International Journal for Computational Civil and Structural Engineering, 20(1) 116–123(2024)

DOI:10.22337/2587-9618-2024-20-1-116-123

ON ERRORS WHEN REPLACING NON-BIFURCATIONAL STABILITY PROBLEMS FOR ELASTIC FRAMES WITH BIFURCATIONAL ONES

Gaik A. Manuylov, Sergey B. Kosytsyn, Maxim M. Begichev
Russian University of Transport, Moscow, RUSSIA

Abstract: the method of replacing the existing distributed load with an equivalent load in the form of concentrated forces applied at the nodes is often used in frame stability problems. This replacement of the initial flexural equilibrium with an unbended tension-compression equilibrium introduces certain errors in the calculated values of critical loads. These errors can be of different signs (ie, the calculated "pseudo-bifurcation" critical force can be either greater than the actual one or less than it). The paper demonstrates the errors of such a load change on the example of frame stability problems.

Keywords: stability, geometric nonlinearityfinite element method, bifurcation, limit point

ОБ ОШИБКАХ ПРИ ЗАМЕНЕ НЕБИФУРКАЦИОННЫХ ЗАДАЧ УСТОЙЧИВОСТИ УПРУГИХ РАМ НА БИФУРКАЦИОННЫЕ

Г.А. Мануйлов, С.Б. Косицын, М.М. Бегичев
Российский университет транспорта, г. Москва, РОССИЯ

Аннотация: При расчетах рам на устойчивость достаточно часто используется прием замены действующей распределенной нагрузки на эквивалентную нагрузку в виде сосредоточенных сил, приложенных в узлах. Такая замена начального изгибного равновесия на безизгибное равновесие растяжения-сжатия вносит определенные ошибки в вычисленные значения критических нагрузок. Эти ошибки могут быть разных знаков (т.е. вычисленная «псевдобифуркационная» критическая сила может оказаться как больше действительной, так и меньше её). В работе продемонстрированы ошибки такой замены нагрузки на примере задач устойчивости рам.

Ключевые слова: устойчивость, геометрическая нелинейность, метод конечных элементов, бифуркация, предельная точка

It is known that when calculating frames for stability, the method of replacing the existing distributed load with an equivalent nodal load in the form of concentrated nodal forces is often used [1, 2]. The distributed load causes flexural stress-strain state, and the frame equilibrium stability problem either becomes non-bifurcational, or is reduced to the construction of a stable equilibrium curve without singular points. If the initial post-bifurcation equilibrium of the frame is unstable, then the frame turns out to be sensitive to initial (subcritical) imperfections in the form of deflections from a distributed load. The above-mentioned sensitivity is manifested in a sharp decrease in the critical load at the limiting points in comparison with the bifurcation load even at relatively small amplitudes of these deflections.

Reduction of the non-bifurcation stability problem to the bifurcation problem by replacing the initial flexural equilibrium with an unbending stress-compression equilibrium introduces certain errors in the values of critical loads. These errors can be of different signs (the calculated "pseudo-bifurcational" critical force can be either greater than the actual one or less than it).

The idea of transition to a bifurcation problem when calculating the stability of frames appeared due to the fact that in the recent past, the engineers did not have the opportunity to obtain sufficiently reliable solutions of geometrically nonlinear problems such as of construction of equilibrium curves up to the limiting point. Bifurcation problems allow for the possibility of linearized solution of stability loss problems. If the stability problem is non-bifurcational (loss of stability at the limiting point), then there is no linearized solution for such a problem. It can be solved only with the help of a geometrically nonlinear formulation of the equilibrium stability problem. The possibility of such a solution appeared after the creation of computational FEM complexes (NASTRAN, ANSYS, etc.) [3]. However, until now students are taught the transition from the real geometrically nonlinear problem of the frame stability loss at the limiting point to the “pseudo-bifurcational” one, followed by the search for the first zero of the characteristic determinant.

