов, метод конечных элементов, В-сплайны, численное решение, изотропная пластина, изгиб изотропной пластины

## INTRODUCTION

As we have already mentioned [1, 2], the B-spline in a given simple knot sequence can be constructed by employing piecewise polynomials between the knots and joining them together at the knots [1–3]. For instance, compared with commonly used Daubechies wavelets [4–8] B-spline wavelet on interval (BSWI) has explicit expressions, facilitating the calculation of coefficient integration and differentiation [1–3]. Besides, the multiresolution and localization properties of BSWI can also supply some superiority for engineering structural analysis [1–3]. The early applications of spline can be found in papers of H. Antes [9], J.G. Han [10, 11, 27], Y. Huang [10, 11], W.X. Ren [10, 11]. The spline wavelet finite element method was further developed in papers of D.P. Chen [28], X.F. Chen [12, 13, 15–18, 23, 24, 26], H.B. Dong [23], J.G. Han [25], Y.M. He [17], Z.H. He [18], Z.J. He [12, 13, 15–17, 23, 24, 26], Y. Huang [25, 27], Z.S. Jiang [22], B. Li [13, 15, 17, 23], M. Liang [19, 21], J.Q. Long [20], G. Ma [20], T. Matsumoto [20, 22], S.T. Mau [30], H.H. Miao [15], Q,M. Mo [18], T.H.H. Pian [28–30], K.Y. Qi [23], W.X. Ren [25, 27], K. Sumihara [29], P. Tong [30], Y.W. Wang [22], J.W. Xiang [12–14, 17–22], Z.B. Yang [15, 16, 24], X.W. Zhang [16, 24, 26], Y.H. Zhang [12], Y.T. Zhong [14].

As is known, generally the structural analysis normally require accurate computer-intensive calculations using numerical (discrete) methods. The field of application of discrete-continual finite element method (DCFFEM), proposed by A.B. Zolotov [33] and P.A. Akimov [31–33] comprises structures with regular (in particular, constant or piecewise constant) physical and geometrical parameters in some dimension (so-called "basic" direction (dimension)). Considering problems remain continual along "basic" direction while along other directions DCFEM presupposes finite element approximation. Solution of corresponding resultant multipoint boundary problems [34] for systems of ordinary differential equations with piecewise constant coefficients and immense number of unknowns is the most time-consuming stage of the computing, especially if we take into account the limitation in performance of personal computers, contemporary software and necessity to obtain correct semianalytical solution in a reasonable time.

High-accuracy solution at all points of the model is not required normally, it is necessary to find only the most accurate solution in some pre-known domains. Generally the choice of these domains is a priori data with respect to the structure being modelled. Designers usually choose domains with the so-called edge effect (with the risk of significant stresses that could potentially lead to the destruction of structures, etc.) and regions which are subject to specific operational requirements. It is obvious that the stress-strain state in such domains is of paramount importance. Specified factors along with the obvious needs of the designer or researcher to reduce computational costs by application of DCFEM cause considerable urgency of constructing of special algorithms for obtaining local solutions (in some domains known in advance) of boundary problems. Wavelet analysis provides effective and popular tool for such researches. Solution of the considering problem within multilevel wavelet analysis is represented as a composition of local and global components. Wavelet-based DCFEM is presented in papers of P.A. Akimov [35–42], M. Aslami [38–40], T.B. Kaytukov, M.L. Mozgaleva [35–42] and O.A. Negrozov [38–40].

The distinctive paper is devoted to numerical solution of the problem of isotropic plate analysis with the use of B-spline DCFEM.

## 1. FORMULATIONS OF THE PROBLEM

In accordance with [1] let the constancy of the parameters of the problem be in the direction corresponding to $x_2$ (main direction). The operational formulation of the problem with the use of so-called method of extended domain [43], taking into account the selection of the main direction, is determined by the equation:

$$L\,y = \widetilde{F},\ 0 \le x_1 \le \ell_1,\ 0 \le x_2 \le \ell_2 \qquad (1.1)$$

where we have

$$L = -L_4\,\partial_2^4 + L_2\,\partial_2^2 + L_0\,; \qquad (1.2)$$

$$L_4 = \theta D\,; \qquad (1.3)$$

$$L_2 = -[\partial_1^2\theta D\nu + 2\partial_1\theta D(1-\nu)\partial_1 + \theta D\nu\partial_1^2]\,; \qquad (1.4)$$

$$L_0 = -\partial_1^2\theta D\partial_1^2\,; \qquad (1.5)$$

$$\widetilde{F} = \theta F + \delta_\Gamma f\,; \qquad (1.6)$$

$$\theta(x_1,x_2) = \begin{cases} 1, & (x_1,x_2)\in\Omega \\ 0, & (x_1,x_2)\notin\Omega; \end{cases} \qquad (1.7)$$

$$\delta_\Gamma(x_1,x_2) = \partial\theta/\partial\overline{n}\,; \qquad (1.8)$$

$\Omega$ is the domain, occupied by plate;

$$\Omega = \{(x_1, x_2): 0 < x_1 < \ell_1; 0 < x_2 < \ell_2\}; \qquad (1.9)$$

$\ell_1$, $\ell_1$ are corresponding dimensions of extended domain (linear dimensions of plate); $x = (x_1, x_2)$; $x_1$, $x_2$ are Cartesian coordinates; $\theta(x_1, x_2)$ is characteristic function of domain $\Omega$; $\delta_\Gamma = \delta_\Gamma(x_1, x_2)$ is the delta function of boundary $\Gamma = \partial\Omega$; $\overline{n} = [n_1, n_2]^T$ is boundary normal vector; $y$ is deflection of plate; $D$ is flexural rigidity of plate; $\nu$ is Poisson's ratio of plate; $F$ is the load in domain $\Omega$; $f$ is the corresponding boundary load; $\partial_s = \partial/\partial x_s$, $s = 1,2$. Let us introduce the following notations

$$y_1 = y, \quad y_2 = \partial_2 y = y_1', \quad y_3 = \partial_2^2 y = y_2', \quad y_4 = \partial_2^3 y = y_3'. \qquad (1.10)$$

Thus we can rewrite (1.1):

$$-L_4 y_4' + L_2 y_3 + L_0 y_1 = \widetilde{F}. \qquad (1.11)$$

Finally we obtain system of differential equations with operational coefficients:

$$\begin{bmatrix} y_1' \\ y_2' \\ y_3' \\ y_4' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ L_4^{-1}L_0 & 0 & L_4^{-1}L_2 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -L_4^{-1}\widetilde{F} \end{bmatrix}, \qquad (1.12)$$

or

$$\overline{U}' = \widetilde{L}\,\overline{U} + \overline{\overline{F}}, \qquad (1.13)$$

where

$$\widetilde{L} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ L_4^{-1}L_0 & 0 & L_4^{-1}L_2 & 0 \end{bmatrix}; \qquad (1.14)$$

$$\overline{\overline{F}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -L_4^{-1}\widetilde{F} \end{bmatrix}; \quad \overline{U} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}. \qquad (1.15)$$

The system of equations (1.12) is supplemented by boundary conditions, which are set in sections with coordinates $x_2^1 = 0$ and $x_2^2 = \ell_2$.

## 2. SOME ASPECTS OF THE CONSTRUCTION OF NORMALIZED BASIS FUNCTIONS OF THE B-SPLINE

The construction of B-spline basic functions is determined by the recursive Cox-de Boer formulas [1]:

$$k=1: \quad \varphi_{i,1}(t) = \begin{cases} 1, & x_i \le t < x_{i+1} \\ 0, & t < x_i \vee t \ge x_{i+1} \end{cases}, \qquad (2.1)$$

$$k \ge 2: \quad \varphi_{i,k}(t) = \frac{(t-x_i)\varphi_{i,k-1}(t)}{x_{i+k-1}-x_i} + \frac{(x_{i+k}-t)\varphi_{i+1,k-1}(t)}{x_{i+k}-x_{i+1}}. \qquad (2.2)$$

We will consider such a construction for the case $x_i = i$ are integers. Let us note that,

$$\varphi_{i,k}(t) = \varphi_{0,k}(t-i)$$

and therefore, recursive formulas (2.1)–(2.2) can be represented in the form

$$k=1: \qquad \varphi_{0,1}(t)=\begin{cases}1\,, & 0\le t<1\\ 0\,, & t<0 \vee t\ge 1;\end{cases} \qquad (2.3)$$

$$k\ge 2: \qquad \varphi_{0,k}(t)=\frac{1}{k-1}[t\cdot\varphi_{0,k-1}(t)+ \\ +(k-t)\varphi_{0,k-1}(t-1)]. \qquad (2.4)$$

The function $\varphi_{0,1}$(t) can be represented by formula

$$\varphi_{0,1}(t)=\frac{1}{2}[\operatorname{sign}(t)-\operatorname{sign}(t-1)]\,. \qquad (2.5)$$

Let us denote by $\Delta_1$ the operator of the first difference. Then we have

$$\varphi_{0,1}(t)=-\frac{1}{2}\Delta_1\operatorname{sign}(t)\,. \qquad (2.6)$$

We can substitute formula (2.5) into (2.4) in order to determine $\varphi_{0,2}(t)$:

$$\varphi_{0,2}(t)=1\cdot[t\cdot\varphi_{0,1}(t)+(2-t)\varphi_{0,1}(t-1)]= \\ =\frac{1}{2}\{t\cdot[\operatorname{sign}(t)-\operatorname{sign}(t-1)]+ \\ (2-t)[\operatorname{sign}(t-1)-\operatorname{sign}(t-2)]\}= \\ =\frac{1}{2}[t\operatorname{sign}(t)-2(t-1)\operatorname{sign}(t-1)+ \\ (t-2)\operatorname{sign}(t-2)]=\frac{1}{2}[|t|-2|t-1|+|t-2|.$$

Let us denote by $\Delta_2$ the operator of the second difference. Then we have

$$\varphi_{0,2}(t)=\frac{1}{2}[|t|-2|t-1|+|t-2|=\frac{1}{2}\Delta_2\;|t-1|\,. \qquad (2.7)$$

We can define function $\varphi_{0,3}(t)$:

$$\varphi_{0,3}(t)=\frac{1}{2}[t\cdot\varphi_{0,2}(t)+(3-t)\varphi_{0,2}(t-1)]\,.$$

Omitting intermediate calculations, we get

$$\varphi_{0,3}(t)=\frac{1}{4}[t\cdot|t|-3(t-1)|t-1|+ \\ +3(t-2)|t-2|-(t-3)|t-3|]= \\ =-\frac{1}{2!}\frac{1}{2}\Delta_1\Delta_2((t-1)|t-1|)\,. \qquad (2.8)$$

Based on formulas (2.8) and (2.4), we can define the function

$$\varphi_{0,4}(t)=\frac{1}{3}[t\cdot\varphi_{0,3}(t)+(4-t)\varphi_{0,3}(t-1)]\,.$$

Omitting intermediate calculations, as a result we get

$$\varphi_{0,4}(t)= \\ =\frac{1}{2\cdot 3}\cdot\frac{1}{2}[t^2\cdot|t|-4(t-1)^2|t-1|+ \\ +6(t-2)^2|t-2|-4(t-3)^2|t-3|+ \\ +(t-4)^2|t-4|]= \\ =\frac{1}{3!}\frac{1}{2}(\Delta_2)^2((t-2)^2|t-2|)\,. \qquad (2.9)$$

It can be proved that for even $k = 2m$ we have

$$\varphi_{0,k}(t)=\frac{1}{(2m-1)!}\frac{1}{2}(\Delta_2)^m((t-m)^{2m-2}|t-m|) \qquad (2.10)$$

and for odd (uneven) $k = 2m + 1$ we have

$$\varphi_{0,k}(t)=-\frac{1}{(2m)!}\frac{1}{2}\Delta_1(\Delta_2)^m((t-m)^{2m-1}|t-1|)\,. \qquad (2.11)$$

Note that $\varphi_{0,k}(t)$ is a polynomial of degree $k - 1$ with bounded support and, as follows from the difference operator, this support is equal to the interval [0, $k$].

In addition, we should note the following property of B-spline basis functions:

$$\sum_i \varphi_{0,k}(t-i) \equiv 1 \text{ for arbitrary } t. \qquad (2.12)$$

## 3. SOME GENERAL ASPECTS OF FINITE ELEMENT APPROXIMATION

The discrete component of the numerical solution is represented by the direction along the axis corresponding to $x_1$. The fulfillment within an element (interval) for all components of a vector function $\overline{U}$ (see (1.15)) is the same. Therefore, let us use the following notation for simplicity:

$$x = x_1,\ \ell = \ell_1,\ y = y_1,\ j = 1,2,3,4. \qquad (3.1)$$

Let us divide the interval $(0, \ell)$ segment into $N_e$ parts (elements). Therefore $h_e = \ell/N_e$ is the length of the element. Besides, let us also divide each element into $N_k$ parts (see Figures 3.1-3.3), for example, $N_k = 5$ (see Figure 3.1).

Let us use the following notation system: $i_e$ is the element number; $N_p = N_k + 1 = 6$ is the number of nodes within the element; $x_1(i_e)$ is the coordinate of the starting point of the $i_e$-th element; $x_6(i_e)$ is the coordinate of the end point of the $i_e$-th element. We take $y_i$ and $y'_i = \partial_1 y(x_i)$, $i = 1,6$ as unknowns at the boundary points. Besides, we take $y_i$, $i = 2,3,4,5$ as unknowns at the inner points. Thus, the number of unknowns per element with such approximation is equal to

$$N_{ie} = N_k - 1 + 2 \cdot 2 = N_k + 3 = 8.$$

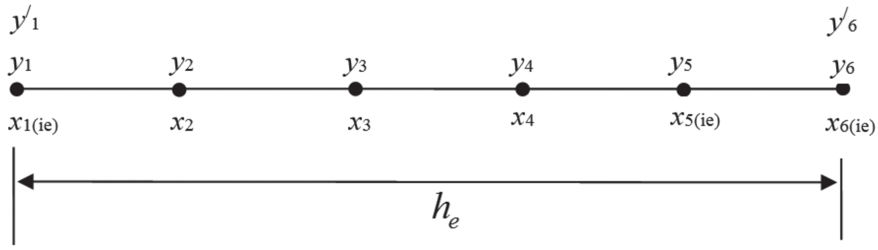



*Figure 3.1. Finite element discretication for $N_k$ = 5 (sample).*

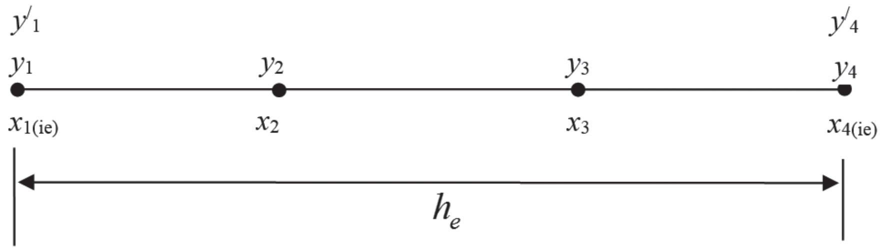



*Figure 3.2. Finite element discretication for $N_k$ = 3 (sample).*

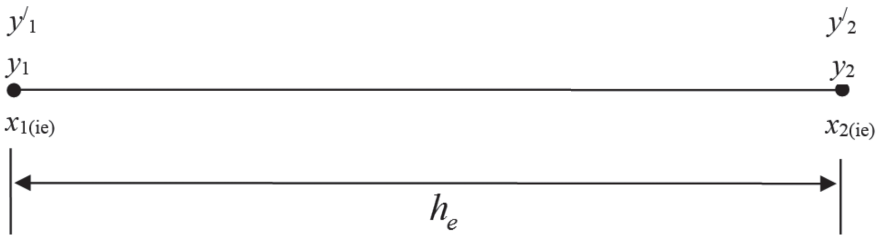



*Figure 3.3. Finite element discretication for $N_k$ = 1 (sample).*

For the elements of localization we can take reduced number of $N_k$. For instance, if we take $N_k = 3$ (Figure 3.2) we get $N_p = N_k + 1 = 4$ and the number of unknowns per element with such approximation is equal to

$$N_{ie} = N_k - 1 + 2 \cdot 2 = N_k + 3 = 6;$$

$x_1(i_e)$ is the coordinate of the starting point of the $i_e$-th element; $x_4(i_e)$ is the coordinate of the end point of the $i_e$-th element; $y_i$ and $y'_i$, $i = 1,4$ are unknowns at the boundary points; $y_i$, $i = 2,3$ are unknowns at the inner points.
Besides, let us consider the case with (Figure 3.3). Therefor we have and the number of unknowns per element with such approximation is equal to

$$N_{ie} = N_k - 1 + 2 \cdot 2 = N_k + 3 = 4;$$

$x_1(i_e)$ is the coordinate of the starting point of the $i_e$-th element; $x_2(i_e)$ is the coordinate of the end point of the $i_e$-th element; $y_i$ and $y'_i$, $i = 1,2$ are unknowns at the boundary points.

## 4. LOCAL CONSTRUCTIONS FOR ARBITRARY FINITE ELEMENT

Let us introduce local coordinates:

$$t = (x - x_{1(ie)}) / h_e, \quad x_{1(ie)} \le x \le x_{N_p(ie)}, \; 0 \le t \le 1. \tag{4.1}$$

In this case, we have the following relations:

$$x = x_i \Rightarrow t_i = (x_i - x_{1(ie)}) / h_e, \quad i = 1, ..., N_p; \tag{4.2}$$

$$\frac{d^p}{dx^p} = \frac{1}{h_e^p}\frac{d^p}{dt^p}; \quad dx = h_e \cdot dt. \tag{4.3}$$

Since the number of unknowns on the element is equal to $N = 8$, we use a B-spline of the seventh degree in order to represent the unknown deflection function.

Let us use the following notation:

$$\varphi(t) = \varphi_{0,8}(t+4);$$

$$\varphi(t) = \frac{1}{7!}\frac{1}{2}(\Delta_2)^4(t^6\,|\,t\,|) = \frac{1}{2\cdot 7!}[(t+4)^6\,|\,t+4\,| - 8(t+3)^6\,|\,t+3\,| + 28(t+2)^6\,|\,t+2\,| - 56(t+1)^6\,|\,t+1\,| + 70t^6\,|\,t\,| - 56(t-1)^6\,|\,t-1\,| + 28(t-2)^6\,|\,t-2\,| - 8(t-3)^6\,|\,t-3\,| + (t-4)^6\,|\,t-4\,|]. \tag{4.4}$$

This function is a B-spline, symmetric with respect to $t = 0$ and its support is defined by an interval [–4, 4] (Figure 4.1).
We take the following eight functions as basis functions on the unit interval (Figures 4.2, 4.3):

$$\varphi_1(t) = \varphi(t+3), \quad \varphi_2(t) = \varphi(t+2), \quad \varphi_3(t) = \varphi(t+1), \quad \varphi_4(t) = \varphi(t), \quad \varphi_5(t) = \varphi(t-1), \quad \varphi_6(t) = \varphi(t-2), \quad \varphi_7(t) = \varphi(t-3), \quad \varphi_8(t) = \varphi(t-4), \quad 0 \le t \le 1. \tag{4.5}$$

Since the number of unknowns on the element is equal to $N = 6$, we use a B-spline of the fifth degree in order to represent the unknown deflection function.
Let us use the following notation:

$$\varphi(t) = \varphi_{0,6}(t+4);$$

$$\varphi(t) = \frac{1}{5!}\frac{1}{2}(\Delta_2)^3(t^4\,|\,t\,|) = \frac{1}{2\cdot 5!}[(t+3)^4\,|\,t+3\,| - 6(t+2)^4\,|\,t+2\,| + 15(t+1)^4\,|\,t+1\,| - 20t^4\,|\,t\,| + 15(t-1)^4\,|\,t-1\,| - 6(t-2)^4\,|\,t-2\,| + (t-3)^4\,|\,t-3\,|]. \tag{4.6}$$

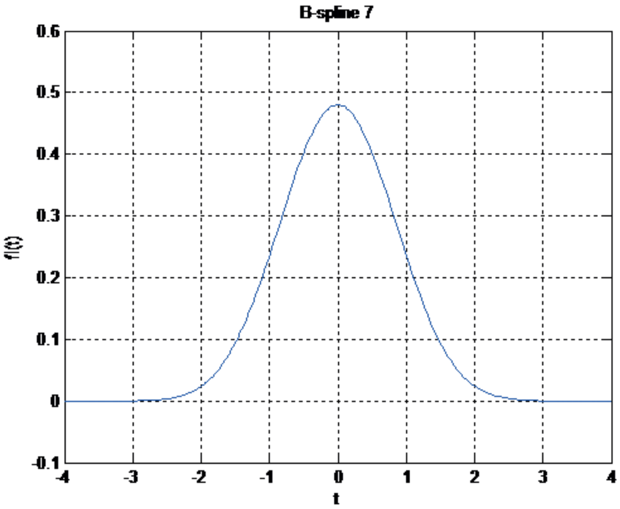



Figure 4.1. B-spline of the seventh order $\varphi(t) = \varphi_{0,8}(t + 4)$.

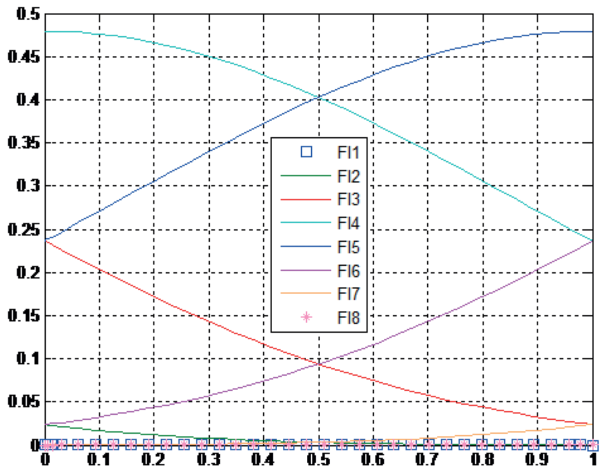



Figure 4.2. Basis functions $\varphi_k(t)$, $k = 1,2,\ldots,8$.

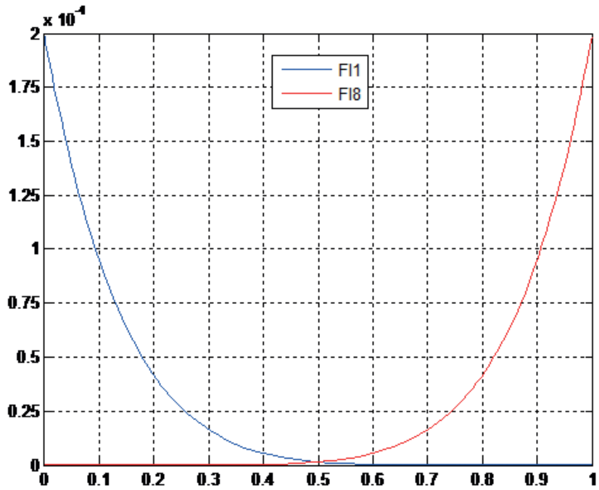



Figure 4.3. Basis functions $\varphi_1(t)$ and $\varphi_8(t)$.

This function is a B-spline, symmetric with respect to $t$ = 0 and its support is defined by an interval [–3,3] (Figure 4.4).

We take the following six functions as basis functions on the unit interval (Figures 4.5, 4.6):

$$\varphi_1(t) = \varphi(t+2), \quad \varphi_2(t) = \varphi(t+1),$$
$$\varphi_3(t) = \varphi(t), \quad \varphi_4(t) = \varphi(t-1),$$
$$\varphi_5(t) = \varphi(t-2), \quad \varphi_6(t) = \varphi(t-3),$$
$$0 \le t \le 1. \quad (4.7)$$

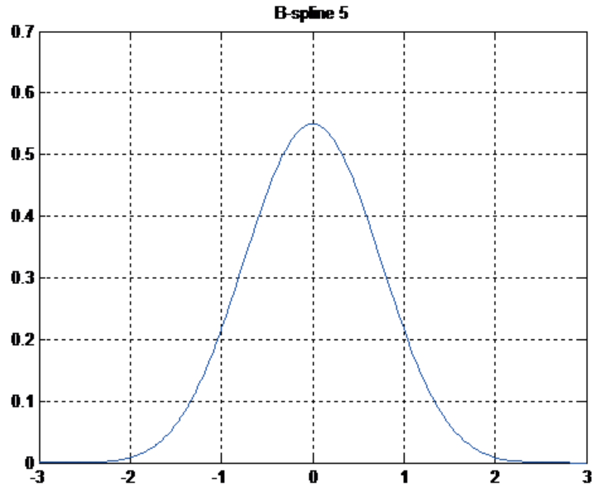



*Figure 4.4. B-spline of the fifth order $\varphi(t) = \varphi_{0,6}(t + 3)$.*

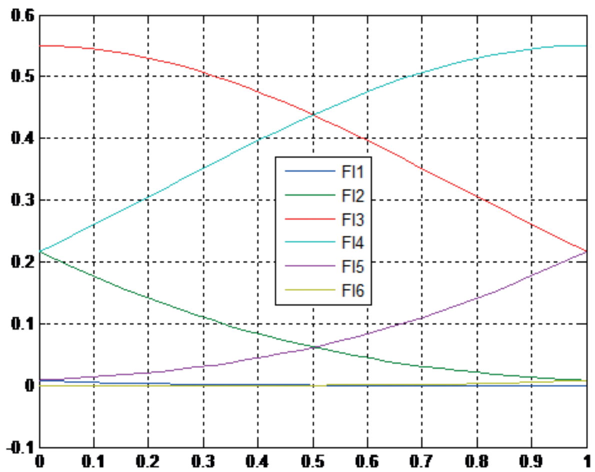
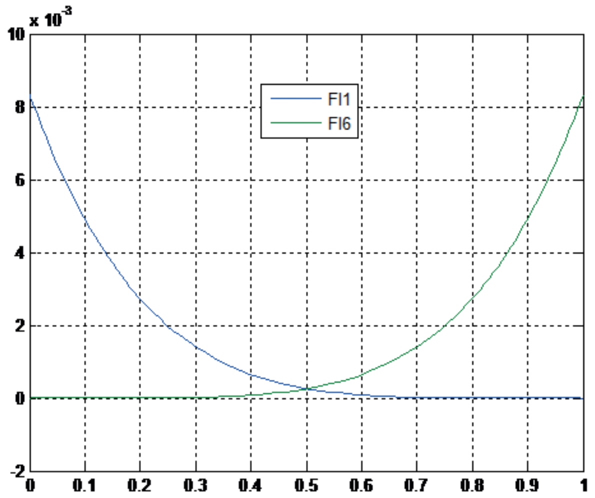



*Figure 4.5. Basis functions $\varphi_k(t)$, $k$ = 1,2,...,3.*



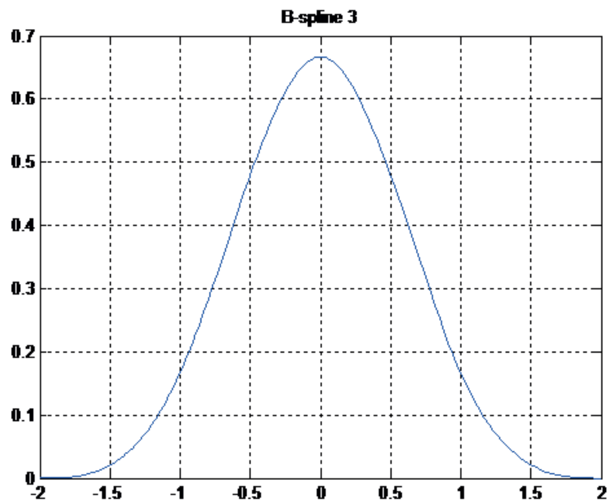

*<u>Figure 4.6.</u> Basis functions $\varphi_1(t)$ and $\varphi_6(t)$.*

Since the number of unknowns on the element is equal to $N = 4$, we use a B-spline of the third degree in order to represent the unknown deflection function.
Let us use the following notation:

$$\varphi(t) = \varphi_{0,4}(t + 4) ;$$

$$\varphi(t) = \frac{1}{3!}\frac{1}{2}(\Delta_2)^2(t^2\,|\,t\,|) = \frac{1}{2\cdot 3!}[(t+2)^4\,|\,t+2\,| - 4(t+1)^4\,|\,t+1\,| + 6t^2\,|\,t\,| - 4(t-1)^2\,|\,t-1\,| + (t-2)^2\,|\,t-2\,|]. \quad (4.8)$$

This function is a B-spline, symmetric with respect to $t = 0$ and its support is defined by an interval [–2, 2] (Figure 4.7).




*<u>Figure 4.7.</u> B-spline of the fifth order $\varphi(t) = \varphi_{0,6}(t + 3)$.*

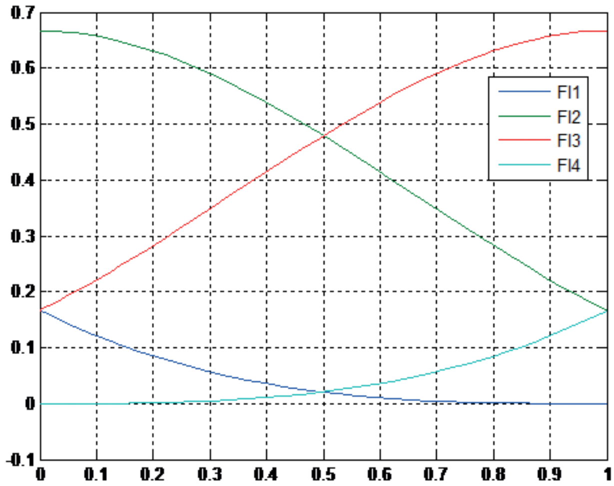



Figure 4.8. Basis functions $\varphi_k(t)$, $k = 1,2,...,4$.

We take the following four functions as basis functions on the unit interval (Figures 4.8):

$$\varphi_1(t) = \varphi(t+1),\ \ \varphi_2(t) = \varphi(t)\,,$$
$$\varphi_3(t) = \varphi(t-1),\ \ \varphi_4(t) = \varphi(t-2)\,,$$
$$0 \le t \le 1. \quad (4.7)$$

We represent the unknown deflection function $y(x)$ within the element number $i_e$ in the form

$$y(x) = w(t) = \sum_{k=1}^{N} \alpha_k \varphi_k(t)\,, \quad x_{1(i_e)} \le x \le x_{6(i_e)}\,,$$
$$0 \le t \le 1. \quad (4.8)$$

We have to consider bilinear forms with allowance for relations (4.2) in order to construct local stiffness matrices corresponding to the operators $L_0$, $L_2$ and $L_4$ (see (1.3)–(1.5)):

$$B_0(y,z) = < L_0 y, z > = -\int_{x_{1(i_e)}}^{x_{5(i_e)}} \frac{d^2}{dx^2}\theta D \frac{d^2 y}{dx^2}\cdot z\,dx =$$
$$= -\theta_{i_e} D_{i_e} \int_{x_{1(i_e)}}^{x_{5(i_e)}} \frac{d^2 y}{dx^2}\cdot\frac{d^2 z}{dx^2}\,dx =$$
$$= -\frac{1}{h_e^3}\theta_{i_e} D_{i_e} \int_0^1 \frac{d^2 w}{dt^2}\cdot\frac{d^2 v}{dt^2}\,dt = B_0(w,v)\,; \quad (4.9)$$

$$B_4(y,z) = < L_4 y, z > = \theta_{i_e} D_{i_e} \int_{x_{1(i_e)}}^{x_{5(i_e)}} y\cdot z\,dx =$$
$$= h_e \theta_{i_e} D_{i_e} \int_0^1 w\cdot v\,dt = B_4(w,v)\,; \quad (4.10)$$

$$B_2(y,z) = < L_2 y, z > = < L_{21} y, z > +$$
$$+ < L_{22} y, z > + < L_{23} y, z >, \quad (4.11)$$

where

$$< L_{21} y, z > = -\theta_{i_e} D_{i_e} \nu_{i_e} \int_{x_{1(i_e)}}^{x_{5(i_e)}} \frac{d^2 y}{dx^2} z\,dx =$$
$$= -\frac{1}{h_e}\theta_{i_e} D_{i_e} \nu_{i_e} \int_0^1 \frac{d^2 w}{dt^2}\cdot v\,dt = B_{21}(w,v)\,; \quad (4.12)$$

$$< L_{23} y, z > = -\theta_{i_e} D_{i_e} \nu_{i_e} \int_{x_{1(i_e)}}^{x_{5(i_e)}} y\cdot\frac{d^2 z}{dx^2}\,dx =$$
$$= -\frac{1}{h_e}\theta_{i_e} D_{i_e} \nu_{i_e} \int_0^1 w\cdot\frac{d^2 v}{dt^2}\,dt = B_{23}(w,v)\,; \quad (4.13)$$

$$< L_{22} y, z > = -2\int_{x_{1(i_e)}}^{x_{5(i_e)}} \frac{d}{dx}\theta D(1-\nu)\frac{dy}{dx}\cdot z\,dx =$$
$$= 2\theta_{i_e} D_{i_e} (1-\nu_{i_e}) \int_{x_{1(i_e)}}^{x_{5(i_e)}} \frac{dy}{dx}\cdot\frac{dz}{dx}\,dx =$$
$$= \frac{1}{h_e} 2\theta_{i_e} D_{i_e} (1-\nu_{i_e}) \int_0^1 \frac{dw}{dt}\cdot\frac{dv}{dt}\,dt = B_{22}(w,v)\,. \quad (4.14)$$

for the following type of functions

$$y(x) = w(t) = \sum_{k=1}^{N_{ie}} \alpha_k \varphi_k(t),$$

$$z(x) = v(t) = \sum_{k=1}^{N_{ie}} \beta_k \varphi_k(t),$$

$$x_{1(ie)} \le x \le x_{N_p(ie)}, \quad 0 \le t \le 1 \quad (4.15)$$

Let us substitute (4.15) into (4.9)–(4.14):

$$B_0(w,v) = -\frac{1}{h_e^3}\theta_{i_e} D_{i_e} \int_0^1 \frac{d^2 w}{dt^2} \cdot \frac{d^2 v}{dt^2} dt =$$

$$= -\frac{\theta_{i_e} D_{i_e}}{h_e^3} \sum_{i=1}^{N}\sum_{j=1}^{N} \alpha_i \beta_j \int_0^1 \varphi_i''(t)\varphi_j''(t)\, dt =$$

$$= -\frac{\theta_{i_e} D_{i_e}}{h_e^3} (K_{\alpha\beta}^{0} \overline{\alpha}, \overline{\beta}), \quad (4.16)$$

where

$$K_{\alpha\beta}^{0}(i,j) = \int_0^1 \varphi_i''(t)\varphi_j''(t)\, dt; \quad \varphi'' = \frac{d^2\varphi}{dt^2}; \quad (4.17)$$

$$B_4(w,v) = h_e \theta_{i_e} D_{i_e} \int_0^1 w \cdot v\, dt =$$

$$= \theta_{i_e} D_{i_e} h_e \sum_{i=1}^{N}\sum_{j=1}^{N} \alpha_i \beta_j \int_0^1 \varphi_i(t)\varphi_j(t)\, dt =$$

$$= h_e \theta_{i_e} D_{i_e} (K_{\alpha\beta}^{4} \overline{\alpha}, \overline{\beta}), \quad (4.18)$$

where

$$K_{\alpha\beta}^{4}(i,j) = \int_0^1 \varphi_i(t)\varphi_j(t)\, dt; \quad (4.19)$$

$$B_{21}(w,v) = -\frac{1}{h_e}\theta_{i_e} D_{i_e} \nu_{i_e} \int_0^1 \frac{d^2 w}{dt^2} \cdot v\, dt =$$

$$= -\frac{\theta_{i_e} D_{i_e} \nu_{i_e}}{h_e} \sum_{i=1}^{N}\sum_{j=1}^{N} \alpha_i \beta_j \int_0^1 \varphi_i''(t)\varphi_j(t)\, dt =$$

$$= -\frac{\theta_{i_e} D_{i_e} \nu_{i_e}}{h_e} (K_{\alpha\beta}^{21} \overline{\alpha}, \overline{\beta}), \quad (4.20)$$

where

$$K_{\alpha\beta}^{21}(i,j) = \int_0^1 \varphi_i''(t)\varphi_j(t)\, dt; \quad (4.21)$$

$$B_{23}(w,v) = -\frac{1}{h_e}\theta_{i_e} D_{i_e} \nu_{i_e} \int_0^1 w \cdot \frac{d^2 v}{dt^2} dt =$$

$$= -\frac{\theta_{i_e} D_{i_e} \nu_{i_e}}{h_e} \sum_{i=1}^{N}\sum_{j=1}^{N} \alpha_i \beta_j \int_0^1 \varphi_i(t)\varphi_j''(t)\, dt =$$

$$= -\frac{\theta_{i_e} D_{i_e} \nu_{i_e}}{h_e} (K_{\alpha\beta}^{23} \overline{\alpha}, \overline{\beta}), \quad (4.22)$$

where

$$K_{\alpha\beta}^{23}(i,j) = \int_0^1 \varphi_i(t)\varphi_j''(t)\, dt = K_{\alpha\beta}^{21}(j,i); \quad (4.23)$$

$$B_{22}(w,v) = \frac{1}{h_e} 2\theta_{i_e} D_{i_e} (1-\nu_{i_e}) \int_0^1 \frac{dw}{dt} \cdot \frac{dv}{dt} dt =$$

$$= 2\frac{\theta_{i_e} D_{i_e} (1-\nu_{i_e})}{h_e} \sum_{i=1}^{N}\sum_{j=1}^{N} \alpha_i \beta_j \int_0^1 \varphi_i'(t)\varphi_j'(t)\, dt =$$

$$= 2\frac{\theta_{i_e} D_{i_e} (1-\nu_{i_e})}{h_e} (K_{\alpha\beta}^{22} \overline{\alpha}, \overline{\beta}), \quad (4.24)$$

where

$$K_{\alpha\beta}^{22}(i,j) = \int_0^1 \varphi_i'(t)\varphi_j'(t)\, dt, \quad \varphi' = \frac{d\varphi}{dt}. \quad (4.25)$$

Let us define the parameters $\alpha_k$ and $\beta_k$ through the nodal unknowns on the element:

$$y_i = w(t_i) = \sum_{k=1}^{N_{ie}} \alpha_k \varphi_k(t_i),$$

$$t_i = (x_i - x_{1(ie)})/h_e, \quad i = 1,\ldots, N_p; \quad (4.26)$$

$$\frac{dy_i}{dx} = \frac{1}{h_e} w'(t_i) = \frac{1}{h_e} \sum_{k=1}^{N_{ie}} \alpha_k \varphi_k'(t_i),$$

$$t_i = (x_i - x_{1(ie)}) h_e, \; i = 1, N_p. \quad (4.27)$$

For the case $N_{ie} = 8$ we have

$$\overline{y}^{ie} = T_8 \overline{\alpha}\,, \tag{4.28}$$

where

$$\overline{y}^{i_e} = [y_1 \;\; \frac{dy_1}{dx} \;\; y_2 \;\; y_3 \;\; y_4 \;\; y_5 \;\; y_6 \;\; \frac{dy_6}{dx}]^{\mathrm{T}}\,; \tag{4.29}$$

$$\overline{\alpha} = [\alpha_1 \;\; \alpha_2 \;\; \alpha_3 \;\; \alpha_4 \;\; \alpha_5 \;\; \alpha_6 \;\; \alpha_7 \;\; \alpha_8]^{\mathrm{T}}\,; \tag{4.30}$$

$$D = diag(1 \;\; 1/h_e \;\; 1 \;\; 1 \;\; 1 \;\; 1 \;\; 1 \;\; 1/h_e)\,. \tag{4.31}$$

$$T = D\begin{bmatrix} \varphi_1(0) & \varphi_2(0) & \varphi_3(0) & \varphi_4(0) & \varphi_5(0) & \varphi_6(0) & \varphi_7(0) & \varphi_8(0) \\ \varphi_1'(0) & \varphi_2'(0) & \varphi_3'(0) & \varphi_4'(0) & \varphi_5'(0) & \varphi_6'(0) & \varphi_7'(0) & \varphi_8'(0) \\ \varphi_1(0.2) & \varphi_2(0.2) & \varphi_3(0.2) & \varphi_4(0.2) & \varphi_5(0.2) & \varphi_6(0.2) & \varphi_7(0.2) & \varphi_8(0.2) \\ \varphi_1(0.4) & \varphi_2(0.4) & \varphi_3(0.4) & \varphi_4(0.4) & \varphi_5(0.4) & \varphi_6(0.4) & \varphi_7(0.4) & \varphi_8(0.4) \\ \varphi_1(0.6) & \varphi_2(0.6) & \varphi_3(0.6) & \varphi_4(0.6) & \varphi_5(0.6) & \varphi_6(0.6) & \varphi_7(0.6) & \varphi_8(0.6) \\ \varphi_1(0.8) & \varphi_2(0.8) & \varphi_3(0.8) & \varphi_4(0.8) & \varphi_5(0.8) & \varphi_6(0.8) & \varphi_7(0.8) & \varphi_8(0.8) \\ \varphi_1(1) & \varphi_2(1) & \varphi_3(1) & \varphi_4(1) & \varphi_5(1) & \varphi_6(1) & \varphi_7(1) & \varphi_8(1) \\ \varphi_1'(1) & \varphi_2'(1) & \varphi_3'(1) & \varphi_4'(1) & \varphi_5'(1) & \varphi_6'(1) & \varphi_7'(1) & \varphi_8'(1) \end{bmatrix}, \tag{4.32}$$

For the case $N_{ie} = 6$ we have

$$\overline{y}^{ie} = T_6 \overline{\alpha}\,, \tag{4.33}$$

where

$$\overline{y}^{i_e} = [y_1 \;\; \frac{dy_1}{dx} \;\; y_2 \;\; y_3 \;\; y_4 \;\; \frac{dy_4}{dx}]^{\mathrm{T}}\,; \tag{4.34}$$

$$\overline{\alpha} = [\alpha_1 \;\; \alpha_2 \;\; \alpha_3 \;\; \alpha_4 \;\; \alpha_5 \;\; \alpha_6]^{\mathrm{T}}\,; \tag{4.35}$$

$$D = diag(1 \;\; 1/h_e \;\; 1 \;\; 1 \;\; 1 \;\; 1/h_e)\,. \tag{4.36}$$

$$T_6 = D_6\begin{bmatrix} \varphi_1(0) & \varphi_2(0) & \varphi_3(0) & \varphi_4(0) & \varphi_5(0) & \varphi_6(0) \\ \varphi_1'(0) & \varphi_2'(0) & \varphi_3'(0) & \varphi_4'(0) & \varphi_5'(0) & \varphi_6'(0) \\ \varphi_1(1/3) & \varphi_2(1/3) & \varphi_3(1/3) & \varphi_4(1/3) & \varphi_5(1/3) & \varphi_6(1/3) \\ \varphi_1(2/3) & \varphi_2(2/3) & \varphi_3(2/3) & \varphi_4(2/3) & \varphi_5(2/3) & \varphi_6(2/3) \\ \varphi_1(1) & \varphi_2(1) & \varphi_3(1) & \varphi_4(1) & \varphi_5(1) & \varphi_6(1) \\ \varphi_1'(0) & \varphi_2'(0) & \varphi_3'(0) & \varphi_4'(0) & \varphi_5'(0) & \varphi_6'(0) \end{bmatrix}; \tag{4.37}$$

For the case $N_{ie} = 4$ we have

$$\overline{y}^{ie} = T_4 \overline{\alpha}, \tag{4.38}$$

where

$$\overline{y}^{i_e} = [y_1 \;\; \frac{dy_1}{dx} \;\; y_2 \;\; \frac{dy_2}{dx}]^{\mathrm{T}}; \tag{4.39}$$

$$\overline{\alpha} = [\alpha_1 \;\; \alpha_2 \;\; \alpha_3 \;\; \alpha_4]^{\mathrm{T}}\,; \tag{4.35}$$

$$D = diag(1 \;\; 1/h_e \;\; 1 \;\; 1/h_e)\,; \tag{4.40}$$

$$T_4 = D_4\begin{bmatrix} \varphi_1(0) & \varphi_2(0) & \varphi_3(0) & \varphi_4(0) \\ \varphi_1'(0) & \varphi_2'(0) & \varphi_3'(0) & \varphi_4'(0) \\ \varphi_1(1) & \varphi_2(1) & \varphi_3(1) & \varphi_4(1) \\ \varphi_1'(1) & \varphi_2'(1) & \varphi_3'(1) & \varphi_4'(1) \end{bmatrix}. \tag{4.41}$$

Similarly, we get

$$\overline{z}_{ie} = T_{N_{ie}} \overline{\beta} \tag{4.42}$$

for $N_{ie} = 8$, $N_{ie} = 6$, $N_{ie} = 4$.

From (4.28), (4.33), (4.38) and (4.42) it follows

$$\overline{\alpha} = T_{N_{i_e}}^{-1} \overline{y}^{i_e}\,; \quad \overline{\beta} = T_{N_{i_e}}^{-1} \overline{z}^{ie}\,. \tag{4.43}$$

We have the following chain of equalities

$$(K_{\alpha\beta}\overline{\alpha}, \overline{\beta}) = (K_{\alpha\beta} T_{N_{ie}}^{-1} \overline{y}^{ie}, T_{N_{ie}}^{-1} \overline{z}^{ie}) = \\ = ((T_{N_{ie}}^{-1})^{\mathrm{T}} K_{\alpha\beta} T_{N_{ie}}^{-1} \overline{y}^{ie}, \overline{z}^{ie})\,. \tag{4.44}$$

Therefore, substituting (4.43) sequentially in (4.16), (4.18), (4.20), (4.22), (4.24), we obtain local stiffness matrices $K_0^{i_e}$, $K_4^{i_e}$, $K_{21}^{i_e}$, $K_{23}^{i_e}$, $K_{22}^{i_e}$, $K_2^{i_e}$, corresponding to the operators $L_0$, $L_4$, $L_{21}$, $L_{23}$, $L_{22}$, $L_2$.

## 5. EXAMPLE OF ANALYSIS

### *5.1. Formulation of the problem.*

Let us consider the problem of the bending of a thin plate rigidly fixed along the side faces under the influence of a load concentrated in the center as an example (Figure 5.1).

Let us consider the following geometric parameters: $L_1 = 0.9$ m, $L_2 = 1.0$ m, $h = 0.05$ m is the thickness.

Let us consider the following design parameters of material of plate: coefficient of elasticity $E$ = $3000 \cdot 10^4$ kN/m$^2$, Poisson's ratio $\nu = 0.16$.

Let external load parameter be equal to $P = 1$ kN.

### *5.2. Structural analysis with allowance for localization.*

Let the number of elements be equal to $N_e = 6$.

Then we have the following element length:

$$h_e = L_1/N_e = 0.9/6 = 0.15.$$

Let's define localization in the load area.

For the first element and for the sixth element we have $N_k = 1$ and third-order spline; distance between the coordinates of the nodes of the first element and the sixth element is equal to

$$h_1 = h_6 = 0.15/1 = 0.15.$$

For the second element and for the fifth element we have $N_k = 3$ and fifth-order spline; distance between the coordinates of the nodes of the second element and the fifth element is equal to

$$h_2 = h_5 = 0.15/3 = 0.05.$$

For the third element and for the fourth element we have $N_k = 5$ and seventh-order spline; distance between the coordinates of the nodes of the third element and the fourth element is equal to

$$h_3 = h_4 = 0.15/5 = 0.03.$$

The total number of inner nodes is equal to

$$N_p = 2(4 + 2 + 0) = 12.$$

The total number of boundary nodes is equal to

$$N_b = N_e + 1 = 7.$$

The total number of nodes for all elements is equal to

$$N_x = N_p + N_b = 12 + 7 = 19.$$

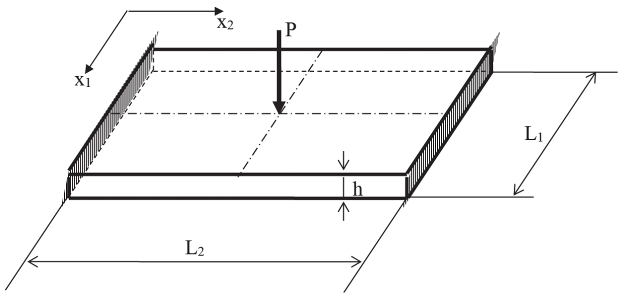


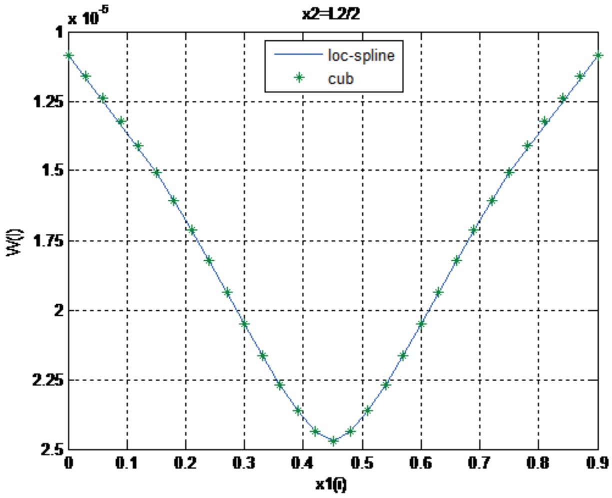

*Figure 5.1. Example of analysis.*

The number of nodal unknowns for each component of the vector function $y_j$, $j = 1,2,3,4$ is equal to

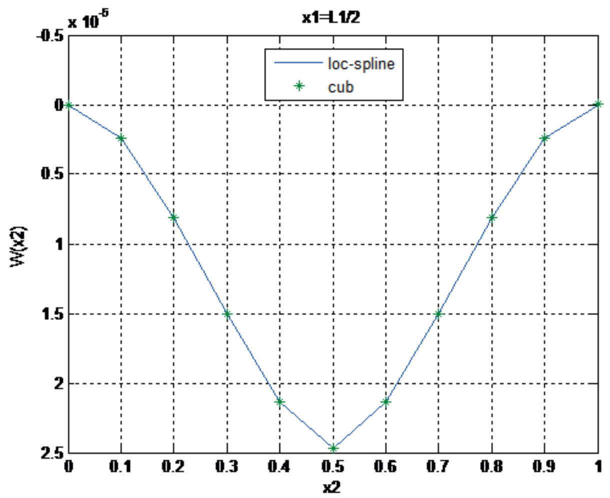
$$N_g = N_p + 2N_b = 12 + 2 \cdot 7 = 26.$$

The total number of unknowns is equal to

$$N_U = 4N_g = 4 \cdot 26 = 104.$$




*Figures 5.2. Comparison of the results of analysis in the middle sections along $x_1$ direction (discrete direction).*




*Figures 5.3. Comparison of the results of analysis in the middle sections along $x_2$ direction (continual direction).*

***5.3. Structural analysis without localization.*** In this case, we will consider only the standard cubic fulfilment. In this case, the length of the element is taken equal to the minimum distance between the nodes, i.e. $h_e = 0.03$. Then the number of elements is equal to

$$N_e = 0.9/0.03 = 30$$

and the total number of nodes is equal to $N_x = 31$. In this case, the number of nodal unknowns for each component of the vector function $y_j$, $j = 1,2,3,4$ is equal to

$$N_g = 2 \cdot N_x = 2 \cdot 31 = 62$$

and the total number of nodal unknowns is equal to

$$N_U = 4N_g = 4 \cdot 62 = 248.$$

Graphical comparison of corresponding results of analysis is presented at Figures 5.2 and 5.3

(`loc-spline` are nodal values computed with allowance for localization; `cub` are nodal values computed without localization).

As researcher can see, the results obtained are almost completely identical. Besides, the use of localization based on application of B-splines of various degrees leads to a significant decrease in the number of unknowns. The difference for this example is equal to

$$N_{Ucub} - N_{Uloc} = 248 - 104 = 144.$$

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# STRENGTHENING OF THE FOUNDATIONS OF RENOVATED BUILDINGS WITH INJECTION PILES

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**Abstract**. The paper describes the technique of strengthening shallow foundations of reconstructed buildings using injection piles. First, the constructive solution of the existing foundations, the structural scheme of the building, as well as the loads transferred to the building structures before and after its reconstruction are established. At the same time, an assessment of the soil conditions of the construction site of the reconstructed building is carried out; a bearing soil layer is revealed for deepening the lower ends of injection piles. Based on the data obtained, the loading of the base of the foundations of the reconstructed building is assessed and the need for their reinforcement (or further operation without reinforcement) is established. In the case of strengthening the foundations of the building, the method of transferring the additional load to the injection piles is selected. Then their bearing capacity and design loads allowed on the piles are substantiated. The construction of foundations is carried out, taking into account their reinforcement with injection piles, which are hereinafter called combined. Verification calculations of the base of the combined foundations are performed for the first and second groups of limit states. In accordance with the regulatory documents, strength calculations of the main structural elements of foundations are carried out, which are necessary to ensure their full operation, taking into account the reinforcement. At the final stage, working documentation is developed to strengthen the foundations of the reconstructed building. The stages of the design of strengthening the foundations of reconstructed buildings using injection piles presented in the work allow to properly and consistently organize the work of specialists.

**Keywords**: reconstruction of buildings, soil conditions, shallow foundations, injection piles, calculation and design methods, foundation reinforcement, combined foundations.

# УСИЛЕНИЕ ФУНДАМЕНТОВ РЕКОНСТРУИРУЕМЫХ ЗДАНИЙ ИНЪЕКЦИОННЫМИ СВАЯМИ

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**Аннотация**. Излагается методика усиления фундаментов мелкого заложения реконструируемых зданий с использованием инъекционных свай. Вначале устанавливается конструктивное решение существующих фундаментов, конструктивная схема здания, а также нагрузки, передаваемые на строительные конструкции до и после его реконструкции. Одновременно выполняется оценка грунтовых условий строительной площадки реконструируемого здания; выявляется несущий слой грунта для заглубления нижних концов инъекционных свай. На основе полученных данных производится оценка загружения основания фундаментов реконструируемого здания и устанавливается необходимость их усиления (или дальнейшей эксплуатация без усиления). В случае усиления фундаментов здания выбирается способ передачи дополнительной нагрузки на инъекционные сваи. Затем обосновывается их несущая способность и расчетные нагрузки, допускаемые на сваи. Производится конструирование фундаментов с учетом их усиления инъекционными сваями, которые в дальнейшем называются комбинированными. Выполняются поверочные расчеты основания комбинированных фундаментов по первой и второй группам предельных состояний. В соответствии с нормативными документами ведутся расчеты основных конструктивных элементов фундаментов на прочность, необходимых для обеспечения их полноценной работы с учетом усиления. На заключительном этапе разрабатывается рабочая документация по усилению фундаментов реконструируемого здания. Представленные в работе этапы проектирования усиления фундаментов реконструируемых зданий с использованием инъекционных свай позволяют правильно и последовательно организовать работу специалистов.