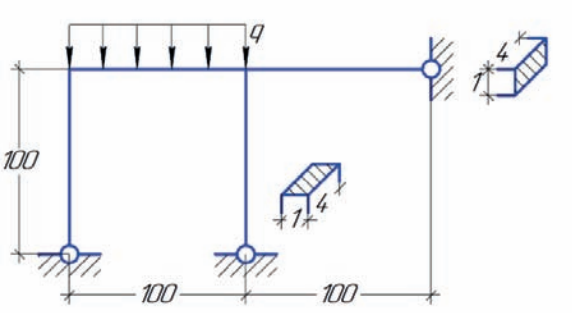


Figure 1. 2-L-shaped frame

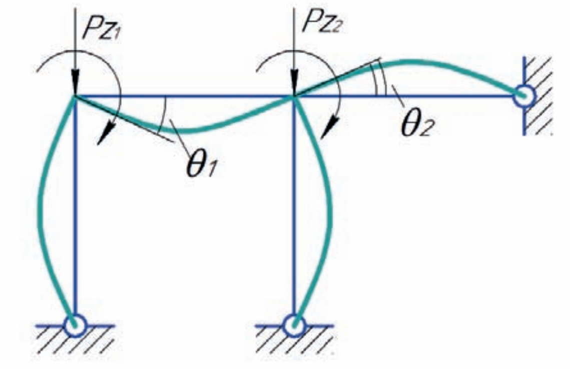


Figure 2. 2-L-shaped frame loaded with two concentrated forces

Let us show the example of such error (because of a "reduction") by the example of a simple 2-L-shaped frame loaded in the left span of the girder with distributed load (Fig. 1).

The frame elements had the same length l = 100 cm and the same cross-sections (rectangular 4 × 1 cm). Material - St. 3, $E = 2.1\times10^6$ kg/cm^2, $\nu = 0.3$. The model of each terminal consisted of 100 beam FE. Solving this stability problem with NASTRAN (the buckling option) gives $(ql)_{cr} = 1890\,kg$. This is the value of the critical load in the linear approximation (according to the distribution of longitudinal forces according to the solution of the static equilibrium problem).

If, instead of a distributed load, this frame is loaded with two concentrated forces $P = ql/2$ (Fig. 2), then we obtain the bifurcation stability problem. Its solution by the displacement method leads to a stiffness matrix [4]

$$r = \frac{EJ}{l}\begin{bmatrix} 4+3\varphi_1(\nu) & 2 \\ 2 & 7+3\varphi_1(\nu) \end{bmatrix}$$

where the function $\varphi_1(\nu)$ is a special function by A.F. Smirnov [5], taking into account changes in the flexural stiffness of compressed beams as the compressive load increases.

The parameter $\nu = l\sqrt{P/(EJ)}$ for the first time turns out to be critical if $\nu = \nu_{cr} = 3{,}726$ $(det\, r\,(3{,}726) = 0)$. The form frame’s loss of stability through nodal turns ($\theta_1 = 0{,}597$ and $\theta_2 = 0{,}297$) is shown on Fig. 2 and corresponds to the critical load $(ql)_{cr} = 2P_{cr} = 1943\,kg$. This value is slightly larger than the linearized FEM solution.

The actual frame’s loss of stability from the action of a distributed load according to a geometrically nonlinear solution according to FEM (NASTRAN) occurs at the limit point with a significantly lower critical load $(ql)_{cr} = 1473\,kg$. The difference between this load and the "pseudo-bifurcational" (1943 kg) was ~ +32%.

$$\Delta P_{cr}\% = \frac{1943 - 1473}{1473}100\% \cong 32\%$$

For this frame the initial post-bifurcation equilibrium is unstable (the frame "jumps" into a distant strongly curved equilibrium). Consequently, this frame is sensitive to initial imperfections (which are deflections from a distributed load). However, this does not lead to the conclusion that when replacing any distributed load with nodal forces, the pseudo-bifurcation critical load will always be greater than the actual critical load at the limiting point. The next example is an illustration of the opposite situation, when the pseudo-bifurcation load turned out to be less than the actual one.

Let us consider the same 2-L-shaped non-free frame, loaded with a distributed load q = const throughout the girder (Fig. 3). The critical load at the limiting point was ~ 12.8 kN/m. The sequence of the frame's deformed state development is shown in Fig. 4.