**Ключевые слова**: реконструкция зданий, фундаменты мелкого заложения, методика усиления фундаментов, инъекционные сваи, комбинированные фундаменты.

## BASIC PROVISIONS

During the reconstruction of buildings, the loads on the building structures often increase and the pressure along the bottom of the existing foundations increases. Such operating conditions of buildings can lead to a loss of the bearing capacity of the base, significant settlement of foundations, the development of cracks in walls and other building structures. Therefore, from time to time there is a need to strengthen the foundations and harden the soils of the base of the reconstructed buildings.

One of the ways to increase the bearing capacity of the foundation of shallow foundations is to change their operation scheme by transferring part of the load from the building to additionally arranged piles [1–5]. At the same time, in order to strengthen shallow foundations in clayey soils, injection cylindrical piles are increasingly being developed. They are arranged by pressing a metal injector into clay soil, followed by feeding a fine-grained mobile concrete mixture under pressure and pressing the wells [6–12].

## FOUNDATION REINFORCEMENT DESIGN METHODOLOGY

The design technique for reinforcing shallow foundations using injection piles includes the following main stages:

1. Based on the results of the survey and the use of archival documentation, the constructive solution of the existing foundations, the structural scheme of the building, as well as the loads transferred to the building structures before and after the reconstruction of the building are established;
2. Assess the soil conditions of the construction site of the building to be reconstructed and identify the load-bearing soil layer for deepening the lower ends of injection piles used to strengthen the foundations;
3. Assess the loading of the shallow foundations of the building to be reconstructed and establish the need for their reinforcement (or further operation without reinforcement);
4. Choose a method for installing injection piles and transferring an additional load to them in case of strengthening the foundations of the building;
5. Substantiate the bearing capacity of the injection piles, as well as the design loads allowed on these piles.
6. Carry out the design of the foundations, taking into account their reinforcement with injection piles.
7. Perform verification calculations of the base of the reinforced foundations (combined) for the first and second groups of limiting states (for bearing capacity and deformations). The settlements of the foundations established in this case after their reinforcement and the transfer of an additional load should not exceed the permissible values;
8. Perform calculations of the main structural elements of foundations for strength (selection of reinforcement of reinforcement elements, designation of sections of thrust beams, calculation of foundations for punching, etc.), necessary to ensure their full operation, taking into account reinforcement; the solution to these issues is carried out in accordance with the regulatory documents (SP 16.13330.2017, SP 63.13330.2018, etc.).
9. Prepare working documentation for strengthening the foundations of the re-constructed building.

## MAIN DESIGN RESULTS AND THEIR APPLICATION

***Constructive solution of existing foundations***. The constructive solution of the foundations of the buildings under reconstruction is established, as a rule, according to the results of the examination of building structures and soils of the foundation. In this case, it is recommended to use design, executive and archival documentation for the building being reconstructed (conclusions, technical reports, inspection reports, etc.). Inspection of foundations is carried out more often from pits, which are laid in the most characteristic places of the building plan. Other methods, instruments and equipment are also used to inspect foundations, base soils. When examining foundations in prepared pits, their geometrical parameters are determined (depth, base width, thickness of the slab part, dimensions of the pocket foundation or wall part of the foundation, etc.) At the same time, during the examination, the strength characteristics of the foundation materials are determined and their defects are identified (the presence of concrete chips, brickwork, cracks, corrosion of reinforcement, violation of waterproofing, etc.). The strength of the foundation material is established by destructive or non-destructive methods.

Based on the results of a survey of foundations (including base soils), taking into account the identified defects and damages, an assessment of their technical condition is given in accordance with the recommendations of GOST 31937–2011. Details of the assessment methods can be found in the papers [13, 14, 16].

At the design stage of foundation reinforcement, it is important to determine the loads transmitted to the foundations of the building before and after its reconstruction. Loads, as a rule, are installed at the level of the foundation cutoff, the external planning mark or the basement floor level (basement, technical floor). Constant loads (weight of bearing and enclosing structures, weight and pressure of soil, etc.), transferred to the foundations of the building before its reconstruction, are determined by the results of the survey of the building and the use of archival data (including measurements, identification of types building materials). Temporary loads are established depending on the purpose of the building (taking into account the weight of equipment and devices, people, furniture, etc.) and the area of its construction (snow, wind, seismic, etc.). The loads transferred to the foundations of the building after its reconstruction are usually established according to the data of the adopted design decisions, taking into account the technical condition of the design assignment.

***Assessment of soil conditions.*** Assessment of the soil conditions of the construction site of a building being reconstructed is understood as a generalization of the results of studies of soil properties and composition of underground water performed at the stage of surveys (geotechnical, geotechnical, including archival, etc.), and identifying the possibility of their use as foundation base. Assessment of soil conditions of construction sites of reconstructed buildings is carried out based on the results of studying materials of geomorphology, lithological structure and engineering-geological sections, as well as physical and mechanical properties of soils and hydro-geological conditions of construction, which are given in reports (conclusions) for engineering and geological surveys. Assessment of soil conditions usually begins with an analysis of archived soil data from the site in question. The reliability and efficiency of the decisions on the foundations and foundations of the reconstructed buildings and structures taken in the project largely depend on the quality and completeness of the survey materials [14, 15].

One of the main issues in assessing the soil conditions of the construction site of a building being reconstructed is the choice of the bearing layer of the foundation soil for injection piles, into which it is assumed that their lower ends are buried. As a rule, this layer should be the strongest, less compressible in comparison with other engineering-geological elements (soil layers) within the length of the piles and the compressible thickness of the base of the pile foundations.

***Assessment of the loading of the foundation base***. The assessment of the loading of the base of the foundations (shallow foundation) is understood as the analysis of the initial data and the results of the calculation, in which the correspondence of the size of the base of the considered foundation to the existing loads (N, M, Q) and the soil conditions of the construction site of the reconstructed building is revealed. When assessing the loading of the base of the foundations, the conditions are checked (Fig. 1):

$$p_{max} \leq 1{,}2R_{cs}$$
$$p_{min} \geq 0, \qquad (1)$$
$$p \leq R_{yn}$$

where $p$ – average pressure along the foot of the foundation (before and after reconstruction of the building), kPa; $R_{cs}$ – design resistance of the compacted soil of the base, kPa; $p_{max}$, $p_{min}$ – respectively, the maximum and minimum pressure along the foot of the foundation before and after the reconstruction of the building, kPa.

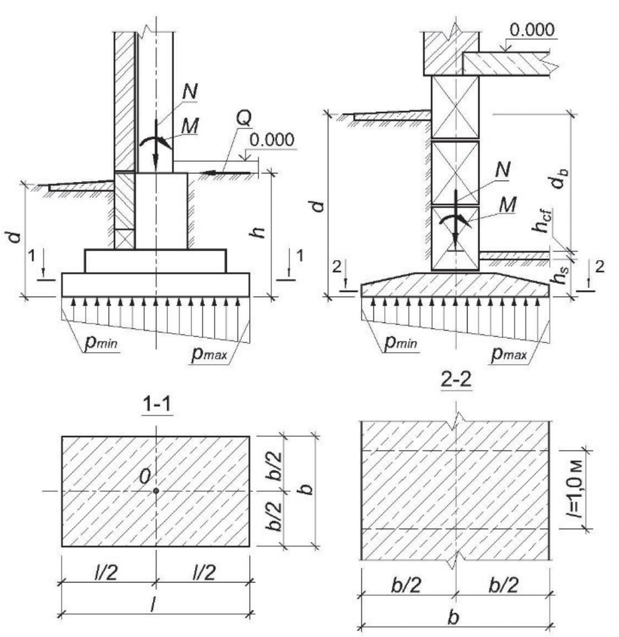


*Figure 1. Design diagrams of the projected foundations of reconstructed buildings: a – a free-standing foundation for a building without a basement; b – strip foundation for a building with a basement; $p_{max}$, $p_{min}$ – respectively, the maximum and minimum pressure on the base of the foundation, kPa; l, b – respectively, the length and width of the base of the foundation, m; d – foundation depth, m; h – foundation height, m; Q, N, M – loads acting on the foundation; $h_s$ – thickness of the soil layer above the foot of the foundation from the basement side, m; $h_{cf}$ basement floor structure thickness, m; $d_b$ – basement depth, m*

Parameter $R_{cs}$ – it is such a pressure on the ground of the foundation along the foot of the foundation, which takes into account the effect of its compression by the load from the weight of the building. In order to determinate $R_{cs}$, we use the method the basis of which is formula (5.7) from SP 22.13330.2016 with the existing prerequisites and restrictions. The method takes into account the changes in soil properties caused by compaction and watering (soaking), which lie at the base of the foundations of long-term buildings. In general, the design resistance of the compacted soil of the base Rcs is determined by the formula [16]:

$$R_{\text{уп}} = \frac{\gamma_{c1} \cdot \gamma_{c2}}{K} \cdot (M_\gamma \cdot K_z \cdot b \cdot \gamma_{\text{II}} \cdot K_\gamma + M_q \cdot d_1 \cdot \gamma'_{\text{II}} + (M_q - 1) \cdot d_b \cdot \gamma'_{\text{II}} + M_c \cdot c_{\text{II}} \cdot K_c) \cdot K_s, \qquad (2)$$

where $K_\gamma$, $K_c$ coefficients taking into account changes in density characteristics $\rho$ (specific gravity $\gamma_{\text{II}}$) and specific cohesion $c$ of base soils under the foundation during the period of operation of the building; $M_\gamma$, $M_q$, $M_c$ – coefficients adopted according to SP 22.13330.2016 depending on the characteristics of the angle of internal friction of the soil $\overline{\varphi_{\text{II}}}$ base, compacted by pressure $p$ from the building in use:

$$\overline{\varphi_{\text{II}}} = \varphi_{\text{II}} \cdot K_\varphi, \qquad (3)$$

$\varphi_{\text{II}}$ – angle of internal friction of natural (non-compacted) foundation soil, degrees; $K_\varphi$ – coefficient taking into account the change in the characteristic of the angle of internal friction for the period of building operation; $K_s$ – coefficient taking into account the degree of implementation (use) of the maximum foundation settlement during the period of the building's operation. The rest of the notation is the same as in formula (5.7) SP 22.13330.2016.

Coefficient values $K_\gamma$, $K_\varphi$, $K_c$ established experimentally for clayey soils. They are selected depending on the ratio of the average pressure p along the base of the operating foundation to the design resistance of the natural (uncompacted) soil of the foundation R, which was taken during the initial design of the object (p/R). Values of coefficients $K_\gamma$, $K_\varphi$, $K_c$ accepted by [13, table 16.3; 16, table 6.1].

The value of the coefficient $K_s$ varies from 1 to 1.4, depending on how fully the ultimate foundation settlement $s$ for the building under consideration is realized during its operation. The largest value of the $K_s$ coefficient is taken when the calculated (actual) settlement of the foundation s of the existing building, established at the actual pressure p, is less than 20% of the maximum permissible $s_u$ ($s < 0.2s_u$). If the calculated (actual) foundation settlement $s$ is more than 70% of the maximum permissible su ($s > 0.7\,s_u$), then the value of the $K_s$ coefficient is taken to be one ($K_s = 1$). The values of the coefficients $K_s$ are taken according to [13, table. 16.4; 16, table 6.2].

In the event that one or two of the conditions (1) are not met, then a decision is made on the need to strengthen the foundation (strengthening the base) of the reconstructed building.

***Method for setting up injection piles***. After assessing the loading of the base of the foundation (shallow foundation) and making a decision on the need to strengthen it, one of the two main methods

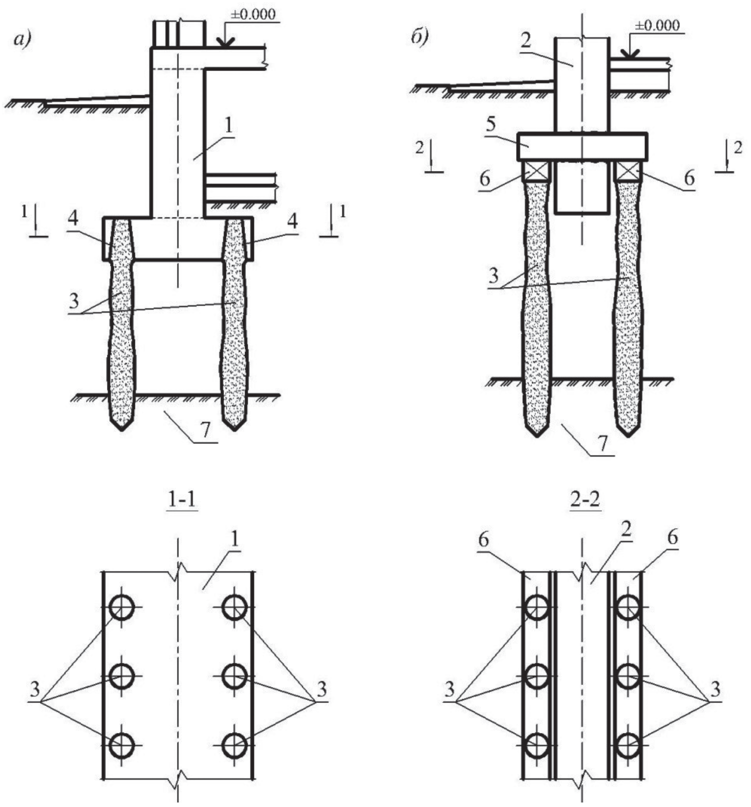


*Figure 2. Methods for arranging injection piles when reinforcing strip foundations of reconstructed buildings: a, b – respectively, under the sole of the existing strip foundation and adjacent to its perimeter; 1 – existing strip monolithic reinforced concrete foundation; 2 – existing strip rubble (brick) foundation; 3 – injection pile; 4 – a tapered hole in the slab part of the existing foundation; 5 – transverse metal beam, arranged through a hole in the wall part of the existing strip foundation; 6 – longitudinal reinforced concrete beam; 7 – bearing layer of solid (low-compressive) soil*

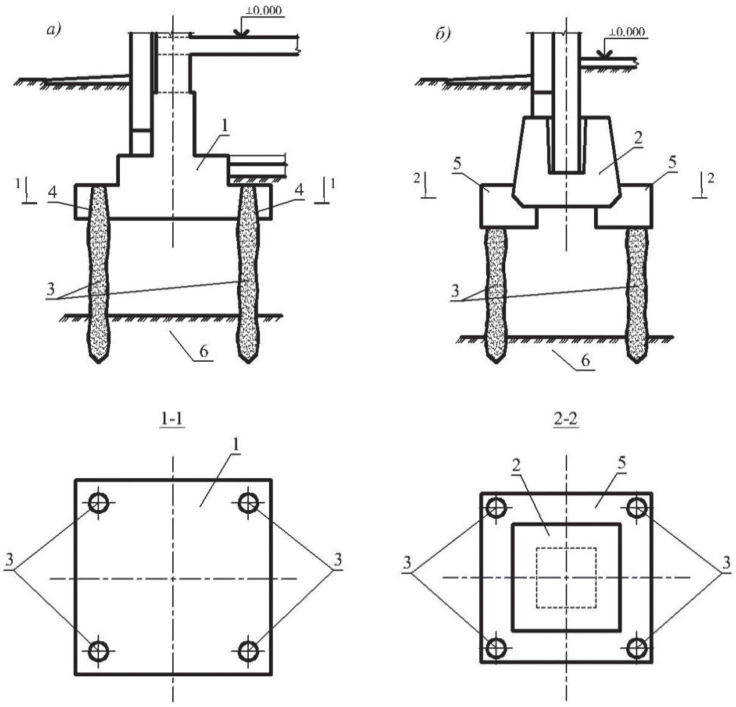


*Figure 3. Methods for installing injection piles when strengthening individual foundations of reconstructed buildings: a, b – respectively, under the sole of the existing separate foundation and adjacent to its perimeter; 1 – existing separate monolithic reinforced concrete foundation; 2 – existing separate precast reinforced concrete foundation; 3 – injection pile; 4 – a tapered hole in the slab part of the existing foundation; 5 – reinforced concrete cage, arranged along the perimeter of the foundation; 6 – bearing layer of solid (low-compressive) soil*

of installing injection piles and transferring to them and below the soils of the base part of the load from overhead building structures is selected. Such piles can be installed directly under the base of the existing foundation (first method) or adjacent along its perimeter (second method) (Fig. 2, 3). The foundation arranged in this way is often called combined. In this case, the combined foundation can be single-pile (if one pile is arranged under a column or pillar), cluster (if two or more piles are arranged) and in the form of a pile field (if ten or more piles are arranged under the existing foundation).

In the first method, injection piles, arranged directly under the base of the foundation, transfer part of the external load to the bearing layer of the solid foundation soil within the contour (perimeter) of the base of the existing foundation (Fig. 2a, 3a).

According to the second method, piles are arranged along the contour or along the perimeter of the sole of the existing foundation (strip and separate). Part of the external load on the supporting solid layer of the foundation soil is transferred to the injection piles adjacent to the foundation. The junction of the piles with the body of the foundation (the base of the foundation) is usually arranged rigid and equal in strength (Fig. 2b, 3b) [16, 17].

***Justification of the bearing capacity of piles.*** Let us consider the assessment of the bearing capacity of injection piles. Their bearing capacity $F_d$ is understood as the limiting resistance of the near-pile basement soil to their movement under load. To estimate $F_d$, the bearing layer of the foundation soil is selected according to the engineering-geological section, in which the lower ends of the piles are arranged (see above). The choice is made in favor of the most durable soil layer, less compressible in comparison with other upstream engineering-geological elements. Weak soils (varieties of clayey soils of fluid-plastic and fluid consistency, loose sands, peat, etc.) are not recommended to be used as a bearing layer of the base of combined foundation.

The value of the bearing capacity of the hanging pile $F_d$ can be pre-determined analytically by the physical and mechanical characteristics of the base soil using the formula (7.8) according to SP 24.13330.2011 – Pile foundations:

$$F_d = \gamma_c \cdot \left( \gamma_{cR} \cdot R \cdot A + u \cdot \sum_{i=1}^{n} \gamma_{cf} \cdot f_i \cdot h_i \right). \quad (4)$$

Considering the data for determining the bearing capacity of injection piles $F_d$ according to condition (4), the following should be noted. At the initial stage of performing work on their device, a metal injector is pressed into the ground without its excavation, which is similar to the method of immersing piles by indentation. But the supply of fine-grained mobile concrete under pressure into the well (through the injector) and its further pressure testing have common features of the technological processes of the injection pile device. Therefore, the peculiarities of the injection pile device do not allow to fully use the values of the coefficients $\gamma_{cR}$ and $\gamma_{cf}$ provided in the regulatory and other technical literature. Thus, formula (4) for injection piles often leads to significant reserves of their bearing capacity, and, often, does not reflect the features of the device technology. Consequently, the values of $F_d$ established by the calculation according to (4) should be considered as preliminary, which need additional verification. It should also be noted that the method for determining $F_d$ according to (4) is used more often for conditions of new construction and does not take into account the stress-strain state of the soil of the foundation of the building being reconstructed. In recent years, methods for determining the bearing capacity $F_d$ of injection piles for foundations of reconstructed buildings have been developed, which are based on the results of experimental and theoretical studies. The results of these studies can be found in the papers [7, 9, 10, 13].

Foundation design. When designing foundations, taking into account their reinforcement, first analyze the data on loads $N_1$ and $N_2$ at the floor elevations of the first (basement) floor before and after the re-construction of the building (Fig. 4). The planned number of injection piles required to strengthen the foundation, as well as the choice of the method of transferring additional load ΔN from the building $\Delta N = N_2 - N_1$, to them significantly depends on the

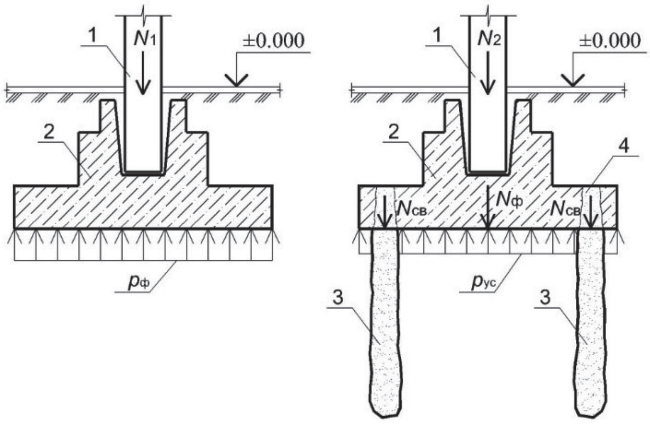


*Figure 4. Scheme of distribution of external load from the reconstructed building between the elements of the foundation: a – before its strengthening; b – after reinforcement with injection piles; 1 – column; 2 – separate foundation; 3 – injection pile; $N_1$ is the load on the foundation before the reconstruction of the building; $N_2$ – load on the foundation after the re-construction of the building; $N_f$ is the load transferred to the slab part of a separate foundation under conditions of building reconstruction; $N_{sv}$ – the same, transferred to injection piles; $p_f$ – pressure on the sole of a separate foundation before its strengthening; $p_{yc}$ – pressure along the bottom of a separate foundation after its strengthening*

difference between these loads $N_1$ and $N_2$. Then, the calculation of the proportion of loads transmitted to the base soil by the base of the foundation $N_f$ and injection piles $N_w$. The share of $N_f$ is calculated taking into account the strength (bearing capacity) of the foundation soil and the pressure pf (load $N_1$) from the building before its reconstruction [12, 17]. Next, they begin to design the foundations, taking into account its reinforcement with injection piles (Fig. 4). Assign the geometric dimensions of additional structural elements of the reinforcement (beams, clips, grillages, struts, tides, etc.). In the presence of significant defects in the pocket foundation (for individual shallow foundations) or in the wall, slab parts of the foundation (for strip foundations), their preliminary restoration is envisaged.

***Verification calculations of the base of foundations.*** Verification calculations of the base of foundations, taking into account their reinforcement by injection piles, are performed according to the first and second groups of limiting states (in terms of bearing capacity and deformations). The complexity of such calculations lies in the fact that the base of the existing foundation is in a stressed state, as it experiences pressure from the reconstructed building, which must be taken into account in the design. At present, a limited number of methods for calculating the settlement [8, 18] and the bearing capacity [2, 6, 10] of the base of foundations, taking into account their strengthening, have been developed, which have not yet been included in the regulatory documents of Russia. Therefore, quite often in the design of the foundations of reconstructed buildings, taking into account their strengthening, regulatory and technical literature is used, prepared for the design of foundations in conditions of new construction [2, 3, 5, 19].

***Strength calculation of foundation elements and their design.*** The structural elements of the reinforced foundation and the nodes of their interface are calculated in accordance with the methods given in the normative and technical literature (SP 63.13330.2018; SP 16.13330.2017; SP 14.13330.2018; Reference book of geotechnics, 2016, etc.), taking into account those in force on these elements are actual loads and actions. Such structural elements include reinforced concrete clips, stiffening belts, beams, building elements, as well as metal (steel) struts, struts, shelves, anchors, etc. Taking into account the calculations performed, they are designed [17, 20, 21].

Preparation of working documentation. Design of reinforcement of foundations of reconstructed buildings with injection piles provides for the preparation of working documentation. The documentation includes detailed drawings for the performance of work, specifications for the materials used, a work production project (WPP), etc. The working documentation also indicates measures for unloading the building structures of the aboveground part of the building, as well as safety measures during production work to strengthen the foundations.

## CONCLUSION

It has been established that in the conditions of reconstruction of buildings in clay soils, a method of strengthening (increasing the bearing capacity) of shallow foundations by transferring part of the load from aboveground building structures to additionally arranged injection piles is widely developed. At the same time, their device is performed by pressing a metal injector into the ground, followed by injecting a fine concrete mixture into the well. Fine-grained mobile concrete mixture is supplied to the wells formed in this way under pressure, which performs pressure testing. The experience of the reconstruction of buildings shows that by now the main approaches to the design of reinforcement of shallow foundations known from technical and regulatory literature have been developed. However, for the conditions of reconstruction and restoration of buildings, the issues of justifying the bearing capacity of injection piles used to strengthen, design combined foundations, methods of their verification calculations have not yet received proper development. The solution of these issues will make it possible to properly and adequately organize the work of specialists.

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# RELIABILITY ANALYSIS OF RHS STEEL TRUSSES JOINTS BASED ON THE P-BOXES APPROACH

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**Abstract**: Reliability is one of the main indicators of structural elements mechanical safety. The choice of stochastic models is an important task in reliability analysis for describing the variability of random variables with aleatory and epistemic uncertainty. The article proposes a method for the reliability analysis of RHS (rectangular hollow sections) steel truss joints based on p-boxes approach. The p-boxes consist of two boundary distribution functions that create an area of possible distribution functions of a random variable. The using of p-boxes make possible to model random variables without making unreasonable assumptions about the exact cumulative distribution functions (CDF) or the exact values of the CDF parameters. The developed approach allows to give an interval estimate of the non-failure probability of the truss joints, which is necessary for a comprehensive (system) reliability analysis of the entire truss.

**Keywords**: reliability, truss, joint, p-box, random variable, safety, uncertainty, probabilistic design

# МЕТОД ОЦЕНКИ НАДЕЖНОСТИ УЗЛОВ СТАЛЬНЫХ ФЕРМ ИЗ ГНУТОСВАРНЫХ ПРОФИЛЕЙ (ГСП) НА ОСНОВЕ P-БЛОКОВ

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**Аннотация**: Надежность является одним из главных показателей механической безопасности элементов строительных конструкций. При вероятностной оценке надежности важной задачей является выбор стохастических моделей для описания изменчивости случайных величин с учетом алеаторной и эпистемологической неопределенности. В статье предложен метод расчета надежности узлов стальных ферм из гнутосварных профилей на основе p-блоков (p-boxes). P-блоки состоят из двух граничных функций распределения, создающих область возможных функций распределения случайной величины. Использование p-блоков позволяет моделировать случайные величины без необоснованных предположений о точном вероятностном распределении или точных значениях параметров функций распределения вероятностей. Разработанный подход позволяет дать в интервальной форме оценку вероятности безотказной работы узла фермы, которая необходима для комплексной (системной) оценки надежности всей фермы.

**Ключевые слова**: надежность, ферма, узел, p-блок, случайная величина, безопасность, неопределенность, вероятностное проектирование

## INTRODUCTION

Ensuring safety and reliability is one of the key aims in the design and inspection of structural elements. Structural reliability analysis is a powerful tool for evaluating the structural safety level. As noted in Eurocode 0 "Basis of structural design", reliability is "an ability of a structure or a structural member to fulfil the specified requirements, including the design working life, for which it has been designed. Reliability is usually expressed in probabilistic terms, and it covers safety, serviceability and durability of a structure".

The paper [1] also notes that theory and methods for structural reliability have been developed substantially in the last few years and they are actually a useful tool for evaluating rationally the safety of complex structures or structures with unusual designs. Recent evolution allows to anticipate that their application will gradually increase, even in the case of common structures".

The existence of the randomness in structural parameters and loads makes the reliability analysis essential to structural safety [2].

One of the main tasks in structural reliability analysis is the choice of accurate stochastic models for random variables, taking into account all statistical uncertainties. The article [3] notes, that structural reliability analysis requires addressing all sources of uncertainty. To design reliable structural products, different forms of uncertainty in engineering provide a challenge for reliability analysis and carry higher requirements on uncertainty analysis.

In accordance with [3, 4, etc.], there are two fundamentally distinct forms of uncertainty in the structural probabilistic design. The first is variability that arises from environmental stochasticity, inhomogeneity of materials, fluctuations in time, variation in space. Variability is sometimes called Type I uncertainty or aleatory uncertainty to emphasize its relation to the randomness in gambling and games of chance. It is also sometimes called irreducible uncertainty because, in principle, it cannot be reduced by further empirical study (although it may be better characterized). The second kind of uncertainty is the incertitude that comes from scientific ignorance, measurement uncertainty, censoring, or other lack of knowledge. This is sometimes called Type II uncertainty or epistemic uncertainty. In contrast with aleatory uncertainty, epistemic uncertainty is sometimes called reducible uncertainty because it can generally be reduced by additional empirical effort at least in principle.

There are different approaches in structural reliability analysis for uncertainty modeling. One of the most common approaches is p-box (probability box) approach [4, 5]. P-boxes were introduced as interval-type bounds on cumulative distribution functions by Williamson and Downs [6] in 1990. In general, p-box is an area bounded by cumulative distribution functions which is covers the real cumulative distribution function (unknown due to the aleatory and epistemic uncertainties).

Trusses are the common part of many structures. The safety and reliability of whole building or structure depends from the trusses reliability. The complex truss reliability consist of truss elements (bars and joints) reliabilities. It is necessary to have the reliability values of the bars and joints of the truss for a comprehensive assessment of the entire truss reliability.

The article presents the approaches for RHS (rectangular hollow sections) steel trusses joints reliability analysis based on p-boxes for uncertainty modeling.

## METHODS

Four limit state criteria for each joint element should be taken into account in reliability analysis of typical RHS truss: bearing capacity of the chord wall to which the brace element is adjacent (I); buckling of the chord side wall (II); bearing capacity of the brace element near the abutment to the chord (III); strength of welds (IV).

The Appendix L of SP 16.13330.2011 «Steel structures» presents the following inequality for bearing capacity of the chord wall to which the brace element is adjacent:

$$\left(N+\frac{1.5M}{d_b}\right)\frac{(0.4+1.8g/b)\cdot f\cdot\sin\alpha}{\gamma_c\cdot\gamma_d\cdot\gamma_D\cdot R_y\cdot t^2\cdot\left(b+g+\sqrt{2Df}\right)}\le 1, \quad (1)$$

where $N$ is an initial force in a brace element; $M$ is a bending moment in a brace element; $R_y$ is a ultimate steel stress (design yield strength); is a force factor: $\gamma_d$ = 1.0 if a brace element is compressed, $\gamma_d$ = 1.2 if a brace element in tension; $\gamma_c$ is a factor of operation conditions; $\gamma_D$ is a factor of force influence in the chord; other parameters in (1) are shown on Fig. 1.

Bending moment $M$ occurs in most types of truss structures from the truss self-weight load and misalignments (eccentricities) of the truss member's connections. As the forces in the truss bars from the self-weight are a small variable value, the bending moment $M$ is considered as a deterministic (constant) value.

In case if the condition $\dfrac{F}{A\cdot R_y}>0.5$ met, then $\gamma_D = 1.5 - F/(R_y A)$ and the mathematical model of the limit state (1) based on SP 16.13330.2011 can be written as:

$$\left(\tilde{N}+\frac{1.5M}{d_b}\right)\cdot k_I+\frac{\tilde{F}}{A}\le 1.5R_y, \quad (2)$$

where $k^I=\dfrac{(0.4+1.8g/b)\cdot f\cdot\sin\alpha}{\gamma_c\cdot\gamma_d\cdot t^2\cdot\left(b+g+\sqrt{2Df}\right)}$.

Wave line above the symbols is the indicator of random variables in mathematical models of limit states.

As the force in brace member $\tilde{N}$ and in chord member $\tilde{F}$ depends on the joint load $\tilde{P}$ and the truss geometric parameters, then inequality (2) can be written as:

$$\left(\delta_N\tilde{P}+\frac{1.5M}{d_b}\right)\cdot k_I+\frac{\delta_F\tilde{P}}{A}\le 1.5R_y, \quad (3)$$

where $\delta_N$ and $\delta_F$ are factors for internal forces in truss bars depending on the truss geometric parameters. The inequality (2) can be converted to the next form:

$$\tilde{P}\le P_{ult}^{I}, \quad (4)$$

where $P_{ult}^{I}=\left(1.5R_y-\dfrac{1.5M}{d_b}k_I\right)/\left(\delta_N\cdot k_I+\dfrac{\delta_F}{A}\right)$.

In other cases, the reliability analysis of the truss joints according to the criterion of the bearing capacity of the chord wall can be carried out using the mathematical model:

$$\delta_N\tilde{P}\le\frac{R_y}{k_I}-\frac{1.5M}{d_b}\ \text{ or }\ \tilde{P}\le P_{ult}^{I'}, \quad (5)$$

where $P_{ult}^{I'}=\dfrac{1}{\delta_N}\left(\dfrac{R_y}{k_I}-\dfrac{1.5M}{d_b}\right)$.

The geometric parameters (Fig. 1) in the factor $k_I$, can be adopted as constant values on the structural design stage. After the construction of the truss structure, the reliability can be recalculated taking into account changes in the parameter $k_I$ by the actual values of geometric parameters after manufacturing and installation of the truss in site. The mathematical model for the truss's joint reliability analysis according to the buckling of the chord side wall criterion (II) can be written as:

$$\tilde{P}\le P_{ult}^{II}, \quad (6)$$

where $P_{ult}^{II}=\dfrac{1}{\delta_N}\left(\dfrac{2\gamma_c\gamma_t kR_y td_b}{\sin^2\alpha}\right)$.

The mathematical model for the RHS truss's joint reliability analysis according to the criterion of bearing capacity of the brace element near the abutment to the chord (III) can be written as:

$$\tilde{P}\le P_{ult}^{III}, \quad (7)$$

where $P_{ult}^{III}=\dfrac{1}{\delta_N}\left[\dfrac{\gamma_c\cdot\gamma_d\cdot k\cdot R_{yd}\cdot A_d}{(1.4+0.018D/t)\cdot\sin\alpha}-\dfrac{M}{2d_b}\right]$.

And, the mathematical model for the RHS truss's joint reliability analysis according to the criterion of strength of welds (IV) is:

$$\widetilde{P} \leq P_{ult}^{IV}, \tag{8}$$

where

$$P_{ult}^{IV} = \frac{1}{\delta_N}\left(\frac{4 \cdot \beta_f \cdot k_f \cdot d_b \cdot \gamma_c \cdot R_{wf}}{\left[1 + 0.01\left(3 + \frac{5d}{D} - \frac{0.1 d_b}{t_d}\right)\frac{D}{t}\right] \cdot \sin\alpha} - \frac{M}{2d_b}\right),$$

$R_{wf}$ is a design resistance of the welding corner seams to the shear; $k_f$ is a leg of a fillet weld; $\beta_f$ is a fillet weld factor.

The design mathematical models (4)–(8) contain joint loads of $P$ (Fig. 1). To account for deterministic non-joint loads (self-weight and bracing/purlin weight), an equivalent joint load $P^{eq}$ is selected, which creates similar forces $F$ and $N$ in the truss elements.

There are possible some different limit state equations. For example, "Design guide for rectangular hollow section (RHS) joints under predominantly static loading" [7] proposed 5 limit state criterion: chord face plastification, local yielding of brace, chord punching shear, chord shear, chord side wall failure. In general, there are the same shape of presented in [7] equations and equations (4)–(8). A similar observation is true for the equations in the paper [8]. The reliability analysis approach presented below can be easily adapted to the design equations in [7, 8, etc.].

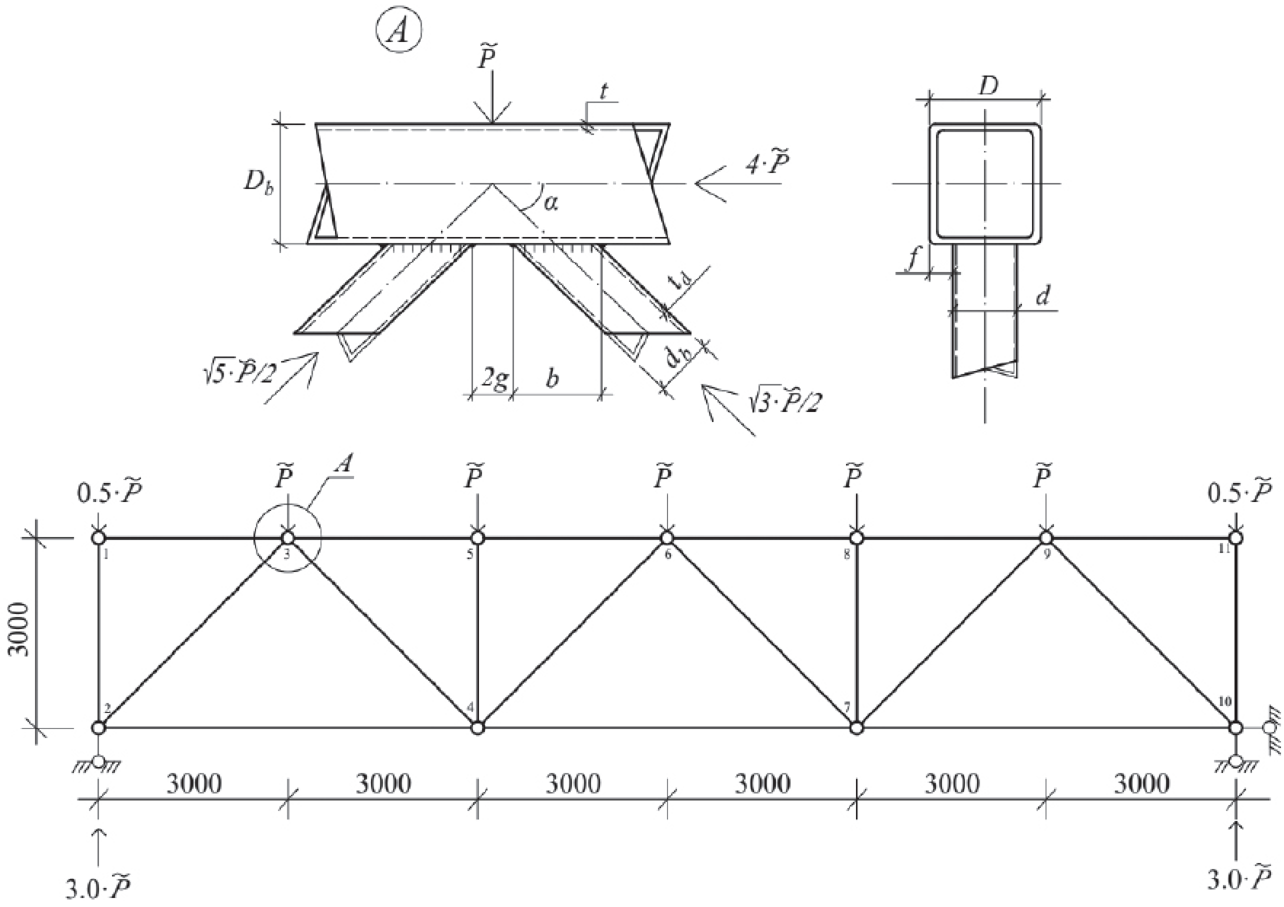


*Figure 1. Design scheme of truss and joint No. 3* *Figure 1. Design scheme of truss and joint No. 3*

## RESULTS AND DISCUSSION

Table 1 presents the forces in the truss (Fig. 1) bars. As noted above, there are different p-box models for random variables depending on quantity and quality of statistical data. For example, if the cumulative distribution function of a random variable is known – normal distribution, but there are interval estimations of expected value $m_X \in [\underline{m}_X;\ \overline{m}_X]$ and standard deviation $S_X \in [\underline{S}_X;\ \overline{S}_X]$, the following boundary distribution functions can be used for the p-box model:

$$\underline{F}_X(x) = \begin{cases} F^{norm}\left(\underline{m}_X,\ \overline{S}_X\right) & if\ x < \underline{m}_X \\ F^{norm}\left(\underline{m}_X,\ \underline{S}_X\right), & if\ x \ge \underline{m}_X \end{cases}, \quad (9)$$

$$\overline{F}_X(x) = \begin{cases} F^{norm}\left(\overline{m}_X,\ \underline{S}_X\right) & if\ x < \overline{m}_X \\ F^{norm}\left(\underline{m}_X,\ \overline{S}_X\right), & if\ x \ge \overline{m}_X \end{cases}, \quad (10)$$

where $F^{norm}()$ is a cumulative distribution function of normal distribution.

If there are extremely small statistical data about random variable, the fuzzy distribution functions can be used [9]:

$$\overline{F}_X(x) = \begin{cases} 1 - \exp\left[-\left(\dfrac{x - a_X}{b_X}\right)^2\right] & if\ x > a_X \\ 0\ otherwise \end{cases}, \quad (11)$$

$$\underline{F}_X(x) = \begin{cases} \exp\left[-\left(\dfrac{x - a_X}{b_X}\right)^2\right] & if\ x < a_X \\ 1\ otherwise \end{cases}, \quad (12)$$

where $a_X$ is a "mean" value which is calculated as $a_X = 0.5(X_{max} + X_{min})$; $b_X$ is a measure of variance which is calculated as $b_X = 0.5(X_{max} - X_{min})/\sqrt{-\ln\alpha}$, where $X_{max}$ and $X_{min}$ is a maximum and a minimum value in the set of $\{X\}$; $\alpha$ is a cut (risk) level [9]. The Gumbel distribution (or Generalized Extreme Value distribution Type-I) is often used for stochastiseveral c modeling of the snow load [10, 11]. If there is uncertainty in estimation of snow load statistical parameters $m_X \in [\underline{m}_X;\ \overline{m}_X]$ and $S_X \in [\underline{S}_X;\ \overline{S}_X]$, the following boundary distribution functions (p-box model) can be used:

$$\underline{F}_X(x) = \begin{cases} \exp\left[-\exp\left(\dfrac{\left[\underline{m}_X - 0.45\cdot\overline{S}_X\right] - x}{0.78\cdot\overline{S}_X}\right)\right] & if\ x < \underline{m}_X \\ \exp\left[-\exp\left(\dfrac{\left[\underline{m}_X - 0.45\cdot\underline{S}_X\right] - x}{0.78\cdot\underline{S}_X}\right)\right] & if\ x \ge \underline{m}_X \end{cases},$$

$$\overline{F}_X(x) = \begin{cases} \exp\left[-\exp\left(\dfrac{\left[\overline{m}_X - 0.45\cdot\underline{S}_X\right] - x}{0.78\cdot\underline{S}_X}\right)\right] & if\ x < \overline{m}_X \\ \exp\left[-\exp\left(\dfrac{\left[\overline{m}_X - 0.45\cdot\overline{S}_X\right] - x}{0.78\cdot\overline{S}_X}\right)\right] & if\ x \ge \overline{m}_X \end{cases}.$$

*Table 1. Forces in the truss (Fig.1) bars*

| Bar | Force | $\delta_N$ | Bar | Force | $\delta_N$ |
|---|---|---|---|---|---|
| 1-2, 10-11 | $-0{,}5\widetilde{P}$ | -0.5 | 4-5, 7-8 | $-\widetilde{P}$ | -1 |
| 1-3, 9-11 | 0 | 0 | 4-6, 6-7 | $-\widetilde{P}/\sqrt{2}$ | $-1/\sqrt{2}$ |
| 2-3, 9-10 | $-\sqrt{5}\widetilde{P}/2$ | $-\sqrt{5}/2$ | 4-7 | $+4.5\cdot\widetilde{P}$ | +4.5 |
| 2-4, 7-10 | $+2.5\widetilde{P}$ | +2.5 | 3-4, 7-9 | $+\sqrt{3}\widetilde{P}/2$ | $+\sqrt{3}/2$ |
| 3-5, 8-9, 5-6, 6-8 | $-4\widetilde{P}$ | -4 | | | |

*Table 2. Statistical data about loads on the truss*

| No. | Joint load | Distribution | Expected value of force $P$, kN | Deviation of force $P$, kN |
|---|---|---|---|---|
| 1 | Load from the truss self-weight and weight of the braces ($P^{eq}$) | Constant value | 4.5 | - |
| 2 | Ribbed reinforced concrete slabs weight | Normal, (9)-(10) | [70.0; 73.0] | [1.0; 3.0] |
| 3 | Structural insulation weight | Fuzzy, (11)-(12) | $a_x$=45.0 | $b_x$=2.6 |
| 4 | Roof membrane weight | Normal, (9)-(10) | [2.5; 3.0] | 0.3 |
| 5 | Snow load | Gumbel [10] | $\alpha \in \left[\underline{\alpha};\ \overline{\alpha}\right] = \left[35.0;\ 40.0\right]$ | $\beta \in \left[\underline{\beta};\ \overline{\beta}\right] = \left[8.0;12.0\right]$ |

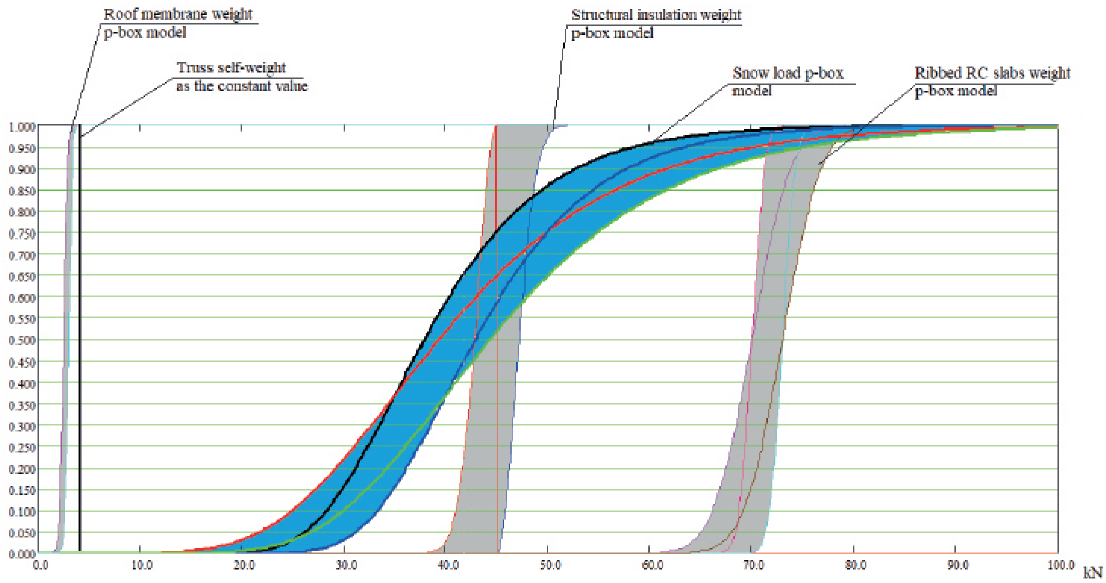


*Figure 2. P-boxes plots of truss loads by the Table 2*

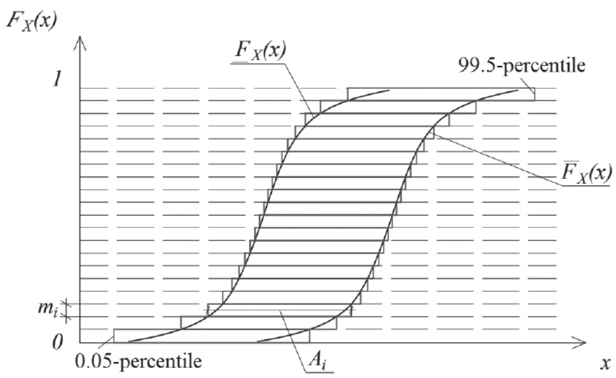


*Figure 3. P-box discretization on focal elements $A_i$ with basic probabilities mi*

Interval estimates for statistical parameters are refined for snow regions or individual localities depending on the available statistical sample of weather station data on yearly snow cover height maxima (or the snow weight directly).

Fig. 2 presents the p-boxes plots of joint loads by statistical data of the Table 2. There is a problem of different type p-boxes addiction for stochastic estimation of the total load.

For p-boxes addiction, it is necessary to transform it into a Dempster-Shafer type structure [4]. Continuous boundary cumulative distribution functions in p-boxes are sampled into a certain number of blocks (Fig. 3). The distributions during discretization are limited to 0.05 and 99.5 percentiles [4].

The basic probability (discretization level) $m_i$ is usually takes as $m_i = 0.01$ [4], and $\sum_{i=1}^{n} m_i = 1$.

In such case, we get 100 interval estimations $A_i$ of random variable $X$ (Fig. 3). For example, one hundred intervals can be obtained by discretization the p-box of the joint load from the snow weight: $\left[\underline{P}_1^{snow};\ \overline{P}_1^{snow}\right]$, $\left[\underline{P}_2^{snow};\ \overline{P}_2^{snow}\right]$, … $\left[\underline{P}_{100}^{snow};\ \overline{P}_{100}^{snow}\right]$. Similarly, one hundred intervals can be obtained for joint loads, for example, joint load form reinforced concrete slabs weight: $\left[\underline{P}_1^{r};\ \overline{P}_1^{r}\right]$, $\left[\underline{P}_2^{r};\ \overline{P}_2^{r}\right]$, … $\left[\underline{P}_{100}^{r};\ \overline{P}_{100}^{r}\right]$.

After p-boxes discretization, it is possible to summarize the p-boxes [4, 6] from different joint loads to the one common p-box. The table form (Table 1) is useful for two p-boxes addiction. Boundary distribution functions of p-boxes after discretization are can be presented as Dempster-Shafer [12] structures. Detailed information on mathematical operations with Dempster-Shafer structures is given in the research [4].

*Table 3. Addiction of two Dempster-Shafer structures [4]*

| | $\left[\underline{P}_1^{snow};\ \overline{P}_1^{snow}\right]$, 0.01 | $\left[\underline{P}_2^{snow};\ \overline{P}_2^{snow}\right]$, 0.01 | … | $\left[\underline{P}_{100}^{snow};\ \overline{P}_{100}^{snow}\right]$, 0.01 |
|---|---|---|---|---|
| $\left[\underline{P}_1^{r};\overline{P}_1^{r}\right]$, 0.01 | $\left[\underline{P}_1^{snow}+\underline{P}_1^{r};\overline{P}_1^{snow}+\overline{P}_1^{r}\right]$, 0.0001 | $\left[\underline{P}_2^{snow}+\underline{P}_1^{r};\overline{P}_2^{snow}+\overline{P}_1^{r}\right]$, 0.0001 | … | $\left[\underline{P}_{100}^{snow}+\underline{P}_1^{r};\overline{P}_{100}^{snow}+\overline{P}_1^{r}\right]$, 0.0001 |
| $\left[\underline{P}_2^{r};\ \overline{P}_2^{r}\right]$, 0.01 | $\left[\underline{P}_1^{snow}+\underline{P}_2^{r};\overline{P}_1^{snow}+\overline{P}_2^{r}\right]$, 0.0001 | $\left[\underline{P}_2^{snow}+\underline{P}_2^{r};\overline{P}_2^{snow}+\overline{P}_2^{r}\right]$, 0.0001 | … | $\left[\underline{P}_{100}^{snow}+\underline{P}_2^{r};\overline{P}_{100}^{snow}+\overline{P}_2^{r}\right]$, 0.0001 |
| … | … | … | … | … |
| $\left[\underline{P}_{100}^{r};\ \overline{P}_{100}^{r}\right]$, 0.01 | $\left[\underline{P}_1^{snow}+\underline{P}_{100}^{r};\overline{P}_1^{snow}+\overline{P}_{100}^{r}\right]$, 0.0001 | $\left[\underline{P}_2^{snow}+\underline{P}_{100}^{r};\overline{P}_2^{snow}+\overline{P}_{100}^{r}\right]$, 0.0001 | … | $\left[\underline{P}_{100}^{snow}+\underline{P}_{100}^{r};\overline{P}_{100}^{snow}+\overline{P}_{100}^{r}\right]$, 0.0001 |

The elements in Table 1 are written as follows $A_i \in \left[\underline{P}_i;\overline{P}_i\right]$, $m_i$, where $\underline{P}_i$; $\overline{P}_i$ are lower and upper bounds of the focal element $A_i$, $m_i$ is a basic probability for the focal element $A_i$.

The lower and upper boundary distribution functions can be constructed from these intervals based on the provisions of the Dempster-Shafer evidence theory [4, 12]. These boundary distribution functions will create a p-box of joint load $P$. For more than two p-boxes, a combination of the sums of all dispatching options is considered. Information about automating the interval calculation process can be found in [4].

The p-box (Fig. 4) can be obtained by addiction of p-boxes and constant value of different joint loads in the Table 2. Such p-box is presented as empirical boundary distribution functions $\underline{F}_P(P)$ and $\overline{F}_P(P)$ consisting of possible combinations of different joint loads. The probability of non-failure can be calculated as:

$$\underline{P} = \Pr(\widetilde{P} \le P_{ult}) = \underline{F}_P(P_{ult}),$$

$$\overline{P} = \Pr(\widetilde{P} \le P_{ult}) = \overline{F}_P(P_{ult}).$$

For example (Fig, 4), the non-failure probability interval for truss bar 2–3 in joint 3 by the bearing capacity of the brace element near the abutment to the chord is $\underline{P}_{2-3}^{III} = \underline{F}_P\left(P_{ult}^{III}\right)$, $\overline{P}_{2-3}^{III} = \overline{F}_P\left(P_{ult}^{III}\right)$. The reliability is presented in interval form $[\,\underline{P}_{2-3}^{III};\overline{P}_{2-3}^{III}\,]$.

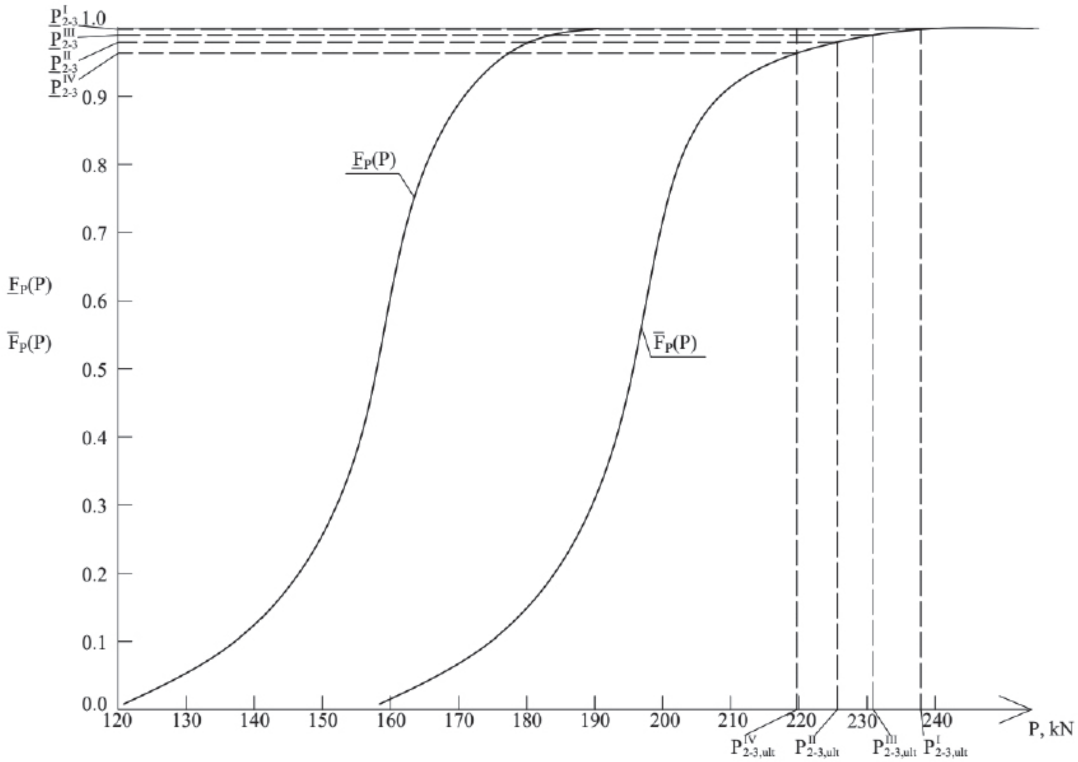


*Figure 4. The p-box obtained by addiction of the b-boxes in the Table 2*

There will be several reliability intervals for every truss joints $[\,\underline{P}_{i-j}^{z};\overline{P}_{i-j}^{z}\,]$, where *z* is an indicator of limit state criterion, *i-j* is an indicator of truss bar.