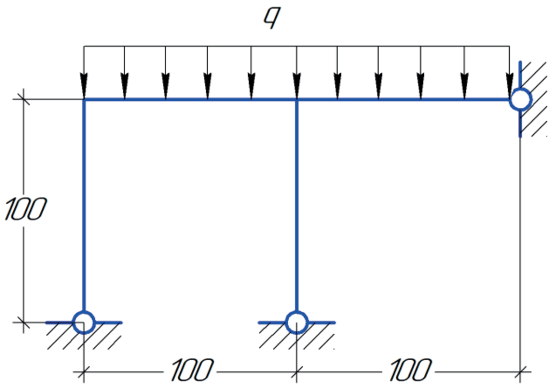


Figure 3. 2-L-shaped non-free frame

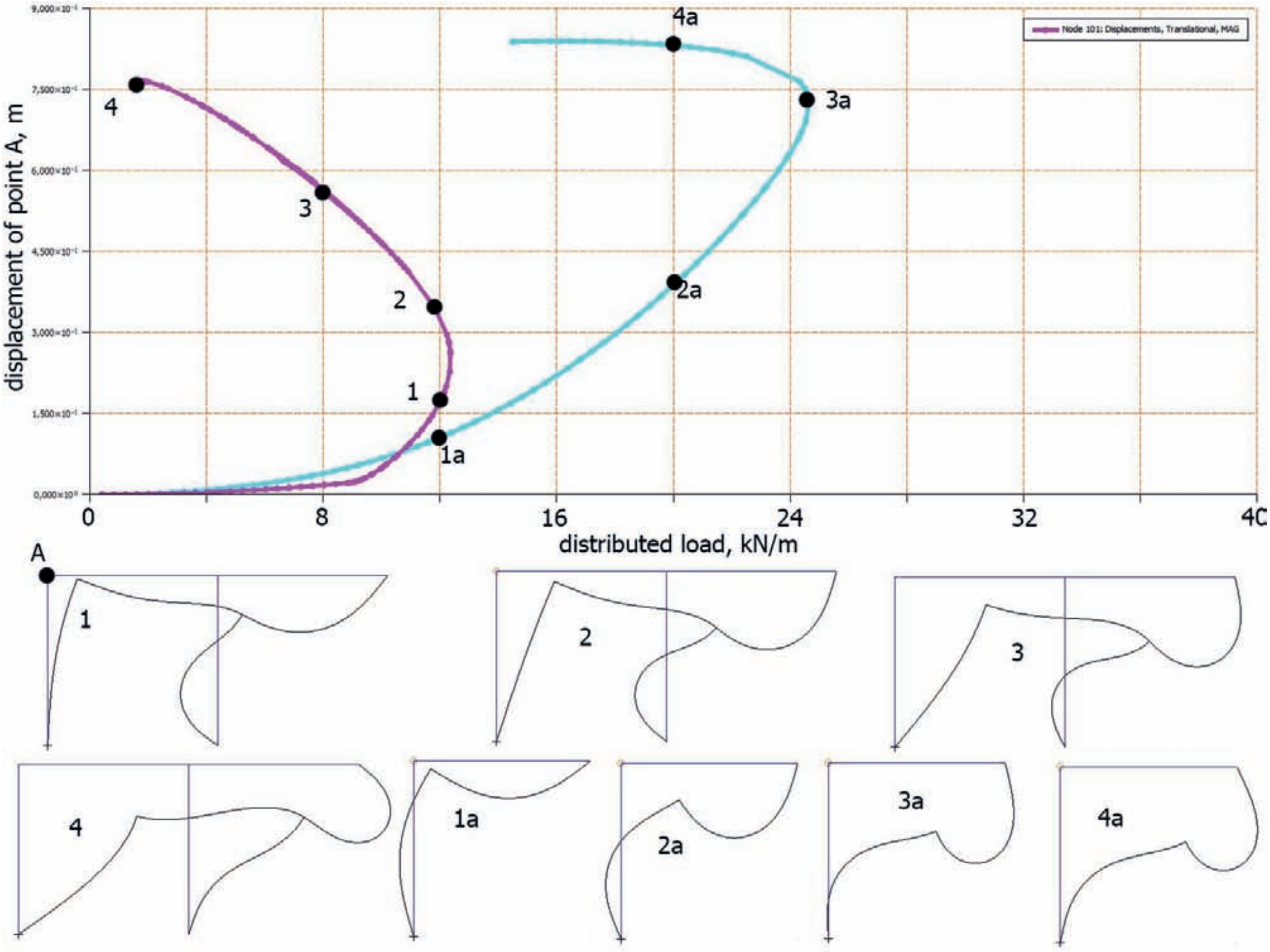


Figure 4. Frame's deformed shape

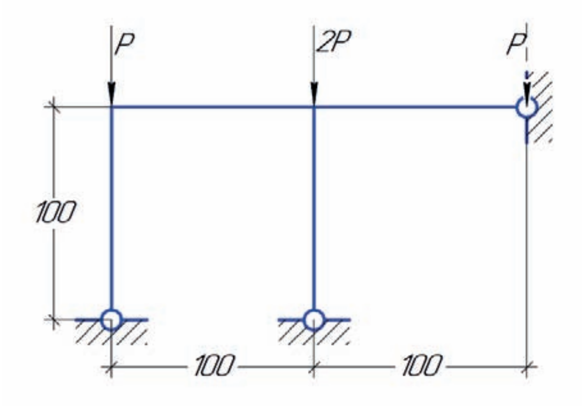


Figure 5. *Frame loaded by concentrated forces*

If the distributed load is replaced by an equivalent nodal load (concentrated forces) $P = ql/2$ and $2P = ql$ (Fig. 5), then the solution of the corresponding bifurcation problem leads to the determination of the first zero of the stiffness matrix's determinant

$$r(\nu) = \begin{bmatrix} 4 + 3\varphi_1(\nu) & 2 \\ 2 & 7 + 3\varphi_1(\sqrt{2}\nu) \end{bmatrix} \frac{EJ}{l},$$

$$\nu = l\sqrt{\frac{P}{EJ}}$$

The φ₁ function argument for the right frame pole is $\sqrt{2}$ times larger than the corresponding argument for the left frame pole. As a result of calculations using tables of special functions by A.F.Smirnov [5], it was found that $\nu_{cr\,1} = \nu_{cr\,min} = 2{,}819$, and the value $P_{cr} \cong 552\ kg$. The total critical load $P_{cr}^{\Sigma} = 4P = 2225{,}1\ kg$, and the critical value of the distributed load is

$$\tilde{q}_{cr} = \frac{P_{cr}^{\Sigma}}{2} = 1112{,}5\ kg/m = 11{,}12\ kg/m$$

This is slightly less than the critical load at the limiting point (~12.8 kN/m). The error was

$$\Delta q_{cr}\% = \frac{11{,}12 - 12{,}8}{12{,}8} 100\% \cong -11{,}7\%$$

The NASTRAN solution for concentrated forces P and 2P gave $\mathrm{P}_{cr}^{lin} = 11{,}6\ kN$ (Fig. 6).
Thus, errors in determining the critical load in the transition to the pseudo-bifurcation problem can indeed be both in the direction of increasing the calculated critical force and in the direction of decreasing it.