A comprehensive assessment of the reliability of the selected truss joint can be obtained from a given subset of intervals $[\,\underline{P}_{i-j}^{z};\overline{P}_{i-j}^{z}\,]$ using the following equations [13]:

$$\begin{cases} \underline{P} = \max\left(0,\ \sum\limits_{i-j}^{n} \underline{P}_{i-j}^{z} - (n-1)\right), & z = \{I, II, III, IV\}, \\ \overline{P} = \min(\overline{P}_{i-j}^{z}) & \end{cases}$$

where n is a number of brace elements in the truss joint.

The advantage of using mathematical models (4)–(8) is that it simple to use different limit state models [7, 8, etc.] in presented algorithm replacing $P_{ult}$.

The mathematical models (4)–(8) can be presented for existing steel trusses as:

$$\widetilde{P} \le \widetilde{P}_{ult}\left(\widetilde{\sigma}_{s,ult}\right),$$

where $\widetilde{\sigma}_{s,ult}$ is an steel ultimate stress obtained by the steel samples tests results.

Such models can be used in cases when there is no information about steel of existing truss or at degradation of physical and mechanical properties of structural steel. Algorithms for analyzing the reliability of truss joints can be taken from paper [14] in such design situations.

Values of truss joints reliability can be used in structural reliability analysis of truss as a mechanical system or in decision making and risk analysis problems [15, 16].

## CONCLUSIONS

1. The p-boxes can be used for more careful modeling of random variables in the structural reliability analysis problems;
2. If the reliability interval [$\underline{P}$; $\overline{P}$] is too wide for decision making, it is necessary to obtain narrower boundary distribution functions by increasing the quantity or quality of statistical data about random variables;
3. The proposed reliability analysis method for HSS truss joints can be used in truss reliability analysis as a structural system.

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# INDOOR AIR QUALITY REQUIREMENTS FOR CIVIL BUILDINGS IN RUSSIAN REGULATIONS IN COMPARISON WITH INTERNATIONAL GREEN STANDARDS

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**Abstract**: A new trend combining the concept of "green" buildings with the idea of preserving and strengthening peoples' health in order to eliminate sick building syndrome and building related illnesses has been observed worldwide. The COVID – 19 pandemic consequences outlined the necessity of updating the regulatory framework considering health preserving built environment principles in order to create sustainable and comfortable living environments. Indoor air quality directly correlates with human health: exposure to polluted air increases the risk of cardiovascular disease, myocardial ischemia, angina pectoris, hypertension and heart disease. It is known that indoor air quality depends not only on ambient air quality, but also on indoor sources of chemical and biological pollutants. Existing regulatory framework does not cover the civil buildings indoor sources of air pollution topic. This article discusses the terms of the Russian national technical and hygienic standards concerning the indoor air quality. A comparative analysis of the Russian Federation regulatory framework that refers the civil buildings indoor air quality with international "green" standards was carried out. Based on the analysis, the necessity to update the Russian regulatory framework is highlighted.

**Keywords**: sustainable development, environmental safety, regulatory framework, green standards, design criteria, comfort, health preserving.

# ТРЕБОВАНИЯ К КАЧЕСТВУ ВОЗДУХА ВНУТРЕННЕЙ СРЕДЫ ПОМЕЩЕНИЙ ГРАЖДАНСКИХ ЗДАНИЙ В РОССИЙСКИХ НОРМАХ В СРАВНЕНИИ С МЕЖДУНАРОДНЫМИ ЗЕЛЕНЫМИ СТАНДАРТАМИ

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**Аннотация**: Сегодня во всем мире формируется новый тренд в строительстве, объединяющий концепцию «зеленого» строительства с идеей сохранения и укрепления здоровья людей, целью которого является борьба с «синдромом больного здания» и «болезней, связанных со зданием». Последствия пандемии COVID – 19 подчеркнули необходимость актуализации нормативно-технической базы с учетом принципов здоровьесбережения для создания устойчивой и комфортной среды жизнедеятельности человека. Качество воздуха внутренней среды зданий напрямую связано со здоровьем: воздействие на человека загрязненного воздуха увеличивает риск сердечно-сосудистых заболеваний, ишемии миокарда, стенокардии, гипертонии и заболеваний сердца. Фактически, известно, что помимо качества наружного воздуха, качество воздуха в помещении также зависит от наличия внутренних источников загрязнения, концентрации химических и биологических загрязнителей. Существующая нормативно – техническая база не охватывает тему внутренних источников загрязнения воздуха в гражданских зданиях.

В статье рассмотрены требования российских нормативно-технических и санитарно-эпидемиологических нормативных документов в отношении качества воздуха внутренней среды гражданских зданий. Проведен сравнительный анализ отечественных нормативно-технических и санитарно-эпидемиологических нормативных документов с международными «зелеными» стандартами, на основании которого показана необходимость актуализации российской нормативно-технической базы с учётом международного опыта «зелёных» стандартов.

**Ключевые слова**: устойчивое развитие, экологическая безопасность, нормативно-техническая база, «зеленые» стандарты, критерии проектирования, комфорт, здоровьесбережение.

## INTRODUCTION

Currently, the professional construction community information field worldwide is largely filled with the topic of creating a comfortable living environment. Within the framework of the renovation of the housing stock under the renovation program, as well as the federal projects "Housing" and "Ensuring a sustainable reduction of unsuitable housing stock" of the national project "Housing and the Urban Environment", the Government of the Russian Federation sets the task of creating a comfortable living environment. Analyzing the regulatory and technical base, research, scientific publications of leading experts in the field of design and construction, as well as feedback from users of buildings, it is noted that the setting of the task of creating a comfortable living environment and the methods of its implementation are interpreted in different ways by different specialists, which, often leads to unsatisfactory results. Thus, the design of space-planning and engineering solutions for a comfortable indoor environment of premises is currently difficult due to the lack of formed conceptual framework that reveals the meaning of the term "comfort" and the concept of "comfortable environment", as well as a criteria list that determines the indoor environment quality in the Russian Federation regulatory and technical framework.

**Sick Building Syndrome and Green Buildings.** According to the World Health Organization (hereinafter WHO), a person spends about 90% of the time indoors. Accordingly, the quality of life, health and well-being of a person directly depends on the quality of the internal environment of the buildings in which we spend a significant part of our life. The main factors forming the indoor environment quality (IEQ) are indoor air quality (IAQ); thermal comfort; water quality; visual comfort; acoustic comfort; spatial comfort (space-planning solutions); finishing materials. IEQ can negatively affect occupants' physical health (e.g., asthma exacerbation and respiratory allergies) through poor air quality, extreme temperatures, excess humidity, and insufficient ventilation and psychological health (e.g., depression and stress) through inadequate lighting, acoustics, and ergonomic design [1-4].

The term "sick building syndrome" (SBS) is used to describe situations in which building occupants experience acute health and comfort effects that appear to be linked to time spent in a building, but no specific illness or cause can be identified. The complaints may be localized in a particular room or zone or may be widespread throughout the building [1, 5, 18]. In contrast, the term "building related illness" (BRI) is used when symptoms of diagnosable illness are identified and can be attributed directly to airborne building contaminants [3, 5]. Studies have shown that employees with such health conditions are absent more often, lose more work hours, and are less productive than employees without these conditions [6-8].

A new trend in construction is now formed worldwide, combining the concept of "green" buildings with the idea of preserving and strengthening people's health. Designing a new building, architects, and engineers strive to take into account not only environmental factors, reduced energy consumption, but also how built environment affects the health, wellbeing, mood, and performance of building users. To solve these problems, interdisciplinary teams are created, in which, in addition to architects, engineers, builders, designers, medical experts and psychologists are involved. Health-preserving approach is a great contributor to designing socially, economically and ecologically sustainable buildings.

The lockdown due to COVID-19 pandemic has outlined that sustainable and comfortable indoor environment is crucial for a person to maintain physical and psychological health during several months of "individual isolation". The consequences of the pandemic, and the imminent risk of its repetition, highlight the necessity to apply a new concept of health, in terms of indoor well-being, to housing policy and building sector [9]. Moreover, COVID-19 pandemic has showed the need to rethink approaches to the design of the human life environment and the need to implement tools for integrating the concept of "health preservation" into the existing design, construction and building operation paradigm.

British Research Establishment Environmental Assessment Methodology (BREEAM) 1990 and Leadership in Energy and Environmental Design (LEED) 1993 were established as constantly improving and dynamically developing systems of design, construction and building operation regulations to construct sustainable, comfortable, energy effective, high performance buildings. To regulate the design of health-saving (health-improving) buildings' indoor environment, in 2014, the United States Green Building Council (USGBC) established the WELL Building Standard based on synergy of medical and engineering science. The Well Standard focuses its attention specifically on building users' health and wellbeing and consists of 115 design and building operation strategies criteria (59 design-referred criteria and 56 operation and management-referred criteria) divided in 10 concepts: air; water; nourishment; thermal comfort; sound; materials; mind; community; innovation. Compared to the widely used BREEAM and LEED building certification schemes the WELL Building Standard is the most precise and scrupulous in covering health and wellbeing topic [10].

In 2018, the Government of the Russian Federation approved the passport of the national project "Housing and Urban Environment", which includes four federal projects: Mortgage, Housing, Formation of a Comfortable Urban Environment, and Ensuring Sustainable Reduction of Unfit Housing. Within the framework of the indicated national project, the goal is to increase the volume of housing construction by 120 million square meters per year, with the following tasks:

• To update, by 2024, 279 units of existing regulatory and technical documents for the introduction of advanced technologies and the establishment of restrictions on the use of obsolete technologies in design and construction.

• Introduce, by 2024, 115 new regulatory and technical documents in construction to phase out the use of outdated technologies in design and construction.

Moreover, to accomplish the national project "Housing and Urban Environment" tasks, the Russian Government established the "Regulatory guillotine" in 2019. The "Regulatory Guillotine" is a tool for large-scale revision and cancellation of regulatory legal acts that negatively affect the overall regulatory environment.

By order of the Russian Federal Agency for Technical Regulation and Metrology dated September 15, 2016 №1315 Technical Committee for Standardization №366 "Green" technologies of the living environment and "green" innovative products" (TK 366) was created. TK 366 acts in order to form "green" standards for the advanced standardization of the promising technological base of the living environment and to ensure advanced technological development and accelerated implementation of scientific developments in production, conduct a full innovation cycle of research and development works, including the creation of samples of a "green" living environment [11]. TK 366 works on accomplishment of the national project "Housing and Urban Environment" tasks of developing new technical documents and updating existing technical regulatory framework. Today, the main TK 366 project is the development of Russian

National “Green” Standards system as a free and accessible tool for implementing sustainability approaches in the Russian building sector. The Russian Federal Agency for Technical Regulation and Metrology approved the development of a GOST R standard “Assessment of built environment parameters’ effects on human health” under the TK 366 by order dated March 18, 2020 №579, which is planned to be a regulatory tool of implementing health- preserving approaches in design, construction and building operation.

**Indoor Air Quality in Russian Regulatory Framework.**

In the Russian Federation, building sector regulatory framework is represented by both technical and hygienic standards. With regard to indoor air quality (IAQ) ventilation rates and air filtration requirements in different premises is regulated by “SP” while thresholds for particulate matters, volatile organic and volatile inorganic compounds is regulated by hygienic guidelines. In this article, a comparative analysis between Russian technical regulations, WELL and LEED concerning office building ventilation rates was conducted. SP 60.13330.2016 regulates minimal ventilation rates for public buildings’ premises. The LEED v4 and WELLv2 regulations concerning indoor air quality are based on ASHRAE 62.1-2010, which is why the analysis office building ventilation rates will be conducted using SP 60.13330.2016 and ASHRAE 62.1-2010. According to LEED v4, ventilation rates should match ASHRAE 62.1-2010. According to WELL v2, ventilation rates have to exceed ASHRAE 62.1-2010 by 60%. The ASHRAE 62.1-2004 standard edition announces a new approach in determining the calculated air exchange for public buildings. The ventilation rate is calculated by summing up the need, taking into account the occupant density, to supply fresh outside air directly for the breathing of a person and for diluting pollutants emitted in the room where a person is located. Oddly enough, the air exchange rate per person for the most typical public buildings’ premises has become lower than recommended in ASHRAE 62.1-1999 [12]. According to retrospective analysis, ventilation rates regulations in ASHRAE 62.1 – 2004/2010/2019 are identical for office premises as shown in table 1 below. In ASHRAE 62.1-1999 the air exchange rate in office premises was accepted as 10 (l/s) which is 36 ($m^3$/h per person) with an area rate of 14 ($m^2$/person) is:

$$36 / 14 = 2.6 \text{ (m}^3\text{/h*m}^2 \text{ of area)} \qquad (1)$$

In the ASHRAE 62.1 – 2004/2010/2019 versions exchange rate in office premises is 2,5 (l/s per person) which is 9 ($m^3$/h), adding 0,3 (l/s per $m^2$ of area) or 1.08 ($m^3$/h per $m^2$ of area), which at the same rate of 14 $m^2$/person is:

$$(9 + 1.08 * 14) = 24 \text{ (m}^3\text{/h per person)} \qquad (2)$$

$$24 / 14 = 1.7 \text{ (m}^3\text{/h*m}^2 \text{ of area)} \qquad (3)$$

Air exchange rate for office premises in ASHRAE 62.1 – 1999 compared to ASHRAE 62.1 – 2004/2010/2019 versions is 1,5 times less.

*Table 1. ASHRAE ventilation rates regulations retrospective analysis.*

| **Premise name** | **Ventilation rate ASHRAE 62.1-1999 $m^3$/(pers*hour)** | **Ventilation rate ASHRAE 62.1-2004 $m^3$/(pers*hour)** | **Ventilation rate ASHRAE 62.1-2010 $m^3$/(pers*hour)** | **Ventilation rate ASHRAE 62.1-2019 $m^3$/(pers*hour)** |
|---|---|---|---|---|
| Office | 36 | 24 | 24 | 24 |
| Breakrooms | 30 | 12 | 12 | 12 |

The trend towards a decrease [13] in the calculated air exchange for most public buildings, which reveals the resource and energy efficiency approach in ASHRAE and LEED standards, is obvious. Considering this background, the air exchange rates in SP 60.13330.2016 are overestimated for office premises. The ventilation rate for office premises in SP (60 $m^3$/h per person) exceeds ASHRAE 62.1-2010/LEED v4 (24 $m^3$/h per person) 2,5 times, which doesn't correspond to the energy efficiency approach. On the contrary, taking into account the building users' health and wellbeing approach, the SP 60.13330.2016 ventilation rate requirement (60 $m^3$/h per person) exceeds WELL v2 (39 $m^3$/h per person) by 1,5 times, which perfectly fits the health- preserving principles (Table 2). However, due to improper distribution of such large volumes of air in office premises workers may feel uncomfortable because of drafts [12]. Often, implementation of green practices results in poor IAQ: energy efficiency strategies that increase indoor pollutants, location of buildings near transportation emissions, and the use of natural ventilation in areas with elevated outdoor pollution [14].

*Table 2. Ventilation rates regulations comparison.*

| **Premise name** | **Ventilation rate SP60.13330.2016 $m^3$/(pers*hour)** | **Ventilation rate WELL v2 $m^3$/(pers*hour)** | **Норма воздухообмена по ASHRAE 62.1-2010 (LEED v4) $m^3$/(pers*hour)** |
|---|---|---|---|
| Office | 60 | 39 | 24 |
| Breakrooms | 60 | 19 | 12 |

According to "Regulatory guillotine" mechanism approved by the Government of the Russian Federation, 108 hygienic standards were decided to be revised, in order to introduce the unified hygienic standard "Hygienic standards of environmental factors" (HS 1.2-20) on 01.01.2021. In particular the hygienic standard "Threshold limit value (TLV) of pollutants in ambient air of urban areas" (HS 2.1.6.3492-17), which regulates threshold limit values (TLV) of particulate matters, volatile organic compounds (VOCs), semi-volatile organic compounds (SVOCs) and volatile organic compounds (VOC) is revised. Ambient air quality and indoor air quality are in direct correlation [14,15]. The migration of dust and toxic substances contained in the atmosphere is due to the aerodynamics of the airflows movement. When ventilating buildings by natural or mechanical ventilation, hazardous contaminants of ambient air enter the premises. Therefore, TLVs of pollutants in ambient air is crucial for IAQ. It is necessary to note that among others the hygienic standard "Threshold limit value (TLV) of pollutants in working zone air" HS 2.2.5.3532-18 is revised due to the "Regulatory guillotine". However, HS 2.2.5.3532-18 regulates IAQ of industrial buildings and cannot be implemented for civil ones, as sources of exposure for these building types are different. Long - term exposure to VOCs cause acute symptoms such as nose, throat, and eyes irritations, headaches, allergic skin reactions, nausea, dizziness. Moreover, several VOCs and SVOCs may not be immediate hazards but can lead to chronic health risks such as liver and kidney damage, the development of cancerous tumors, hematopoietic organs and the central nervous system damage, vascular atherosclerotic changes, chromosomal aberrations [16–21].

*Table 3. Improved and worsened TLVs of airborne contaminants.*

| Compounds | HS 2.1.6.3492-17 ($mg/m^3$) | | HS1.2-20 ($mg/m^3$) | |
|---|---|---|---|---|
| | TLVom | TLVda | TLVom | TLVda |
| Benzene | 0,3 | 0,1 | 0,3 | 0,06 |
| 1,3-Butadiene | 3 | 1 | 3 | 0,02 |
| Hexachloroethane | 0,05 | - | - | 0,05 |
| Tetrachloromethane | 4 | 0,07 | 4 | 0,04 |
| 1-Phenylethanone | 0,01 | - | 0,003 | - |
| Butylethylene | 0,4 | 0,985 | 0,4 | 0,085 |
| 1-Hydroxy-4-chlorobenzene | 0,0015 | 0,003 | 0,015 | 0,003 |
| Dimethylbenzene-1,2-dicarbonate | 0,03 | 0,007 | 0,03 | 0,01 |
| Potassium chloride | 0,03 | 0,01 | 0,3 | 0,1 |
| Ozone | 0,16 | 0,03 | 0,16 | 0,1 |
| Acrolein | 0,03 | 0,01 | 0,1 | 0,04 |
| Allyl acetate | 0,4 | - | 0,04 | - |
| Propyl pentanoate | 0,003 | - | 0,03 | - |
| Mercury | - | 0,0003 | 0,0006 | 0,0004 |
| Selenium dioxide | 0,1 | 0,05 | 0,25 | - |
| Odorant mixture of natural mercaptans with a mass content of ethanethiol 26 – 41%, isopropane-thiol 38 – 47%, sec-butanethiol 7 – 13% | 0,00005 | - | 0,012 | - |
| Tripropylamine | 0,4 | 0,02 | - | 0,25 |
| Chloroform | 0,1 | 0,03 | 0,15 | 0,03 |
| Trichlorofluoromethane | 100 | 1 | 100 | 10 |
| 1,1,1-trichloroethane | 2 | 0,2 | 2 | 1 |
| Carbon monoxide | 5 | 3 (8 h) | 23 | 10 (8 h) |
| Hydrofluoride | 0,02 | 0,005 | 0,02 | 0,014 |
| Chloroprene | 0,02 | 0,002 | 0,02 | 0,007 |
| Chloroethane | - | 0,2 | 30 | - |
| Vinyl chloride | - | 0,01 | 0,18 | 0,04 |

Comparative analysis of hygienic standard HS 1.2–20 addenda 1 “Ambient air of urban areas” and HS 2.1.6.3492-17 was conducted, the results are shown in table 3. TLVs for 15 compounds were introduced. Out of 658 compounds, TLVs were increased for 19 compounds and decreased for 6. TLVs of 618 compounds remained unchanged compared to HS 2.1.6.3492-17. For 62 compounds, average annual TLV (TLVaa) was established based on the additional probability of developing malignant neoplasms (cancer) in an individual throughout his life (an individual carcinogenic risk at the level of 1 x $10^{-4}$, corresponding to 1 additional case of cancer per 10 thousand population). Daily average TLV (TLVda) of benzene was decreased from 0,1 to 0,06 mg/$m^3$, which corresponds to the LEED v4 threshold. TLVda of tetrachloromethane was decreased from 0,07 to 0,04 mg/$m^3$, which corresponds to the LEED v4 threshold, comparative analysis between HS 2.1.6.3492-17, WELL v2 and LEED v4 was conducted in previous researches [21]. TLVda of trichlorofluoromethane was increased from 1 to 10 mg/$m^3$. TLVda of trichlorofluoromethane was increased from 0,2 to 1 mg/$m^3$, however the study [22] showed that 0,02 mg/$m^3$ is the most stringent requirement compared to WELL and LEED. TLVs of carbon monoxide were dramatically increased. One time maximum TLV (TLVom) was increased from 5 to 23 mg/$m^3$. TLVda (8 hour exposure) of carbon monoxide was increased from 3 to 10 mg/$m^3$.

Hygienic standard HS 1.2-20 are developed in coherence with federal law № 52-FL "On the sanitary and epidemiological well-being of the population" dated March 30, 1999. According to Article 38, Clause 2, the development of sanitary rules should provide for conducting comprehensive research, determining sanitary and epidemiological requirements, calculating and assessing risk to human health, establishing safety criteria, analyzing international experience, establishing grounds for revision of hygienic and other standards, forecasting the social and economic consequences of the application of sanitary rules. However, in hygienic standard HS 1.2-20 project no research results justifying changes have been released.

The analyzed hygienic standard regulate “ambient air,” the term has been interpreted as “outdoor air,” or air external to buildings, excluding indoor air. While outdoor air quality can affect the indoors, and indoor air contains pollutants from both outdoor and indoor origin [14]. The indoor air pollutant sources are construction work, furniture, textile, clothes, carpeting, and wood processing household appliances, particle board, painting, plywood. Existing regulatory framework does not cover the indoor air pollutants topic although concentration of chemical and biological pollutants is a crucial IAQ indicator.

Another key indicator of air quality is the concentration of particulate matter PM10 – inhalable particles, with diameters that are generally 10 microns and smaller, and PM2.5 – fine inhalable particles, with diameters that are generally 2.5 microns and smaller. Ammonia, organic and elemental carbon, sulfates, nitrates, chloride ions, sodium, potassium, zinc, calcium, iron, magnesium and copper ions, crustal minerals, particle-bound water, polycyclic aromatic hydrocarbons are the most common chemical components of PM10 and PM2,5. These components are absorbed by alveolar macrophages (as it is one of the main lung functions), which causes inflammation of lung tissue, resulting in chronic hypoxemia. Besides, chronic exposure to PM 2.5 causes the DNA damage itself and influence the DNA repair [23]. Overall, these processes lead to cancerogeneses, which results in increased lung cancer mortality [24, 25]. According to the 2013 WHO report "Health Exposure to Particulate Matter" PM2.5 accounts for approximately 3% of deaths from cardiopulmonary pathology and 5% of deaths from lung cancer. The research [26] found that every 0.01 mg/m3 increase of PM10 stands for respiratory mortality increase by 0.58%. Table 4 shows the result of a comparative analysis of the TLVda of particulate matter in HG 2.1.6.3492-03; HS 2.1.6.3492-17; WELL standard; LEED standard and maximum permissible threshold value recommended by WHO for the concentration of suspended particles PM10; PM2.5. The values of the required TLVs are reduced to a single dimension – mg/$m^3$.

*Table 4. Comparison of TLVs of particulate matters PM10, PM2,5.*

| Particulate Matter Thresholds | PM10 | PM2.5 |
|---|---|---|
| **TLVda in HS 2.1.6.3492-03 (mg/m$^3$)** | Not regulated | Not regulated |
| **TLVda in HS 2.1.6.3492-17 (mg/m$^3$)** | 0,06 | 0,035 |
| **TLVda in HS1.2-20 (mg/m$^3$)** | 0,06 | 0,035 |
| **WELLv.2 (mg/m$^3$)** | 0,02 | 0,01 |
| **TLVda in WHO guidelines (mg/m$^3$)** | 0,05 | 0,025 |

The most stringent requirements for the particulate matter thresholds PM10; PM2.5 are imposed by the WELL Building Standard. TLV regulated by HS1.2-20 PM10 particles is 3 times higher than the WELL requirements, and PM2.5 particles are 3.5 times higher.

## CONCLUSIONS

1. Ambient air quality directly affects indoor air quality. However, indoor air contains both external and internal pollutants and is not dependent only on outdoor air. Emissions from furniture, finishing materials, household chemicals and their impact on the state of the indoor air quality of civil buildings' premises and are currently not standardized by either technical or sanitary - epidemiological regulations in the Russian Federation. The hygiene standards do not consider the indoor air quality of civil buildings as a separate category.
2. Research results justifying changes in threshold limit values of several VOCs and SVOCS have not been released. Threshold limit values of several VOCs and SVOCS should be reconsidered within a view to international "green" standards WELL and LEED. Based on article 38, clause 2 of Federal Law № 52 the development of sanitary rules should provide for conducting comprehensive research, determining sanitary and epidemiological requirements, calculating and assessing risk to human health, establishing safety criteria, analyzing international experience.
3. It is crucial to create interdisciplinary teams consisting of researchers and practitioners, experts of various fields: architects, structural engineers, engineers, builders, medical professionals and psychologists in order to solve the problems of maintaining and improving the health and wellbeing of building users. Interdisciplinary teams should not only be involved in design but also take part in developing building sector regulatory framework.
4. It is important to consider a little shared vocabulary between disciplines and absence of interdisciplinary building sector regulatory documents aimed at regulating design, construction and operation of buildings in accordance with sustainable and health preserving principles, since buildings and are multi-dimensional systems influenced by trends and processes operating at local, national and global levels. Since the building sector regulatory framework in Russia is represented by both technical and hygienic standards, in order to create an accessible tool to design and build comfortable living environments, the Technical Committee for Standardization №366 "Green" technologies of the living environment and "green" innovative products" should work on development of interdisciplinary standards based on health preserving built environment principles.

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# DETERMINATION OF THE CRITERIA OF DEFORMATION IN A SPECIAL LIMITING STATE

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**Abstract**. Constructive measures taken to ensure the integrity of the entire building or its part in emergency situations with design based on the existing criteria of the limiting state method leads to a significant increase of the construction cost. One of the ways to reduce additional costs of construction while the protection design against progressive collapse is the possible use of additional reserves of deformability of load-bearing elements. It leads to redistribution of loads and the use of non-destroyed structures. It also leads to possible changes of limiting states in non-standard emergency design situations, taking into account the peculiarities of the operation of structures in a special limiting state at a stage close to destruction. In the GOST 27751-2014 «Reliability for constructions and foundations. General principles» calculated states of the first and second groups of limiting states are given, and for a special limiting state only the area of its permissible application is indicated. The work of reinforced concrete structures at the stage close to the depletion of the load-bearing capacity is little reflected in the scientific and technical literature; the work of reinforced concrete structures at the unloading stage due to the redistribution of forces is represented in single publications. The article presents theoretical studies based on experimental data on the deformation of bent reinforced concrete beam elements at a stage close to the maximum load-bearing capacity and at the stage of unloading up to the transformation of a structural element into a mechanism. The influence of the longitudinal reinforcement, the class of reinforcement, prestressing and the concrete strength on the deformation of reinforced concrete bending elements is considered in the article. The research of the behavior of structural elements continuation at this stage is relevant and contributes to the development of economical and rational design solutions for protection against progressive collapse and in the design of earthquake-resistant buildings.

**Keywords**: Protection against progressive collapse, special limiting state, deflections, partial destruction of sections, ultimate deformations of concrete, special limiting state criteria

# К ВОПРОСУ ОПРЕДЕЛЕНИЯ КРИТЕРИЕВ ДЕФОРМАТИВНОСТИ В ОСОБОМ ПРЕДЕЛЬНОМ СОСТОЯНИИ

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**Аннотация**. Конструктивные меры, принимаемые для обеспечения целостности всего здания или его части в аварийных ситуациях при проектировании на основе существующих критериев метода предельных состояний приводит к существенному росту стоимости строительства. Одним из способов снижения дополнительных затрат в строительстве при проектировании защиты от прогрессирующего обрушения является возможное расширение использования резервов деформативности несущих элементов, приводящее к большему перераспределению усилий и включению в работу неразрушенных конструкций. Это приводит к возможности изменения границ предельных состояний в нестандартных аварийных расчетных ситуациях, учитывающее особенности работы конструкций в особом предельном состоянии на стадии близкой к разрушению. В ГОСТ 27751 «Надежность строительных конструкций и оснований» приведены расчетные состояния первой и второй групп предельных состояний, а для особого предельного состояния обозначена лишь область его допускаемого применения. В настоящее время в научно-технической литературе мало отражена работа железобетонных конструкций на стадии близкой к исчерпанию несущей способности, а работа железобетонных конструкций на стадии разгрузки за счет перераспределения усилий представлена единичными публикациями. В статье приведены теоретические исследования на базе экспериментальных данных деформирования изгибаемых железобетонных балочных элементов на стадии близкой к максимальной несущей способности, а также в стадии разгрузки вплоть до превращения конструктивного элемента в механизм. Рассматривается влияние количества продольного армирования, класса арматуры, предварительного напряжения, а также прочности бетона на деформирование железобетонных изгибаемых элементов. Продолжение исследований поведения конструктивных элементов на данной стадии является актуальным и способствует разработке экономичных и рациональных конструктивных решений для защиты от прогрессирующего обрушения и при проектировании сейсмостойких зданий.

**Ключевые слова**: защита от прогрессирующего обрушения, особое предельное состояние, прогибы, частичное разрушение сечений, предельные деформации бетона, критерии особого предельного состояния

## INTRODUCTION

Analysis of world practice of design and operation of buildings and structures of the increased level of responsibility or of the normal level with a mass presence of people showed the need of protection against progressive collapse and to ensure the evacuation and preservation of the people in buildings and structures, to increase safety during the operation of objects. Since the end of the twentieth century, the resistance of load-bearing systems of buildings and structures to avalanche-like progressive collapse in emergency situations have been studied in the world design practice.

The protection of buildings and structures from progressive collapse is relevant for newly designed, already constructed objects, and objects under reconstruction; it is regulated by the current normative documents in Russia, including [1] SP 385.1325800.2018 "Protection of buildings and structures against progressive collapse. Design Code. Basic statements".

## THE STATEMENT OF THE PROBLEM

In Russia there is a legal requirement of protection design of the load-bearing objects from the avalanche-like progressive collapse in the hypothetical alternate removal of one load-bearing element. Such calculated situation is unlikely in reality, but if it occurs, it can cause complete or partial object destruction; additional measures excluding catastrophic consequences with the use of limiting states criteria for the operation stage can lead to additional economic costs [2]. It is possible to reduce these costs by using the reserves of load-bearing capacity of structures and buildings in a special limiting state.

It is known that the operation of individual structural elements beyond the normalized maximum load-bearing capacity is possible and it leads to significant redistribution of forces in statically indeterminate systems [3].

The most complete implementation of the reserves of strength and deformability is possible in load-bearing systems with a high degree of constructive interaction of all load-bearing elements and only when changing the criteria of limiting states. It is necessary to go beyond the standard requirements of the serviceability of reinforced concrete structures, rigidly limiting the crack opening and the appearing of sags. The low probability of the calculated situation and the serious consequences of the destruction of the building facilitate it. The local partial destruction of cross-sections and unit connections are must be allowed to keep the building as a whole. But it is necessary to use the unloading parts of the connecting elements, keeping the entire load-bearing system in a stable (equilibrium) state for a certain period.

According to SP 385.1325800.2018, "a special limiting state is the state of structures after exceeding the limit of load-bearing capacity for the first limiting state and deformability for the second limiting state, in which they do not fully fulfil the functional requirements. The further increase of loads and (or) impacts leads to their destruction". In this Building Code some criteria for a special limiting state of reinforced concrete and metal structures are suggested, they are designed based on the existing design practice, operation, experience in examining the technical condition of structures and experimental studies. Due to the limited amount of experimental data, the special limiting state criteria given in the current version of SP 385.1325800.2018 were adopted with some reserve to ensure the safety of building structures. The clarification of the criteria of the special limiting state is necessary to establish the possible further safe exceeding of the limits of the first and second limiting states; it is important for improving the efficiency of design solutions.

## EXPERIMENTAL STUDIES

One of the important research areas is the study of the stress-strain state of bent reinforced concrete elements at the stages close to the maximum of load-bearing capacity and at the stage of destruction. For this purpose, the experiments (performed to solve other problems) of Plotnikov A. I. [5], Gushcha Yu. P. [6], Tamov M. A. [7] were studied in the article.

The diagrams below show the values of relative deflections of bent reinforced concrete elements in sections with the maximum bending moment, depending on the degree of loading (Fig. 1–3).

The results of tests of four series of reinforced concrete beam elements differing in the content of longitudinal stretched reinforcement are presented in the paper [5]. At the same time, one of the samples of each series at the area of pure bending (the middle third of the span) had no longitudinal compressed reinforcement and clamps. In total, 12 beams were manufactured and tested. The cross-section was 100x180 mm and the length was 2000 mm. The beams were reinforced with A400 steel (AIII-35GS). Concrete Rb=27.1 MPa.

Based on the data presented in Fig. 1, samples without longitudinal compressed reinforcement were destroyed more intensively and deflections at the level of 0,7Mult were on average 1/70 of the span L. The longitudinal reinforcement in the compressed area of normal cross-section increases the resistance to the load after the destruction of the compressed area of concrete and deformability before turning the beam structure into a mechanism, it corresponds to the deflections 1/40 of the span. At the same time the maximum deformability before the structural destruction varies inversely with the amount of stretched reinforcement and for samples with a reinforcement coefficient $\mu$=1.95–2.5% and is 1/50 of the span. The increased longitudinal reinforcement of the stretched area of the bent element leads to its rigidity increase, an increase of the compressed area of concrete and decrease of the deformability of the bent element.

Analyzing the diagrams presented in Fig. 2, we can see that using thermally hardened tensile reinforcement of AIV class, structural deformability of plastic stage is reduced to 30% in comparison with similar constructions of AIV regular class. Prestressing of stretched reinforcement structures increases its general rigidity and crack resistance. Also it leads to a decrease of the maximum values of deflections up to 15% in comparison with analogues without prestressing.

The results of static and dynamic tests show that structures loading increase speed before the reduce the load-bearing capacity has little effect on the maximum values of deflections of truss elements (Fig. 3). Also there is an increase of the bearing capacity of dynamic loads up to 15 %.

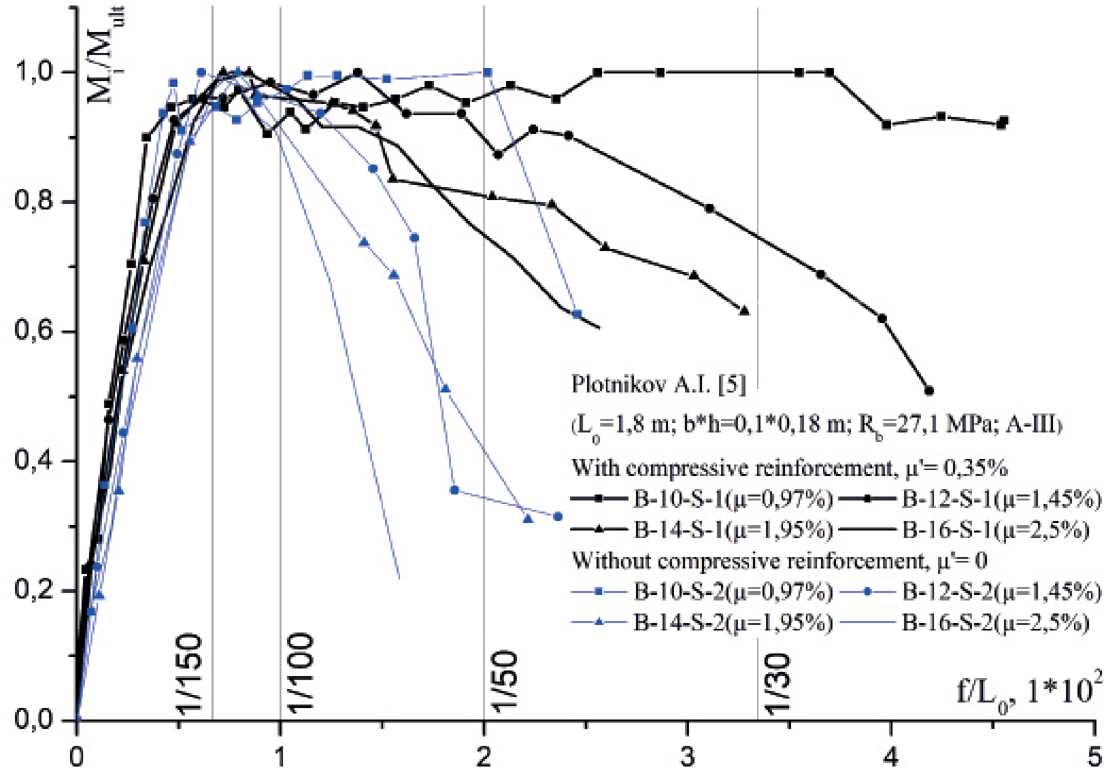


*Figure 1. Diagrams of deformation of bent elements with different amount of longitudinal reinforcement (based on the works of Plotnikov A. I.)*

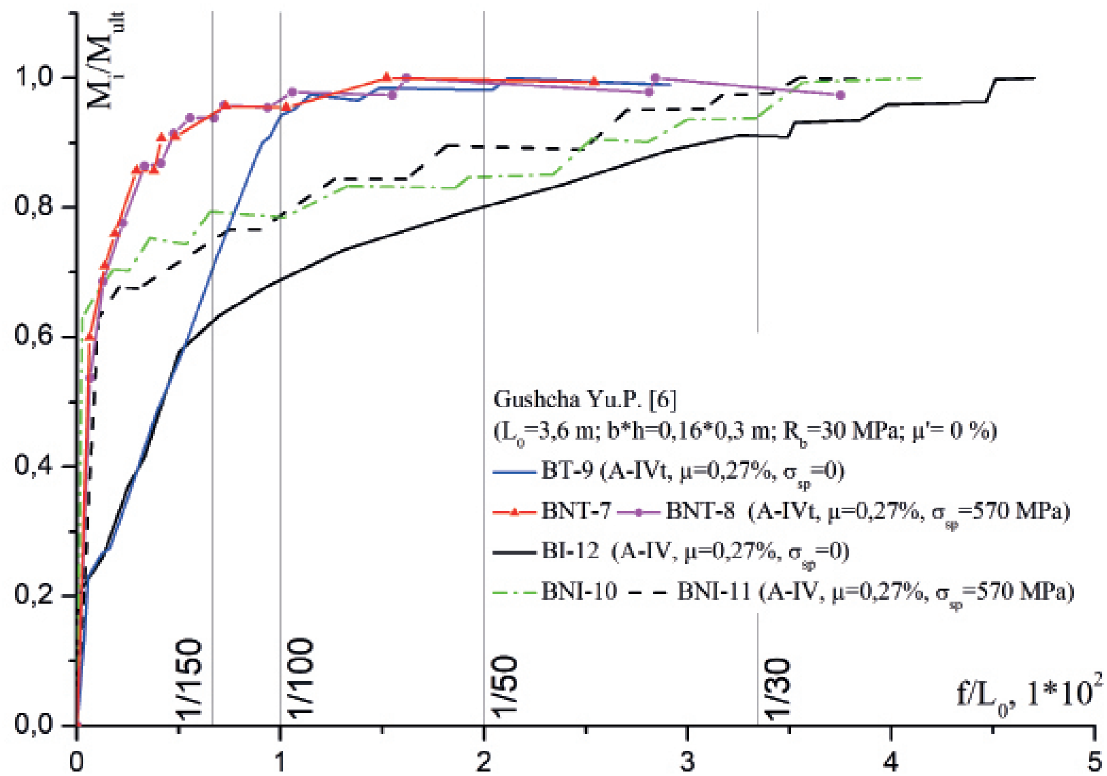


*Figure 2. Diagrams of deformation of bending elements with conventional and thermally strengthened stretched reinforcement of AIV class, with and without prestressing*

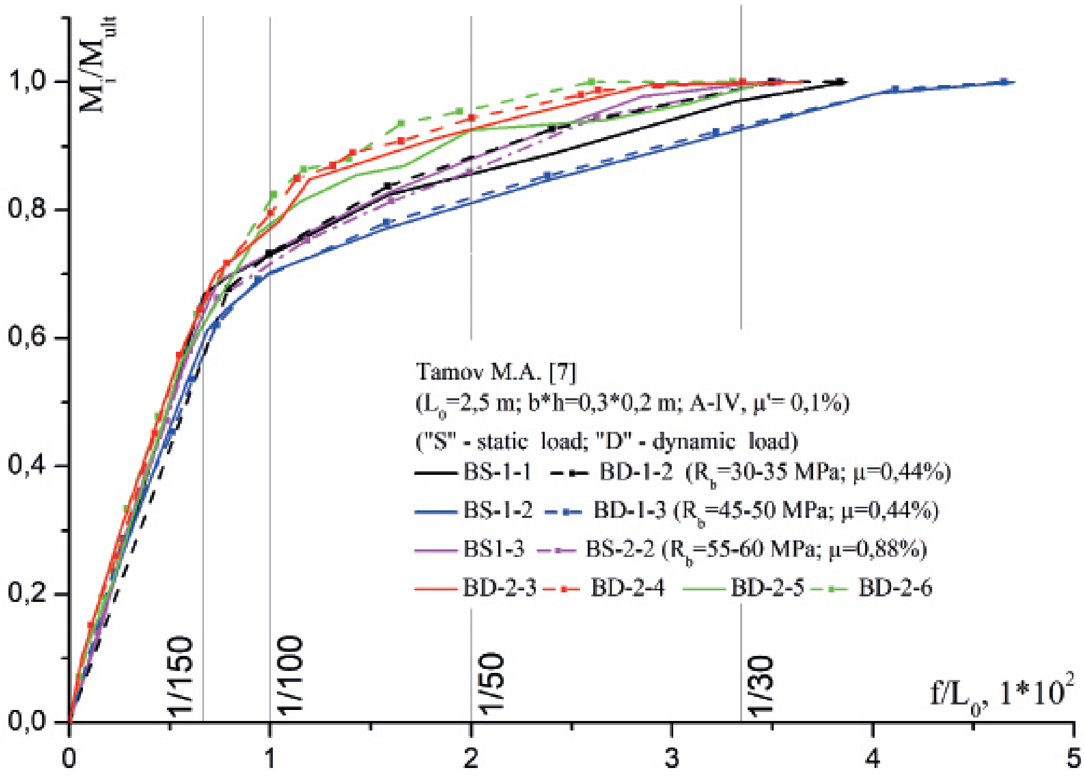


*Figure 3. Diagrams of deformation of bent elements with different strength of concrete and the amount of longitudinal reinforcement, with static and short-term dynamic loads (based on works of Tamov M. A.)*

## THE PROSPECTS OF THE CRITERIA REFINEMENT FOR A SPECIAL LIMITING STATE

The experimental data confirm the presence of reserves for the strength and deformability of structures in the calculation methods of the first and second groups of limiting states.

The present and prospective limits of the deformability criterion are presented in Table 1.

The deformation test in compressed concrete showed that the approved limits in the SP 63.13330.2018 "SNiP 52-01-2003 Concrete and reinforced concrete structures. General provisions" and in European norms match the experimental data. It is not necessary to refine the value of maximum compressibility of concrete for a special limiting state.

The deformation test in stretched reinforcement showed the possibility of increasing the limits of the ultimate deformability of reinforcement, including the hardening stage for steels with a physical yield strength.

The largest reserves of the maximum deformability (deflections) are connected with the possibility of redistributing forces by using the adjacent structures in the process of force redistributing (deformation). To the greatest extent, this mechanism can be used in statically indeterminate structural systems. Additional testing allows to implement the principle of gradation of the criterion of limiting state of deformation (deflection) depending on the type of load-bearing structures and the ability to redistribute loads. It is used for monolithic buildings, prefabricated buildings, depending on the type of reinforcement (physical and conditional yield strength) and availability of the minimum allowable length of the bearing (embedment) area.

*Table 1. Current and prospective limiting state for deformability of reinforced concrete structures*

| | Criteria of I and II group of limiting states according to SP 20.13330.2016, SP 63.13330.2018 | Criteria for a special limiting state according to SP 385.1325800.2018 | Prospective data for the special limiting state after additional testing of the bending elements |
|---|---|---|---|
| The relative deformation of the concrete (below 60), $\varepsilon_{b2}$ | 0,035 (100%) | 0,035 (100%) | 0,035 (100%) |
| Relative deformations of reinforcement (with physical yield strength), $\varepsilon_{s2}$ | 0,025 (100%) | 0,033 (132%) | 0,035…0,038 (140%…152%) |
| The relative deformation of the reinforcement (with a nominal yield strength), $\varepsilon_{s2}$ | 0,015 (100%) | 0,020 (133%) | 0,022…0,025 (147%…167%) |
| Deflections for structures with l span | The maximum deflections and displacements of elements of buildings and structures are not specified in the present and other normative documents; vertical and horizontal deflections and displacements from permanent, long-term and short-term loads must not exceed l/150 (100%) or l/75% of the console overhang. | The deflections of the bending elements of the structural system for a special limiting state, with the minimum allowable length of the bearing (embedment) area must not exceed l/30 (500%) of span length l, excepting the concrete structures reinforced with high-strength reinforcement with nominal yield strength, the deflection of which must not exceed l/50 (300%) of the span length. | The use of the gradation by types of load-bearing structures principle, taking into account the ability to redistribute forces, most fully used in a statically indeterminate structural system, for example, for monolithic buildings, the deflection of the bending element reinforced with a physical yield point, if the minimum permissible length of the support (anchoring) area can be l/25...l/20 (600%...750%) of the span with reinforcement with a conditional yield strength – l/50...l/40 (300%...375%) length. |

## CONCLUSION

Due to the complexity of the reliable assess of stress state of cross sections at the stage of load-bearing capacity limit, and then at the destruction stage (when the hypothesis of flat sections is not tested) it is convenient to use the deformability criterion as the most predictable criterion of a special limiting state when ensuring limited operability of load-bearing reinforced concrete structural elements.

The analysis of normative documents, experimental and theoretical studies of bent reinforced concrete beam elements [8-10] showed that when fulfilling the normative requirements for the strength and deformability of reinforcement and concrete materials, geometric characteristics and the absence of structural defects, the structures have strength reserves up to 25%, and deformability reserves up to 35%. It is confirmed by comparing the experimental data with theoretical data obtained according to the calculations performed on the basis of regulatory documents. The research showed that when performing certain design requirements, including the use of reinforcement with a yield line in sections of the normative (optimal) percentage of reinforcement (not pre-reinforced), at the stage after reaching the limit of load-bearing capacity, there is a limited functional state of the element to reduce load-bearing capacity to (0,5 – 0,9)Mult with the intensive cracks and deflections. The variation of results is due to the limited amount of available experimental data. It is found that at the stage of reducing the load-bearing capacity (unloading) of the bent element, the criteria for a special limiting state in the current version of SP 385.1325800.2018 are presented with some margin and help to fulfil the safety requirements, according to the current regulatory documents. The criteria are confirmed by the analysis of the works of Gushcha Yu. P. [6], Tamov M. A. [7] and Plotnikov A. I. [5]. After additional tests, due to the limited experimental data, the criteria can be refined by using the available reserves. According to the authors, the operation of structures in a special limiting state can be used in earthquake-resistant construction.

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# ON DIFFERENT DEFINITIONS OF STRAIN TENSORS IN GENERAL SHELL THEORIES OF VEKUA-AMOSOV TYPE

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**Abstract**: Several possible definitions of strains in a general shell theory of I.N. Vekua – A.A. Amosov type are considered. The higher-order shell model is defined on a two-dimensional manifold within a set of field variables of the first kind determined by the expansion factors of the spatial vector field of the translation. Two base vector systems are introduced, the first one so-called concomitant corresponds to the cotangent fibration of the modelling surface while the other is defined on a surface equidistant to the modelling one. The distortion appears as a two-point tensor referred to both base systems after covariant differentiation of the translation vector field. Thus, two main definitions of the strain tensor become possible, the first one referred to the main basis whereas the second to the concomitant one. Some possible simplifications of these tensors are considered, and the interrelation between the general theory of A.A. Amosov type and the classical ones is shown.

**Keywords**: hierarchical modeling of shells, dimensional reduction, analytical continuum dynamics, strain tensors, stress tensors

# О РАЗЛИЧНЫХ ОПРЕДЕЛЕНИЯХ ТЕНЗОРОВ ДЕФОРМАЦИИ В ОБЩЕЙ ТЕОРИИ ОБОЛОЧЕК И.Н. ВЕКУА – А.А. АМОСОВА

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**Аннотация**: Рассмотрены различные возможные определения деформации в общей теории оболочек И.Н. Векуа – А.А. Амосова. Модель оболочки высшего порядка определена на двумерном многообразии множеством переменных поля первого рода, определенных коэффициентами разложения пространственного поля вектора перемещения по некоторой системе функций. Введены две системы базисных векторов, а именно сопутствующий базис в касательном расслоении многообразия, соответствующего реперной поверхности оболочки, и основной базис, соответствующей произвольной поверхности, эквидистантной реперной. Введены также двухточечные представления тензора дисторсии в тензорном произведении касательных пространств к реперной и эквидистантной поверхностям, порождаемые градиентом поля вектора перемещения. Таким образом возникают два определения тензора деформации оболочки как трехмерного тела, первый из них отнесен к основному базису, тогда как второй – к сопутствующему. Рассмотрены некоторые упрощенные формы записи тензоров деформации, соответствующие теориям низших порядков, и указано соответствие между теорией оболочек в форме А.А. Амосова и классическими теориями.

**Ключевые слова**: иерархические модели оболочек, редукция пространственной размерности, аналитическая механика континуальных систем, тензоры деформации, тензоры напряжения

## INTRODUCTION

The intrinsic kinematics of the classical shell theory is based on two parameters, the tangent and bending strains of the modelling surface of a shell [1, 2]. These quantities were initially defined using the orthogonal coordinates referred to principal curvature lines of the modelling surface [3] while the modern shell theory is based on the tensor analysis [4]. Various refined theories of shells use different strain definitions [5, 6] referred to the orthogonal coordinates or tensor-based definitions [7–10]. One should note that most of cited works introduce different strain tensor definitions; several theories operates with tensor coefficients with no description of base systems in details [9, 10]. Indeed, at least two base vector triads could be used in generalized shell theories [7, 10–12], the spatial (or main) basis of an arbitrary curvilinear frame normally attached to the modelling surface and the concomitant one [7, 11, 12]; the first one generates a set of space tensors whereas the second one generates surface tensors [7]. Let us note that more complex models use two-point tensor definitions [13–15]. Such a situation leads to different work-conjugated pairs of stress and strain tensors and different formulation of variational principles [16] for a shell theory.

This work is a brief systematization of strain and stress tensor representations in higher-order shell theories. The two-point spatial distortion tensor [15] is used as a background; two main formulations for the spatial strain and four for the stress tensors allowing one to obtain the surface tensors after integration over the thickness are introduced. It is shown that a clear analogy of these quantities and the tensors used in the nonlinear solid mechanics appears if a surface equidistant to the base one is obtained as a result of the imagined motion along normal, and the spatial, or main, basis could be interpreted as "acting" one while the concomitant basis as "initial" one [18]. The appropriate work-conjugated tensor pairs are defined, the strain energy is formulated, and the sets of surface stress and strain tensors defining a higher-order shell theory are described. Finally, the interrelation between the strain tensors in the higher-order theory [15] and the classical shell theories is shown.