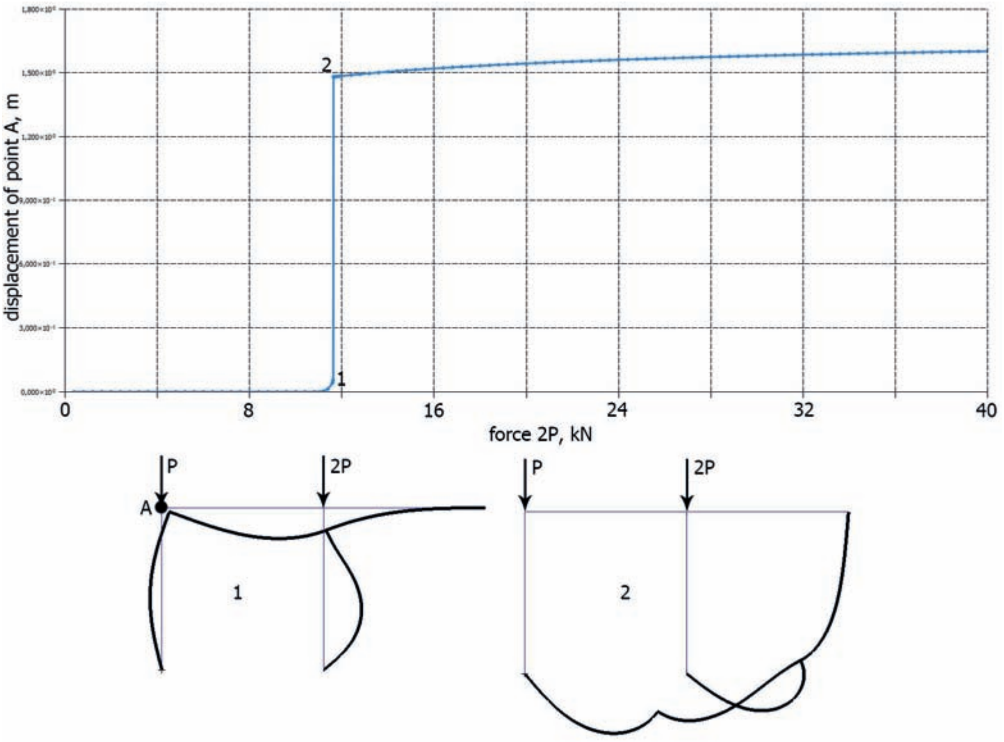
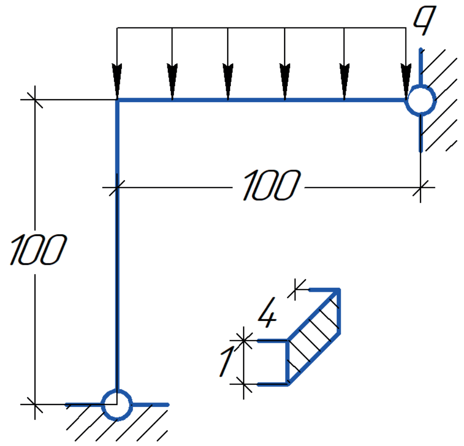


Figure 6. *Stability of 2-L-shaped non-free frame, loaded with forces P and 2P*



Picture 7. L-shaped frame under the action of a distributed load

In another problem, the stability of the L-shaped frame under the action of a distributed load was investigated (Fig. 7). The dimensions of the beams and their cross-sections are the same as in the previous examples. The solution using the FE complex NASTRAN gave the critical load at the limiting point (point 3a on the equilibrium curve, Fig. 4), equal to 24.6 kN/m. This frame also loses its stability "in the large" (Fig. 4).

The NASTRAN solution for the frame with the nodal force P compressing the frame strut gave a critical force $q_{cr} = 24{,}6\ kN/m$. Solution in the linear approximation $P_{cr}^{bif} \cong 9{,}73\ kN$. Then $(ql)_{cr} = 19{,}44\ kN/m$. The error of this approximate solution was -21%. This error is significantly larger than in the previous problem.