## GEOMETRY OF THE MODELLING SURFACE OF A SHELL

Let us consider a two-dimensional manifold:

$$\begin{gathered} S \subset \mathbb{R}^3: \quad \forall M \in S \quad \mathbf{r}(M) = x^i(M)\mathbf{e}_i; \\ \exists \xi^\alpha: \quad S \to D_\xi \subseteq \mathbb{R}^2, \quad i = \overline{1,3}, \alpha = \overline{1,2}; \\ x^i(\xi^\alpha) \in C^{(2)}(D_\xi) \Rightarrow \exists d\mathbf{r}(M) = \mathbf{r}_\alpha d\xi^\alpha, \\ \mathbf{r}_\alpha = \left(\partial x^i / \partial \xi^\alpha\right)\mathbf{e}_i, \quad \mathrm{Rk}\left(\partial x^i / \partial \xi^\alpha\right) = 2; \end{gathered} \tag{1}$$

$\xi^\alpha$ are curvilinear coordinates on $S$. The tangent fibration $TS$ can be defined by:

$$TS = \coprod_{M \in S} T_M S, \quad T_M S = \left\{ y^\alpha \mathbf{r}_\alpha \mid y^\alpha \in \mathbb{R} \right\}. \tag{1.2}$$

The fibration $TS$ is furnished by the metric $\mathbf{a}$

$$\begin{gathered} \mathbf{a}: \quad TS \times TS \to \mathbb{R}_+, \quad \mathbf{a} = a_{\alpha\beta}\mathbf{r}^\alpha \mathbf{r}^\beta, \\ a_{\alpha\beta} = \mathbf{r}_\alpha \cdot \mathbf{r}_\beta, \quad a = \det\left(a_{\alpha\beta}\right), \quad a \neq 0 \quad \Rightarrow \\ \forall M \in S \quad \exists a^{\gamma\beta}, \quad a^{\gamma\beta} a_{\alpha\beta} = \delta^\gamma_\alpha, \quad \mathbf{r}^\gamma = a^{\gamma\beta}\mathbf{r}_\beta, \end{gathered}$$

and the discriminant tensor $\boldsymbol{\tau} = \tau^{\alpha\beta}\mathbf{r}_\alpha\mathbf{r}_\beta$. The cotangent fibration $T^e S$ is given as

$$T^e S = \coprod_{M \in S} T^e_M S, \quad T^e_M S = \{ y_\alpha \mathbf{r}^\alpha \mid y_\alpha \in \mathbb{R} \}. \tag{1.3}$$

The orientation of the manifold $S$ is defined by the normal unit vector $\mathbf{n}$:

$$\mathbf{n}(M) = a^{-1/2}\mathbf{r}^1 \wedge \mathbf{r}^2 = \tau^{\alpha\beta}\mathbf{r}_\alpha \wedge \mathbf{r}_\beta, \tag{1.4}$$

so that we obtain the normal fibration of $S$:

$$NS = \coprod_{M \in S} N_M S, \quad N_M S = \left\{ y\mathbf{n} \mid y \in \mathbb{R} \right\}. \tag{1.5}$$

The differentiation of $\mathbf{r}_\alpha$, $\mathbf{n}$ is defined by

$$\begin{gathered} \partial_\beta \mathbf{r}_\alpha = \Gamma^\gamma_{\alpha\beta}\mathbf{r}_\gamma + b_{\alpha\beta}\mathbf{n}, \quad \partial_\beta \mathbf{r}^\gamma = -\Gamma^\gamma_{\alpha\beta}\mathbf{r}^\beta + b^{\cdot\gamma}_{\beta\cdot}\mathbf{n}, \\ \partial_\beta \mathbf{n} = -b_{\alpha\beta}\mathbf{r}^\alpha. \end{gathered} \tag{1.6}$$

## KINEMATICS OF A DEFORMING MODELLING SURFACE OF A SHELL

Let us define a vector field $\mathbf{u}_s$:

$$\begin{gathered} \mathbf{u}_S : S \times \left(\mathbb{R}_+ \cup \{0\}\right) \to T^e S \oplus NS, \\ \mathbf{u}_S = \overline{u}_\alpha\left(\xi^\beta, t\right)\mathbf{r}^\alpha + w\left(\xi^\beta, t\right)\mathbf{n}; \end{gathered} \tag{1.7}$$

here $\overline{u}_\alpha$ correspond to the tangent translation while $w$ is the deflection of a surface $S$. Introducing Nabla operator $\overline{\nabla} = \mathbf{r}^\beta \partial_\beta$ we have

$$\begin{gathered}\bar{\mathbf{d}} \equiv \bar{\nabla} \otimes \mathbf{u}_S = \bar{\bar{d}}_{\beta\alpha}\mathbf{r}^{\beta}\mathbf{r}^{\alpha} + \vartheta_{\beta}\mathbf{r}^{\beta}\mathbf{n}, \\ \bar{\bar{d}}_{\beta\alpha} = \nabla_{\beta}\bar{u}_{\alpha} - b_{\alpha\beta}w, \quad \vartheta_{\beta} = \partial_{\beta}w + b_{\beta\cdot}^{\cdot\alpha}\bar{u}_{\alpha}.\end{gathered} \tag{1.8}$$

Let us consider the deformed surface $S'$:

$$\forall M \in S \quad \mathbf{r}'(M,t) = \mathbf{r}(M) + \mathbf{u}_S(M,t); \tag{1.9}$$

Thus, we obtain the tangent fibration $TS'$ and normal one $NS'$ for the deformed surface $S'$:

$$\begin{gathered}TS' = \coprod_{M \in S}\left\{y^{\alpha}\mathbf{r}'_{\alpha}(M) \mid y^{\alpha} \in \mathbb{R}\right\}, \\ \mathbf{r}'_{\alpha}(M) = \mathbf{r}_{\alpha}(M) + \bar{d}_{\alpha\cdot}^{\cdot\beta}\mathbf{r}_{\beta} + \vartheta_{\alpha}\mathbf{n};\end{gathered} \tag{1.10}$$

$$\begin{gathered}\mathbf{a}' = a'_{\alpha\beta}\mathbf{r}^{\alpha} \otimes \mathbf{r}^{\beta} = \left(a_{\alpha\beta} + 2\varepsilon_{\alpha\beta}\right)\mathbf{r}^{\alpha}\mathbf{r}^{\beta}, \\ \varepsilon_{\alpha\beta} \approx \tfrac{1}{2}\left(\nabla_{\alpha}\bar{u}_{\beta} + \nabla_{\beta}\bar{u}_{\alpha}\right) - b_{\alpha\beta}w;\end{gathered} \tag{1.11}$$

$$\begin{gathered}NS' = \coprod_{M \in S}\left\{y\mathbf{n}' \mid y \in \mathbb{R}\right\}, \\ \mathbf{n}' = \sqrt{a'/a}\,\mathrm{T}^{\alpha\beta}\mathbf{r}'_{\alpha}\mathbf{r}'_{\beta} \approx \mathbf{n} - \vartheta_{\alpha}\mathbf{r}^{\alpha}.\end{gathered} \tag{1.12}$$

Here $\varepsilon_{\alpha\beta}$ is the surface tangent strain while its bending strain is defined by the tensor **u**:

$$\begin{gathered}\mathbf{b}' = \left(b_{\alpha\beta} + 2\text{ш}_{\alpha\beta}\right)\mathbf{r}^{\alpha}\mathbf{r}^{\beta}, \quad \text{ш}_{\alpha\beta} = \\ = \tfrac{1}{2}\left(\nabla_{\alpha}\vartheta_{\beta} + \nabla_{\beta}\vartheta_{\alpha}\right) + \tfrac{1}{2}\left(b_{\alpha\cdot}^{\cdot\gamma}\bar{d}_{\beta\gamma} + b_{\beta\cdot}^{\cdot\gamma}\bar{d}_{\alpha\gamma}\right),\end{gathered} \tag{1.13}$$

here the second-order terms are neglected as provided by linear shell theories [2].

## GEOMETRY OF THE SHELL AS A THREE-DIMENSIONAL BODY

Let us consider hence the shell as $V \in \mathbb{R}^3$ [15]:

$$\begin{gathered}\forall M_{\zeta} \in V \quad \exists \mathbf{R}(M_{\zeta}) = \mathbf{r}(M) + \zeta h\mathbf{n}(M), \\ \zeta \in D_{\zeta} = [-1,1];\end{gathered} \tag{1.14}$$

here $2h$ denotes the shell thickness. We can define hence an equidistant surface $S_{\zeta}$ for any $\zeta \neq 0$, e.g. a manifold with the fibrations

$$\begin{gathered}TS_{\zeta} = \coprod_{M \in S}\left\{y^{\alpha}\mathbf{R}_{\alpha} \mid y^{\alpha} \in \mathbb{R}\right\}, \\ \mathbf{R}_{\alpha} = \partial_{\alpha}\mathbf{R} = A_{\alpha\cdot}^{\cdot\beta}\mathbf{r}_{\beta}, \quad A_{\alpha\cdot}^{\cdot\beta} = \delta_{\alpha}^{\beta} - \zeta b_{\alpha\cdot}^{\cdot\beta}; \\ NS_{\zeta} = \coprod_{M \in S}\left\{y\mathbf{n} \mid y \in \mathbb{R}\right\} \equiv NS: \\ TV = \coprod_{M \in S}\left\{y^{\alpha}\mathbf{R}_{\alpha} + y\mathbf{n}\right\} = TS_{\zeta} \oplus NS.\end{gathered} \tag{1.15}$$

here the linear transform given by $A_{\alpha\cdot}^{\cdot\beta}$ [7, 15] defines the parallel shifting. This transform is nonsingular if $A = \det(A_{\alpha\cdot}^{\cdot\beta}) \neq 0$, i. e. if [15]

$$A = 1 - 2\zeta hH + \zeta^2h^2K = \left(1 - \zeta hk_1\right)\left(1 - \zeta hk_2\right) \neq 0$$

$$\Leftrightarrow \quad h < \inf_{M \in S} k_{\alpha}^{-1}, \quad \alpha = 1,2; \tag{1.16}$$

where $k_{1,2}$ are principal curvatures of $S$. Accounting for, we could obtain [7, 14]

$$\begin{gathered}\mathbf{r}_{\beta} = A_{\cdot\beta}^{\alpha\cdot}\mathbf{R}_{\alpha}, \quad A_{\cdot\gamma}^{\alpha\cdot}A_{\beta\cdot}^{\cdot\gamma} = \delta_{\beta}^{\alpha}; \\ A_{\cdot\beta}^{\alpha\cdot} = A^{-1}\left(\delta_{\beta}^{\alpha} - \zeta h\mathrm{T}^{\alpha\lambda}\mathrm{T}_{\beta\mu}b_{\lambda\cdot}^{\cdot\mu}\right).\end{gathered} \tag{1.17}$$

The metric **g** on $TS$ is defined by (1.18) [7]:

$$\begin{gathered}g_{\alpha\beta} = A_{\alpha\cdot}^{\cdot\gamma}A_{\beta\cdot}^{\cdot\delta}a_{\gamma\delta} = a_{\alpha\beta} - 2\zeta hb_{\alpha\beta} + \zeta^2h^2c_{\alpha\beta}, \\ g = \det\left(g_{\alpha\beta}\right) = \det\left(A_{\alpha\cdot}^{\cdot\beta}\right)^2 a = A^2a, \\ c_{\alpha\beta} = b_{\alpha\cdot}^{\cdot\gamma}b_{\beta\gamma}. \quad H = \tfrac{1}{2}b_{\alpha\cdot}^{\cdot\alpha}, \quad K = \det\left(b_{\alpha\cdot}^{\cdot\beta}\right).\end{gathered} \tag{1.18}$$

For more details see [14, 15, 17].

## PARALLEL SHIFTING TENSORS

One could have two interpretations of parallel shift tensors. Let us consider a unit tensor **I**:

$$\begin{gathered}\mathbf{I} = \mathbf{r}_{\alpha}\mathbf{r}^{\alpha} = A_{\cdot\alpha}^{\beta\cdot}\mathbf{R}_{\beta}\mathbf{r}^{\alpha} = A_{\gamma\cdot}^{\cdot\alpha}\mathbf{r}_{\alpha}\mathbf{R}^{\gamma} = \\ = A_{\cdot\alpha}^{\beta\cdot}A_{\gamma\cdot}^{\cdot\alpha}\mathbf{R}_{\beta}\mathbf{R}^{\gamma} = \delta_{\gamma}^{\beta}\mathbf{R}_{\beta}\mathbf{R}^{\gamma} = \mathbf{R}_{\beta}\mathbf{R}^{\beta};\end{gathered} \tag{1.19}$$

such an interpretation was introduced by I.N. Vekua in [7]; the left indices of $A_{\alpha\cdot}^{\cdot\beta}$ must be raised and lowered by the metric tensor **g** whereas the right ones – by the metric **a** [7]:

$$\mathbf{I} = A^{\beta\cdot}_{\cdot\alpha}\mathbf{R}_\beta\mathbf{r}^\alpha = A^{\beta\cdot}_{\cdot\alpha}g_{\beta\gamma}\mathbf{R}^\gamma\mathbf{r}^\alpha = A_{\gamma\alpha}\mathbf{R}^\gamma\mathbf{r}^\alpha,$$
$$\mathbf{I} = A^{\beta\cdot}_{\cdot\alpha}\mathbf{R}_\beta\mathbf{r}^\alpha = A^{\beta\cdot}_{\cdot\alpha}a^{\alpha\delta}\mathbf{R}_\beta\mathbf{r}_\delta = A^{\beta\delta}\mathbf{R}_\beta\mathbf{r}_\delta,\ldots$$

The Vekua unit tensors (1.19) allow to define vector fields on the fibrations $T^eS$, $T^e_\zeta S$:

$$\begin{aligned}
&\mathbf{u}_T : V\times\left(\square_+\cup\{0\}\right)\to T^e S_\zeta,\ \mathbf{u}_T = \hat{u}_\alpha\mathbf{R}^\alpha;\\
&\mathbf{u}_T : V\times\left(\square_+\cup\{0\}\right)\to T^e S,\ \mathbf{u}_T = u_\alpha\mathbf{r}^\alpha;\\
&\mathbf{u}_T\cdot\mathbf{I} = A^{\beta\cdot}_{\cdot\alpha}\hat{u}_\gamma\mathbf{R}^\gamma\cdot\mathbf{R}_\beta\mathbf{r}^\alpha = A^{\gamma\cdot}_{\cdot\alpha}\hat{u}_\gamma\mathbf{r}^\alpha = u_\alpha\mathbf{r}^\alpha;\\
&\mathbf{I}\cdot\mathbf{u}_T = A^{\cdot\alpha}_{\gamma\cdot}u_\beta\mathbf{r}^\beta\cdot\mathbf{r}_\alpha\mathbf{R}^\gamma = A^{\cdot\beta}_{\gamma\cdot}u_\beta\mathbf{R}^\gamma = \hat{u}_\gamma\mathbf{R}^\gamma;\\
&\qquad\hat{u}_\gamma = A^{\cdot\beta}_{\gamma\cdot}u_\beta,\quad u_\alpha = A^{\beta\cdot}_{\cdot\alpha}\hat{u}_\beta.
\end{aligned}\tag{1.20}$$

On the other hand we have the tensor $\mathbf{A}(\zeta)$:

$$\begin{aligned}
&\mathbf{A} = \mathbf{R}_\alpha\mathbf{r}^\alpha = A^{\cdot\beta}_{\alpha\cdot}\mathbf{r}_\beta\mathbf{r}^\alpha = A^{\cdot\beta}_{\alpha\cdot}\mathbf{R}_\beta\mathbf{R}^\alpha;\\
&\mathbf{A}^{-1} = \mathbf{r}_\alpha\mathbf{R}^\alpha = A^{\beta\cdot}_{\cdot\alpha}\mathbf{R}_\beta\mathbf{R}^\alpha = A^{\cdot\beta}_{\alpha\cdot}\mathbf{r}_\beta\mathbf{r}^\alpha.
\end{aligned}\tag{1.21}$$

Let us note the analogy of **A** and the position gradients [18] if the vector $\mathbf{r}_\alpha$ is interpreted like "initial basis" and $\mathbf{R}_\alpha$ like "acting basis":

$$\mathbf{A}\cdot\mathbf{r}_\beta = \mathbf{R}_\beta,\quad \mathbf{A}^{-1}\cdot\mathbf{R}_\alpha = \mathbf{r}_\alpha,\quad \ldots\tag{1.22}$$

This tensor provides the linear morphism

$$\begin{aligned}
&\mathbf{A} : TS\to TS_\zeta,\ \mathbf{A} : T^eS_\zeta\to T^eS;\\
&\mathbf{A}^{-1} : T^eS\to T^eS_\zeta,\ \mathbf{A}^{-1} : TS_\zeta\to TS.
\end{aligned}\tag{1.23}$$

## SPATIAL KINEMATICS OF A SHELL

Let $\mathbf{u}(M, \zeta, t)$ be translation vector field:

$$\begin{aligned}
&\mathbf{u} : V\times\left(\square_+\cup\{0\}\right)\to T^eV,\\
&\mathbf{u} = u_\beta\mathbf{R}^\beta + u_3\mathbf{n}.
\end{aligned}\tag{1.24}$$

Since the fibration $TV$ is defined by (1.15) and the morphism (1.23) of bundles exists, $TS_\zeta = TS\times D_\zeta$, we could represent $T^eV$ as

$$T^eV = T^eS_\zeta\oplus NS \,\square\, T^eS\oplus NS\times D_\zeta,\tag{1.25}$$

thus, the vector field **u** can be determined as

$$\begin{aligned}
&\mathbf{u} : S\times D_\zeta\times\left(\square_+\cup\{0\}\right)\to T^eS\oplus NS\times D_\zeta\\
&\qquad\Rightarrow\quad \mathbf{u} = u_\beta\mathbf{r}^\beta + u_3\mathbf{n},
\end{aligned}\tag{1.26}$$

where the basis $\mathbf{r}^\alpha, \mathbf{n}$ does not depend on $\zeta$. Let $\nabla = \mathbf{R}^\alpha\partial_\alpha + \mathbf{n}\partial_\zeta$ be Nabla operator on $TV = TS_\zeta\oplus NS$; thus, we obtain the two-point formulation of the distortion $\mathbf{d} = \nabla\otimes\mathbf{u}$ [14]:

$$\begin{aligned}
&\mathbf{d} = \left(\mathbf{R}^\alpha\partial_\alpha + \mathbf{n}\partial_\zeta\right)\otimes\left(u_\beta\mathbf{r}^\beta + u_3\mathbf{n}\right) =\\
&= \bar{d}_{\alpha\beta}\mathbf{R}^\alpha\mathbf{r}^\beta + \bar{d}_{\alpha3}\mathbf{R}^\alpha\mathbf{n} + \bar{d}_{3\beta}\mathbf{n}\mathbf{r}^\beta + \bar{d}_{33}\mathbf{n}\mathbf{n};
\end{aligned}\tag{1.27}$$

$$\begin{aligned}
&\bar{d}_{\alpha\beta} = \nabla_\alpha u_\beta - b_{\alpha\beta}u_3,\quad \bar{d}_{\alpha3} = \nabla_\alpha u_3 + b^{\cdot\beta}_{\alpha\cdot}u_\beta;\\
&\bar{d}_{3\beta} = \partial_\zeta u_\beta,\quad \bar{d}_{33} = \partial_\zeta u_3.
\end{aligned}\tag{1.28}$$

Taking into account (1.20) we have [15]

$$\begin{aligned}
&\mathbf{d} = d_{\delta\beta}\mathbf{r}^\delta\mathbf{r}^\beta + d_{\delta3}\mathbf{r}^\delta\mathbf{n} + d_{3\beta}\mathbf{n}\mathbf{r}^\beta + d_{33}\mathbf{n}\mathbf{n} =\\
&= \hat{d}_{\alpha\gamma}\mathbf{R}^\alpha\mathbf{R}^\gamma + \hat{d}_{\alpha3}\mathbf{R}^\alpha\mathbf{n} + \hat{d}_{3\gamma}\mathbf{n}\mathbf{R}^\gamma + \hat{d}_{33}\mathbf{n}\mathbf{n};
\end{aligned}\tag{1.29}$$

$$\begin{aligned}
&\hat{d}_{\alpha\gamma} = A^{\cdot\beta}_{\gamma\cdot}\bar{d}_{\alpha\beta},\quad \hat{d}_{\alpha3} = \bar{d}_{\alpha3};\\
&\hat{d}_{3\gamma} = A^{\cdot\beta}_{\gamma\cdot}\bar{d}_{3\beta},\quad \hat{d}_{33} = \bar{d}_{33};
\end{aligned}\tag{1.30}$$

$$\begin{aligned}
&d_{\delta\beta} = A^{\alpha\cdot}_{\cdot\delta}\bar{d}_{\alpha\beta},\quad d_{\delta3} = A^{\alpha\cdot}_{\cdot\delta}\bar{d}_{\alpha3};\\
&d_{3\beta} = \bar{d}_{3\beta},\quad d_{33} = \bar{d}_{33}.
\end{aligned}\tag{1.31}$$

## THREE-DIMENSIONAL REPRESENTATION OF STRAIN TENSORS FOR A SHELL

The formulation (1.29) allows one to obtain the symmetrized tensor, i.e. the strain field:

$$\mathbf{e} = e_{\alpha\beta}\mathbf{r}^{\alpha}\mathbf{r}^{\beta} + e_{\alpha 3}\left(\mathbf{r}^{\alpha}\mathbf{n} + \mathbf{n}\mathbf{r}^{\alpha}\right) + e_{33}\mathbf{n}\mathbf{n} = \quad (1.32)$$

$$= \hat{e}_{\alpha\beta}\mathbf{R}^{\alpha}\mathbf{R}^{\beta} + \hat{e}_{\alpha 3}\left(\mathbf{R}^{\alpha}\mathbf{n} + \mathbf{n}\mathbf{R}^{\alpha}\right) + \hat{e}_{33}\mathbf{n}\mathbf{n}; \quad (1.33)$$

the first formulation (1.32) is referred to the basis $\mathbf{r}^{\alpha}$ in $TS$ that depends not on $\zeta$, and

$$\begin{gathered}\breve{e}_{\alpha\beta} = \tfrac{1}{2}\left(\nabla_{\beta}u_{\alpha} + \nabla_{\alpha}u_{\beta}\right) - \left(b_{\alpha\beta} + \zeta h K a_{\alpha\beta}\right)u_3 - \\ -\tfrac{1}{2}\zeta \tau^{\gamma\lambda} b_{\lambda\cdot}^{\cdot\mu}\left(\tau_{\alpha\mu}\nabla_{\gamma}u_{\beta} + \tau_{\beta\mu}\nabla_{\gamma}u_{\alpha}\right); \\ \breve{e}_{\alpha 3} = \tfrac{1}{2}\left(\nabla_{\alpha}u_3 + b_{\alpha\cdot}^{\cdot\beta}u_{\beta} + \partial_{\zeta}u_{\alpha}\right) + \tfrac{1}{2}\zeta^2 h^2 K \partial_{\zeta}u_{\alpha} - \\ -\tfrac{1}{2}\zeta h\left(\tau^{\gamma\lambda} b_{\lambda\cdot}^{\cdot\mu}\tau_{\alpha\mu}\nabla_{\gamma}u_3 + K u_{\alpha} + 2H\partial_{\zeta}u_{\alpha}\right); \\ \breve{e}_{33} = \left(1 - 2\zeta h H + \zeta^2 h^2 K\right)\partial_{\zeta}u_3; \quad e_{ij} = A^{-1}\breve{e}_{ij};\end{gathered} \quad (1.34)$$

at the same time the second one (1.33) is referred to the main basis $\mathbf{R}^{\alpha}$ in $TS_{\zeta}$:

$$\begin{gathered}\hat{e}_{\alpha\beta} = \tfrac{1}{2}\left(\nabla_{\alpha}u_{\beta} + \nabla_{\beta}u_{\alpha}\right) - \left(b_{\alpha\beta} + \zeta h c_{\alpha\beta}\right)u_3 - \\ -\zeta h\left(b_{\alpha\cdot}^{\cdot\gamma}\nabla_{\beta}u_{\gamma} + b_{\beta\cdot}^{\cdot\gamma}\nabla_{\alpha}u_{\beta}\right); \\ \hat{e}_{\alpha 3} = \tfrac{1}{2}\left(\nabla_{\alpha}u_3 + b_{\alpha\cdot}^{\cdot\beta}u_{\beta} + \partial_{\zeta}u_{\alpha}\right) - \\ -\zeta h b_{\alpha\cdot}^{\cdot\beta}\partial_{\zeta}u_{\beta}; \quad \hat{e}_{33} = \partial_{\zeta}u_3.\end{gathered} \quad (1.35)$$

Let us note that both strain formulations use the translation components $u_{\alpha}$, $u_3$ referred to the basis $\mathbf{r}^{\alpha}$, $\mathbf{n}$ on the fibration $T^{e}S \oplus NS$.
Let us consider hence the tensor $\mathbf{e}_S$:

$$\mathbf{e}_S = \hat{\mathbf{A}}^{\mathrm{T}} \cdot \mathbf{e} \cdot \hat{\mathbf{A}}, \quad \hat{\mathbf{A}} = \mathbf{A} + \mathbf{n}\mathbf{n}; \quad (1.36)$$

$$\mathbf{e}_S = \hat{e}_{\alpha\beta}\mathbf{r}^{\alpha}\mathbf{r}^{\beta} + \hat{e}_{\alpha 3}\left(\mathbf{r}^{\alpha}\mathbf{n} + \mathbf{n}\mathbf{r}^{\alpha}\right) + \hat{e}_{33}\mathbf{n}\mathbf{n}. \quad (1.37)$$

Accounting for (1.36) and (1.37) we have the inverse formula, i. e. $\mathbf{e} = \left(\hat{\mathbf{A}}^{-1}\right)^{\mathrm{T}} \cdot \mathbf{e}_S \cdot \hat{\mathbf{A}}^{-1}$.

## DIMENSIONAL REDUCTION

Let H[–1,1] be a Hilbert space, and let us introduce a base function system on H [15]:

$$\mathrm{H} = \left\{\mathrm{p}_{(k)}(\zeta)\right\} = \left\{\mathrm{p}^{(m)}(\zeta)\right\}, \quad (1.38)$$

$$\left(\mathrm{p}_{(k)}, \mathrm{p}^{(m)}\right) = \delta_{(k)}^{(m)}; \quad (1.39)$$

(1.39) denotes the appropriate scalar product, thus, $p_{(k)}$, $p^{(m)}$ is a biorthogonal base system. Let us introduce a subspace $\mathrm{H}_N = \{p_{(k)}\}_{k=0\ldots N}$, so that $\mathrm{H} = \mathrm{H}_N \oplus \Delta_N$. Taking into account (1.24), (1.38), and (1.39), we obtain [14, 15]

$$\begin{gathered}\mathbf{u} = \mathbf{u}^{(k)}\mathrm{p}_{(k)}(\zeta), \quad \mathbf{u}^{(k)} = u_{\alpha}^{(k)}\mathbf{r}^{\alpha} + u_3^{(k)}\mathbf{n}, \\ \mathbf{u}^{(k)} = \left(\mathbf{u}, \mathrm{p}^{(k)}\right), \quad k = 0\ldots N\end{gathered} \quad (1.40)$$

The formula (1.40) approximates the spatial translation within $N+1$ Fourier coefficient

$$\mathbf{u}^{(k)}: S \times \left(\Box_{+} \cup \{0\}\right) \to T^{e}S \oplus NS \quad (1.41)$$

being the vector fields on the Whitney sums of the cotangent and normal fibration of $S$.
Let us consider hence the strain tensor $\mathbf{e}$ defined by (1.34). Since this tensor is referred to $\mathbf{r}^{\alpha}$, $\mathbf{n}$, we could define them within a set of Fourier coefficients corresponding to (1.38):

$$\mathbf{e} = \mathbf{e}^{(k)}\mathrm{p}_{(k)}(\zeta), \quad \mathbf{e}^{(k)} = \left(\mathbf{e}, \mathrm{p}_{(k)}\right). \quad (1.42)$$

The Fourier coefficients $e^{(k)}$ define tensor fields on the fibration of the manifold $S$:

$$\begin{gathered}\mathbf{e}^{(k)}: S \times \left(\Box_{+} \cup \{0\}\right) \to \\ \to \left(T^{e}S \oplus NS\right) \otimes \left(T^{e}S \oplus NS\right).\end{gathered}$$

On the other hand, let us consider a tensor

$$\mathbf{e}_A = A\mathbf{e}, \quad A = \sqrt{g/a}; \quad (1.43)$$

defined by $\breve{e}_{\alpha\beta}$, $\breve{e}_{\alpha 3}$, $\breve{e}_{33}$; this tensor allows to represent its Fourier coefficients as follows:

$$\begin{aligned}
&\breve{e}_{\alpha\beta} = \tfrac{1}{2}\left(\nabla_\beta u_\alpha^{(k)} + \nabla_\alpha u_\beta^{(k)}\right) - \\
&-\left(b_{\alpha\beta}\delta_{(m)}^{(k)} + hKa_{\alpha\beta}Z_{(m)}^{(k)}\right)u_3^{(k)} - \\
&-\tfrac{1}{2}Z_{(m)}^{(k)}\mathrm{T}^{\gamma\lambda}b_{\lambda\cdot}^{\cdot\mu}\left(\mathrm{T}_{\alpha\mu}\nabla_\gamma u_\beta^{(m)} + \mathrm{T}_{\beta\mu}\nabla_\gamma u_\alpha^{(m)}\right); \\
&\breve{e}_{\alpha 3}^{(k)} = \tfrac{1}{2}\left(\nabla_\alpha u_3^{(k)} + b_{\alpha\cdot}^{\cdot\beta}u_\beta^{(k)} + D_{(n\cdot)}^{(\cdot m)}u_\alpha^{(n)}\right) - \\
&-\tfrac{h}{2}Z_{(m)}^{(k)}\left(\mathrm{T}^{\gamma\lambda}b_{\lambda\cdot}^{\cdot\mu}\mathrm{T}_{\alpha\mu}\nabla_\gamma u_\beta^{(m)} + \right. \\
&\left. + Ku_\alpha^{(m)} + 2HD_{(n\cdot)}^{(\cdot m)}u_\alpha^{(n)}\right) + \\
&+\tfrac{1}{2}Z_{(l)}^{(k)}Z_{(m)}^{(l)}h^2KD_{(n\cdot)}^{(\cdot m)}u_\alpha^{(n)}; \\
&\breve{e}_{33}^{(k)} = \left(\delta_{(m)}^{(k)} - 2Z_{(m)}^{(k)}hH + \right. \\
&\left. + Z_{(l)}^{(k)}Z_{(m)}^{(l)}h^2K\right)D_{(n\cdot)}^{(\cdot m)}u_3^{(n)}.
\end{aligned} \tag{1.44}$$

Accounting for (1.32), (1.34), (1.43)–(1.44), we obtain the Fourier coefficients for $\mathbf{e}_A$:

$$\mathbf{e}_A = \mathbf{e}_A^{(k)}\mathrm{p}_{(k)}, \quad \mathbf{e}_A^{(k)} = \left(\mathbf{e}_A, \mathrm{p}^{(k)}\right). \tag{1.45}$$

The tensor $\mathbf{e}$ referred to the basis $\mathbf{R}^\alpha$, $\mathbf{n}$ and defined by (1.35) cannot be represented by surface tensors being its Fourier coefficients, but we could define the "shifted" tensor $\mathbf{e}_S$:

$$\mathbf{e}_S = \mathbf{e}_S^{(k)}\mathrm{p}_{(k)}, \quad \mathbf{e}_S^{(k)} = \left(\mathbf{e}_S, \mathrm{p}^{(k)}\right); \tag{1.46}$$

the surface tensors $\mathbf{e}_S^{(k)}$ are determined by

$$\begin{aligned}
&\mathbf{e}_S^{(k)} = \hat{e}_{\alpha\beta}^{(k)}\mathbf{r}^\alpha\mathbf{r}^\beta + \hat{e}_{\alpha 3}^{(k)}\left(\mathbf{r}^\alpha\mathbf{n} + \mathbf{n}\mathbf{r}^\alpha\right) + \hat{e}_{33}^{(k)}\mathbf{nn}; \\
&\hat{e}_{\alpha\beta}^{(k)} = \tfrac{1}{2}\left(\nabla_\alpha u_\beta^{(k)} + \nabla_\beta u_\alpha^{(k)}\right) - \\
&-\left(b_{\alpha\beta}\delta_{(m)}^{(k)} + hc_{\alpha\beta}Z_{(m)}^{(k)}\right)u_3^{(m)} - \\
&-Z_{(m)}^{(k)}h\left(b_{\alpha\cdot}^{\cdot\gamma}\nabla_\beta u_\gamma^{(m)} + b_{\beta\cdot}^{\cdot\gamma}\nabla_\alpha u_\alpha^{(m)}\right); \\
&\hat{e}_{\alpha 3}^{(k)} = \tfrac{1}{2}\left(\nabla_\alpha u_3^{(k)} + b_{\alpha\cdot}^{\cdot\beta}u_\beta^{(k)} + D_{(m\cdot)}^{(\cdot k)}u_\alpha^{(m)}\right) - \\
&-hb_{\alpha\cdot}^{\cdot\beta}Z_{(m)}^{(k)}D_{(n\cdot)}^{(\cdot m)}u_\beta^{(n)}; \quad \hat{e}_{33}^{(k)} = D_{(n\cdot)}^{(\cdot m)}u_3^{(m)}.
\end{aligned} \tag{1.47}$$

## STRESS TENSORS FOR A SHELL

Let us introduce the following stress tensors:

$$\boldsymbol{\sigma} = \sigma^{\alpha\beta}\mathbf{r}_\alpha\mathbf{r}_\beta + \sigma^{\alpha 3}\left(\mathbf{r}^\alpha\mathbf{n} + \mathbf{n}\mathbf{r}^\alpha\right) + \sigma^{33}\mathbf{nn} = \tag{1.48}$$

$$= \hat{\sigma}^{\alpha\beta}\mathbf{R}_\alpha\mathbf{R}_\beta + \hat{\sigma}^{\alpha 3}\left(\mathbf{R}^\alpha\mathbf{n} + \mathbf{n}\mathbf{R}^\alpha\right) + \hat{\sigma}^{33}\mathbf{nn}; \tag{1.49}$$

this tensor defines the true stress as the Cauchy one in the nonlinear mechanics [18];

$$\mathbf{s} = s^{\alpha\beta}\mathbf{r}_\alpha\mathbf{r}_\beta + s^{\alpha 3}\mathbf{r}_\alpha\mathbf{n} + s^{3\beta}\mathbf{n}\mathbf{r}_\beta + s^{33}\mathbf{nn} = \tag{1.50}$$

$$= A\left[\hat{\sigma}^{\alpha\beta}\mathbf{r}_\alpha\mathbf{R}_\beta + \hat{\sigma}^{\alpha 3}\left(\mathbf{r}^\alpha\mathbf{n} + \mathbf{n}\mathbf{R}^\alpha\right) + \hat{\sigma}^{33}\mathbf{nn}\right]. \tag{1.51}$$

Let us provide the analogy of the basis $\mathbf{r}^\alpha$ and the initial basis, as well as the basis $\mathbf{R}^\alpha$ and the acting basis in the nonlinear mechanics (i.e. the equidistant surface is obtained by motion of base surface points along normal). Thus, (1.50) is analogous to the Piola tensor;

$$\boldsymbol{\pi} = A\left[\hat{\sigma}^{\alpha\beta}\mathbf{r}_\alpha\mathbf{r}_\beta + \hat{\sigma}^{\alpha 3}\left(\mathbf{r}^\alpha\mathbf{n} + \mathbf{n}\mathbf{r}^\alpha\right) + \hat{\sigma}^{33}\mathbf{nn}\right] \tag{1.52}$$

is analogous to the 2nd Piola stress tensor [18];

$$\mathbf{s} = A\hat{\mathbf{A}}^{-1}\cdot\boldsymbol{\sigma}, \quad \boldsymbol{\pi} = A\hat{\mathbf{A}}^{-1}\cdot\boldsymbol{\sigma}\cdot\left(\hat{\mathbf{A}}^{-1}\right)^{\mathrm{T}}. \tag{1.53}$$

Finally, let us introduce the tensor $\mathbf{t} = A\boldsymbol{\sigma}$:

$$\mathbf{t} = A\left[\sigma^{\alpha\beta}\mathbf{r}_\alpha\mathbf{r}_\beta + \sigma^{\alpha 3}\left(\mathbf{r}^\alpha\mathbf{n} + \mathbf{n}\mathbf{r}^\alpha\right) + \sigma^{33}\mathbf{nn}\right] \tag{1.54}$$

analogous to the Trefftz-Kappus tensor [18].
Let us note that the tensors (1.48), (1.50), (1.52), (1.54) could be represented by their Fourier coefficients being surface tensors:

$$\begin{aligned}
&\boldsymbol{\sigma} = \boldsymbol{\sigma}_{(k)}\mathrm{p}^{(k)}, \quad \boldsymbol{\sigma}_{(k)} = \left(\boldsymbol{\sigma}, \mathrm{p}_{(k)}\right); \\
&\mathbf{s} = \mathbf{s}_{(k)}\mathrm{p}^{(k)}, \quad \mathbf{s}_{(k)} = \left(\mathbf{s}, \mathrm{p}_{(k)}\right); \\
&\boldsymbol{\pi} = \boldsymbol{\pi}_{(k)}\mathrm{p}^{(k)}, \quad \boldsymbol{\pi}_{(k)} = \left(\boldsymbol{\pi}, \mathrm{p}_{(k)}\right); \\
&\mathbf{t} = \mathbf{t}_{(k)}\mathrm{p}^{(k)}, \quad \mathbf{t}_{(k)} = \left(\mathbf{t}, \mathrm{p}_{(k)}\right).
\end{aligned} \tag{1.55}$$

## STRAIN ENERGY FORMULATION

Accordingly to [16], we have,

$$U = \tfrac{1}{2}\int_V \boldsymbol{\sigma} : \mathbf{e}\, dV = \tfrac{1}{2}\int_S \int_{-1}^{1} \boldsymbol{\sigma} : \mathbf{e}\, Ahd\zeta dS. \quad (1.56)$$

Let us define a surface strain energy density:

$$U_S = \int_{-1}^{1} W_S\, hd\zeta, \quad W_S = \tfrac{1}{2} A\boldsymbol{\sigma} : \mathbf{e}; \quad (1.57)$$

Here the strain energy density $W_S$ could be represented using the equivalent formulations, e.g. we have the work-conjugated tensor pairs

$$2W_S = \boldsymbol{\sigma} : \mathbf{e}_A = \mathbf{s} : \mathbf{D} = \boldsymbol{\pi} : \mathbf{e}_S = \mathbf{t} : \mathbf{e}, \quad (1.58)$$

where **D** is the reduced distortion tensor [15]:

$$\begin{gathered} \mathbf{D} = \left(\mathbf{A}^{-1}\right)^{\mathrm{T}} \cdot \mathbf{d} = \\ = \bar{d}_{\alpha\beta}\mathbf{r}^{\alpha}\mathbf{r}^{\beta} + \bar{d}_{\alpha 3}\mathbf{r}^{\alpha}\mathbf{n} + \bar{d}_{3\beta}\mathbf{n}\mathbf{r}^{\beta} + \bar{d}_{33}\mathbf{n}\mathbf{n}. \end{gathered} \quad (1.59)$$

Accounting for (1.42)–(1.47), (1.48)–(1.55) as well as for (1.38) and (1.39), we obtain

$$\begin{gathered} 2U_S = \sigma^{\alpha\beta}_{(k)}\breve{e}^{(k)}_{\alpha\beta} + 2\sigma^{\alpha 3}_{(k)}\breve{e}^{(k)}_{\alpha 3} + \sigma^{33}_{(k)}\breve{e}^{(k)}_{33}, \\ \sigma^{ij}_{(k)} = h\left(\sigma^{ij}, \mathrm{p}_{(k)}\right); \end{gathered} \quad (1.60)$$

$$\begin{gathered} 2U_S = \pi^{\alpha\beta}_{(k)}\hat{e}^{(k)}_{\alpha\beta} + 2\pi^{\alpha 3}_{(k)}\hat{e}^{(k)}_{\alpha 3} + \pi^{33}_{(k)}\hat{e}^{(k)}_{33}, \\ \pi^{ij}_{(k)} = h\left(A\hat{\sigma}^{ij}, \mathrm{p}_{(k)}\right); \end{gathered} \quad (1.61)$$

$$\begin{gathered} 2U_S = t^{\alpha\beta}_{(k)}e^{(k)}_{\alpha\beta} + 2t^{\alpha 3}_{(k)}e^{(k)}_{\alpha 3} + t^{33}_{(k)}e^{(k)}_{33}, \\ t^{ij}_{(k)} = h\left(A\sigma^{ij}, \mathrm{p}_{(k)}\right); \end{gathered} \quad (1.62)$$

$$\begin{gathered} 2U_S = s^{\alpha\beta}_{(k)}\bar{d}^{(k)}_{\alpha\beta} + s^{\alpha 3}_{(k)}\bar{d}^{(k)}_{\alpha 3} + s^{3\beta}_{(k)}\bar{d}^{(k)}_{3\beta} + s^{33}_{(k)}\bar{d}^{(k)}_{33}, \\ s^{ij}_{(k)} = h\left(s^{ij}, \mathrm{p}_{(k)}\right). \end{gathered} \quad (1.63)$$

The formula (1.63) was used to derive the dynamic equations in [14] and [15].

## STRAINS AND STRAIN ENERGY FOR THE FIRST-ORDER SHELL THEORY

Let us consider hence the 1st order shell theory as a particular case of the Nth order:

$$\mathbf{u} = \mathbf{u}^{(0)} + \zeta\mathbf{u}^{(1)}. \quad (1.64)$$

Here $\mathbf{u}^{(0)} = \bar{u}_{\alpha}\mathbf{r}^{\alpha} + w\mathbf{n}$ so that corresponds to (1.7); $\mathbf{u}^{(1)} = \chi_{\alpha}\mathbf{r}^{\alpha} + \psi\mathbf{n}$, here the displacement amplitude $\chi_{\alpha}$ corresponds to the tangent rotation angle, $\varphi^{\alpha} = \mathrm{T}^{\alpha\beta}\chi_{\beta}$, while $\psi$ is the transverse normal strain. Thus, we have

$$\nabla \otimes \mathbf{u}^{(0)} = \mathbf{d}, \quad \mathbf{D} = \mathbf{d} + \psi\mathbf{n}\mathbf{n} + \zeta\mathbf{d}^{(1)};$$

$$\begin{gathered} \mathbf{d}^{(1)} = d^{(1)}_{\alpha\beta}\mathbf{R}^{\alpha}\mathbf{r}^{\beta} + d^{(1)}_{\alpha 3}\mathbf{R}^{\alpha}\mathbf{n}. \\ d^{(1)}_{\alpha\beta} = \nabla_{\alpha}\chi_{\beta} - b_{\alpha\beta}\psi; \; d^{(1)}_{\alpha 3} = \nabla_{\alpha}w + b_{\alpha\cdot}^{\cdot\beta}\bar{u}_{\beta}. \end{gathered} \quad (1.65)$$

Accordingly to (1.11), (1.13), (1.35), (1.65), we obtain the following 1st order approximation of the strain tensor (1.37):

$$\begin{gathered} \hat{e}_{\alpha\beta} = \varepsilon_{\alpha\beta} + \zeta \text{Ш}_{\alpha\beta} - \zeta b_{\alpha\beta}\psi - \\ -\tfrac{1}{2}\zeta^2\left(b^{\gamma}_{\cdot\alpha}d^{(1)}_{\beta\gamma} + b^{\cdot\gamma}_{\beta\cdot}d^{(1)}_{\alpha\gamma}\right); \end{gathered} \quad (1.66)$$

$$\hat{e}_{\alpha 3} = \tfrac{1}{2}\left(\vartheta_{\alpha} + \chi_{\alpha}\right) + \tfrac{1}{2}\zeta\nabla_{\alpha}\psi, \quad \hat{e}_{33} = \psi. \quad (1.67)$$

Thus, the reduced strain tensor (1.37) of the 1st approximation referred to the basis $\mathbf{r}^{\alpha}$, $\mathbf{n}$ on the fibration $TS \otimes NS$ is given by the surface strains, i.e. tangent strain (1.11) and bending strain (1.13) of the surface as well as the averaged transverse shear $\theta_{\alpha}$, splitting shear $\nabla_{\alpha}\psi$ [7] and the normal transverse $\psi$.

Let us consider hence the true strain tensor **e** in terms of the 1st order approximation; it is given by the covariant components (1.66), (1.67). Since this tensor is referred to the basis $\mathbf{R}^{\alpha}$, $\mathbf{n}$ on the fibration $T^{e}_{\zeta}S \otimes NS$ it does not allow one to create a set of surface tensors by the integration over the thickness.

Let us represent this tensor by the particular sum of Taylor series with respect to $\zeta$; since we have the formula for the base vectors $\mathbf{R}^\alpha$:

$$\mathbf{R}^\alpha \approx \mathbf{R}^\alpha\Big|_{\zeta=0} + \zeta\partial_\zeta \mathbf{R}^\alpha\Big|_{\zeta=0} = \left(\delta^\alpha_\beta + \zeta b^{\cdot\alpha}_{\beta\cdot}\right)\mathbf{r}^\beta, \quad (1.68)$$

Use of the Taylor series together with (1.68) gives the following strain tensor $\mathbf{e}$:

$$\mathbf{e} = e_{\alpha\beta}\mathbf{r}^\alpha\mathbf{r}^\beta + e_{\alpha 3}\left(\mathbf{r}^\alpha\mathbf{n} + \mathbf{n}\mathbf{r}^\alpha\right) + e_{33}\mathbf{nn},$$
$$e_{\alpha\beta} \approx \varepsilon_{\alpha\beta} + \zeta\kappa_{\alpha\beta}, \quad (1.69)$$
$$\kappa_{\alpha\beta} = Ш_{\alpha\beta} + b^{\gamma\cdot}_{\cdot\alpha}\varepsilon_{\beta\gamma} + b^{\gamma\cdot}_{\cdot\beta}\varepsilon_{\alpha\gamma};$$

thus, the first-order term obtained by Taylor expansion gives the reduced bending strain tensor $\kappa_{\alpha\beta}$ corresponding to the one used in classical shell theories (e. g. see [2, 3]).

Let us note that the tangent strain $\varepsilon_{\alpha\beta}$ and the reduced bending strain $\kappa_{\alpha\beta}$ definig the tensor $\mathbf{e}_T = e_{\alpha\beta}\mathbf{r}^\alpha\mathbf{r}^\beta$ are two quantities being unknowns in the compatibility equations of the classical theory of thin shells [2]:

$$\nabla_\beta \mathrm{T}^{\beta\gamma}\mathrm{T}^{\alpha\delta}\kappa_{\gamma\delta} - b^{\cdot\alpha}_{\beta\cdot}\nabla_\delta \mathrm{T}^{\beta\gamma}\mathrm{T}^{\alpha\delta}\varepsilon_{\gamma\delta} = 0;$$
$$\nabla_\alpha\nabla_\beta \mathrm{T}^{\alpha\gamma}\mathrm{T}^{\beta\delta}\varepsilon_{\gamma\delta} + b_{\alpha\beta}\mathrm{T}^{\alpha\gamma}\mathrm{T}^{\beta\delta}\kappa_{\gamma\delta} = 0,$$

that provide the static-geometry analogy [2].

Accounting for (1.62), (1.33) and introducing averaged shear $\theta_\alpha = \vartheta_\alpha + \chi_\alpha$, we have hence

$$2U_S = \pi^{\alpha\beta}_{(0)}\varepsilon_{\alpha\beta} + 2\pi^{\alpha 3}_{(0)}\theta_\alpha + \pi^{33}_{(0)}\psi + \\ +\pi^{\alpha\beta}_{(1)}\left(Ш^{(1)}_{\alpha\beta} - b_{\alpha\beta}\psi\right) + 2\pi^{\alpha 3}_{(1)}\nabla_\alpha\psi, \quad (1.70)$$

$$\pi^{\alpha\beta}_{(0)} = h\int_{-1}^{1}\hat{\sigma}^{\alpha\beta}A\,d\zeta,\ \pi^{\alpha\beta}_{(1)} = h\int_{-1}^{1}\hat{\sigma}^{\alpha\beta}A\zeta\,d\zeta, \quad (1.71)$$

$$\pi^{\alpha 3}_{(0)} = h\int_{-1}^{1}\hat{\sigma}^{\alpha 3}A\,d\zeta,\ \pi^{\alpha 3}_{(1)} = h\int_{-1}^{1}\hat{\sigma}^{\alpha 3}A\zeta\,d\zeta; \quad (1.72)$$

$$\pi^{\alpha\beta}_{(2)} = \tfrac{h}{2}\int_{-1}^{1}\hat{\sigma}^{\alpha\beta}A\zeta^2\,d\zeta;\ \pi^{33}_{(1)} = \int_{-1}^{1}\hat{\sigma}^{33}A\,d\zeta. \quad (1.73)$$

Here $\pi^{\alpha\beta}_{(0,1)}$, $\pi^{\alpha 3}_{(0,1)}$, $\pi^{33}_{(0)}$ are expansion factors of the "2nd Piola" tensor $\boldsymbol{\pi}$, or the moments of the contravariant components of the true strain tensor $\hat{\sigma}^{ij}$. Here the tensor given by $\pi^{\alpha\beta}_{(0)}$ works on the tangent strain $\varepsilon_{\alpha\beta}$ (1.11), the tensor given by $\pi^{\alpha\beta}_{(1)}$ works on the bending strain $Ш_{\alpha\beta}$ (1.13) (and $b_{\alpha\beta}\psi$), while $\pi^{\alpha 3}_{(0)}$ works on the averaged shear $\theta_\alpha$. Thus, $\pi^{\alpha\beta}_{(0)}$ defines the work-conjugated tangent force tensor, $\pi^{\alpha\beta}_{(1)}$ defines the work-averaged bending couple, while $\pi^{\alpha 3}_{(0)}$ is the work-averaged shear force and finally $\pi^{\alpha 3}_{(1)}$ is the work-averaged splitting force [7]. Let us note that these quantities differ from the ones working on $\delta\overline{u}_\alpha$, $\delta\chi_\alpha$ and being unknown quantities in the dynamic equations [15]; the last ones were defined as $\mathbf{s}_{(0)}$, $\mathbf{s}_{(1)}$, … being surface tensors on $TS \oplus NS$ [7]. Accounting for the formulae for the contravariant components of the stress tensor $\mathbf{s}$ [15]:

$$s^{\alpha\beta} = AA^{\beta\cdot}_{\cdot\gamma}\sigma^{\alpha\gamma} = AA^{\cdot\alpha}_{\delta\cdot}\hat{\sigma}^{\delta\beta},$$
$$s^{3\beta} = AA^{\beta\cdot}_{\cdot\gamma}\sigma^{3\gamma} = A\hat{\sigma}^{3\beta},\ s^{\alpha 3} = A\sigma^{\alpha 3} = AA^{\cdot\alpha}_{\delta\cdot}\hat{\sigma}^{\delta 3},$$

we obtain the quantities equivalent to the ones introduced in [19]. The stresses used in classical shell theories [2] appears as a result of neglecting of the value A = $\sqrt{g/a}$ (e.g. see [2]), the similar assumption was used by I.N.Vekua for the general theory of thin shallow shells [7]; such a tensor could be obtained as $\mathbf{S} = \mathbf{A}^{-1}\cdot\boldsymbol{\sigma}$, therefore we have

$$S^{\alpha\beta}_{(0)} = h\int_{-1}^{1}A^{\cdot\alpha}_{\gamma\cdot}\hat{\sigma}^{\gamma\beta}d\zeta,\ S^{\alpha\beta}_{(1)} = h\int_{-1}^{1}A^{\cdot\alpha}_{\gamma\cdot}\hat{\sigma}^{\gamma\beta}\zeta\,d\zeta.$$

On the other hand one could consider the strain energy (1.62) with the components of the strain $e^{(k)}_{ij}$ defined by the formula (1.69); this is the formulation that uses the tangent and bending strains accordingly to [2]. Thus, we obtain the following definitions of the tangent force and bending moment:

$$t_{(0)}^{\alpha\beta} = h\int_{-1}^{1} t^{\alpha\beta} d\zeta, \quad t_{(1)}^{\alpha\beta} = h\int_{-1}^{1} t^{\alpha\beta}\zeta d\zeta; \quad (1.74)$$

here the contravariant components $t^{ij} = A\sigma^{ij}$ of the stress tensor referred to the basis $\mathbf{r}_\alpha$, $\mathbf{n}$ in the fibration $TS \otimes NS$ are used contrarily to where the ones referred to the basis $\mathbf{R}^\alpha$, $\mathbf{n}$ in the fibration $T_\zeta S \otimes NS$ were used. The definition of tangent forces and couples tensors corresponding to [2] could be obtained by neglecting the multiplier $A$.

## CONCLUSIONS

One can conclude that several different strain tensors could be considered. These definitions are required, in particular, to derive the constraint equations for extended higher-order shell theories [20] that appears a result of the translation of boundary conditions from faces onto a base surface. The further construction of extended shell theories without the use of constraint multiplier could be based on the constraint equations represented in terms of generalized strains. Moreover, the statement of the boundary value problem of the higher-order shell theory in terms of generalized forces requires the appropriate compatibility equations such as the ones proposed in [22] where only the 1st order shell model was considered; various definitions of strain tensors generate different formulations of the two-dimensional compatibility equations. On the other hand the solution of problems of coupled dynamics of medium-thickness shells and surrounding media (e.g. see [23, 24]) require constraint equations corresponding to the impermeability condition on wetted shell face; such equation could be also expressed in terms of strains, thus, an approach similar to [2] where the strain boundary conditions were defined on shells' contours. Finally, the solution of different static problems [25, 26] could be more efficient if the stress-based problem statement is used as a background.

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# РЕЦЕНЗИЯ

**на монографию «ЖЕЛЕЗОБЕТОННЫЕ СОСТАВНЫЕ КОНСТРУКЦИИ ЗДАНИЙ И СООРУЖЕНИЙ»: Баширов Х. З, Колчунов Вл. И., Федоров В. С., Яковенко И.А. – М.: Издательство АСВ, 2017 – 248 с.**

Актуальность научных исследований железобетонных, в том числе, и составных железобетонных конструкций только возрастает, вследствие получения новых знаний из смежных и других областей науки, технического прогресса, в частности, на основе повышения возможностей современных диагностического и экспериментального оборудования и приборной базы.

Монография представляет собой предметный подход к развитию теории расчета железобетонных составных конструкций, в которой авторы иерархически последовательно дают анализ содержания более полных реальных явлений с учетом ранее не учитываемых и поэтапного преобразования их в физические, расчетные и математические модели механического состояния железобетонных составных конструкций по нормальным и наклонным сечениям.

В обоснования достоверности представленных результатов необходимо отметить, что фундаментом исследований является экспериментальный материал, а теоретические разработки построены с использованием известных положений механики твердого деформируемого тела, теории составных стержней и теории железобетона.

В работе впервые представлен многоуровневый процесс трещинообразования и раскрытия трещин в железобетонных составных конструкциях и предложена иерархия разделения трех типов трещин на характерные веера. В железобетонных составных конструкциях рассмотрена работа зоны сопряжения двух материалов и модернизировано в приложении к железобетонным составным конструкциям уравнение теории составных стержней профессора А. Р. Ржаницына, а именно в отличие от дифференциального уравнения второго порядка было получено дифференциальное уравнение первого порядка, учитывающее физически нелинейное деформирование железобетона с трещинами. Экспериментально установлен и учтен эффект нарушения сплошности бетона в составных железобетонных конструкциях.

Теоретическая значимость работы состоит в том, что на основе обобщения и анализа опытных данных обосновано сформулированы рабочие гипотезы, инварианты и предпосылки для построения расчетных моделей, наиболее полно учитывающие всё многообразие различных типов трещин и отражающие действительное напряженно-деформированное состояние материалов по сечению таких конструкций. Следует отметить, что процессы трещинообразования, деформирования и разрушения железобетонных составных конструкций рассматриваются наиболее полно с учетом вышеуказанных работы сил в шве между бетонами, несовместности деформаций бетона и арматуры и эффекта нарушения сплошности бетона. Предложенный подход к трещинообразованию предопределил построение оригинальной 5и-блочной расчетной схемы.

Практическая значимость работы состоит в том, что разработана единая методика расчета таких конструкций по предельным состояниям первой и второй группам с более полным учетом параметров и особенностей деформирования арматуры и бетона, что позволяет получить в одних случаях более достоверные решения, в других – выявить резервы для эффективного использования материалов. На основе результатов исследования даны предложения по синтезу инновационных решений железобетонных составных конструкции и конструктивных систем для проектируемых и реконструируемых транспортных зданий и сооружений.

Опубликованные результаты являются полезным инструментарием для ученых и специалистов по исследованию железобетона, проектированию сборно-монолитных железобетонных конструкций и других составных железобетонных конструкции, получаемые при усилении в ходе реконструкции.

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