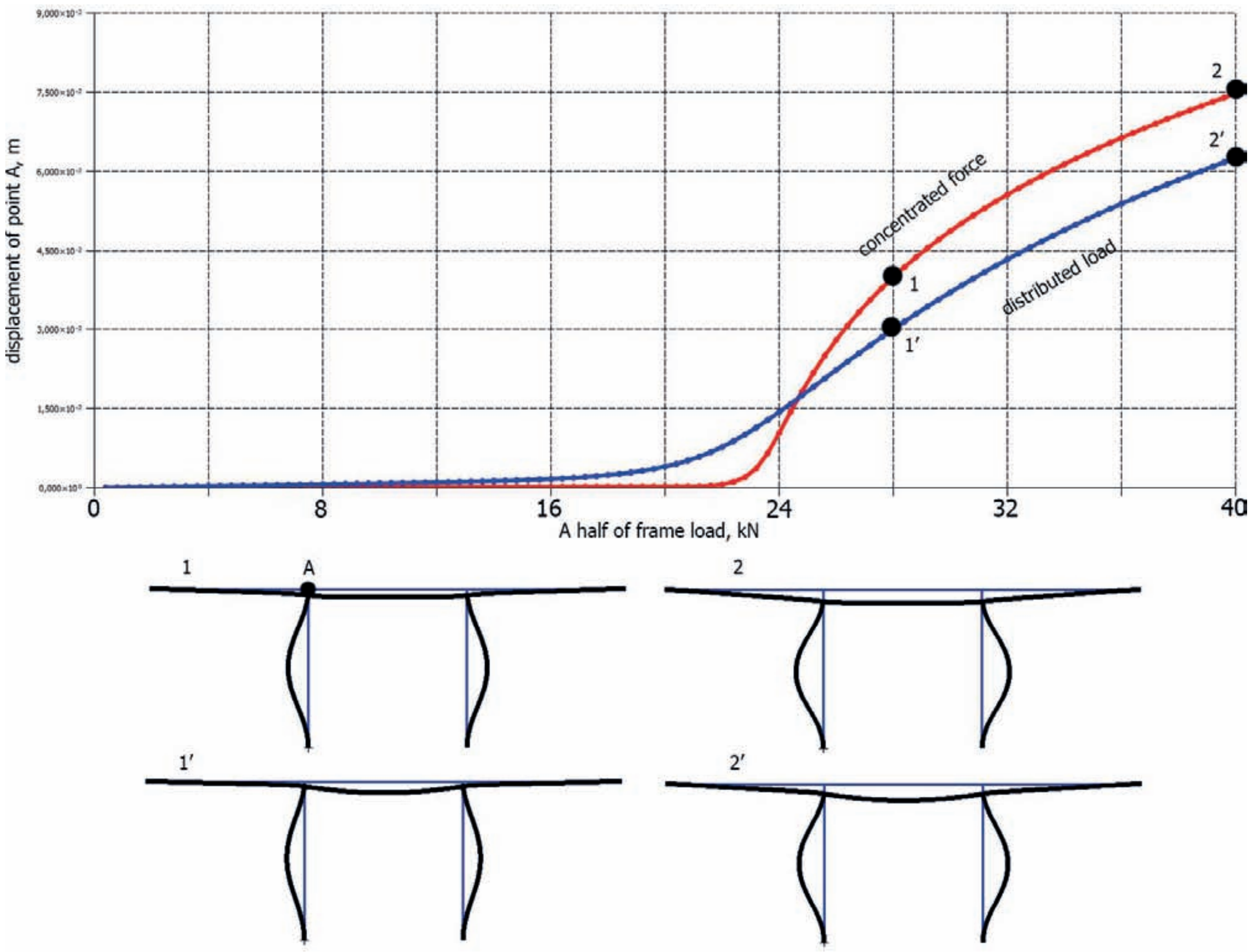


Figure 8. Symmetrically loaded symmetrical frame

For a non-free symmetrical frame symmetrically loaded along the middle span (Fig. 8), the expected loss of stability in a skew-symmetric shape does not occur. The frame deforms smoothly with increasing load, keeping the symmetrical shape of stable deformed equilibrium (point 1, Fig. 8).

If the distributed load is replaced by the nodal one ($P = \frac{ql}{2}$), then the problem becomes bifurcational. When forces P reach a critical value ($P_{cr} \cong 2038\, kg, E = 2 * 10^6\, kg/cm^2,$ $l = 100\, cm,\ rectengular\ cross-section$ 4×1 cm), a stable bifurcation occurs into a symmetric compressed-curved equilibrium (curve 0-1-2 on Fig. 8). In fact, there is no bifurcation under the action of a distributed load q = const. Moreover, the equilibrium from this load can be interpreted as equilibrium with imperfection in the form of symmetric deflections "subordinate" to the post-bifurcation curve from the action of the nodal forces P. Replacing the load with the nodal forces leads to the appearance of an "imposed" bifurcation, which does not actually occur.

In conclusion, we will give an example of a frame for which the error in the critical load in the transition to the initial momentless equilibrium is negligible. Consider a symmetrical Π-shaped frame loaded with an equally distributed load (Fig. 9).

This frame loses its stability as an equilibrium bifurcation, since it experiences symmetric subcritical stress-strain state. This stress-strain state is incomplete, and for it there is an energetically orthogonal complement, which consists of skew-symmetric stress-strain state. The "zero" eigenvector is the vector of skew-symmetric displacements (with the selected ratios of the frame dimensions). This vector, which essentially determines the form of the frame buckling [6, 7], belongs to the aforementioned supplement. The work of external forces q on the displacements set by the eigenvector is equal to zero (Fig. 9). Therefore, under the action of a distributed load, this frame loses its stability as a symmetrical stable bifurcation. The critical load obtained from the solution of the bifurcation problem under nodal loading by forces $P = \frac{ql}{2}$ almost coincides with the critical load caused by the action of the distributed load q.

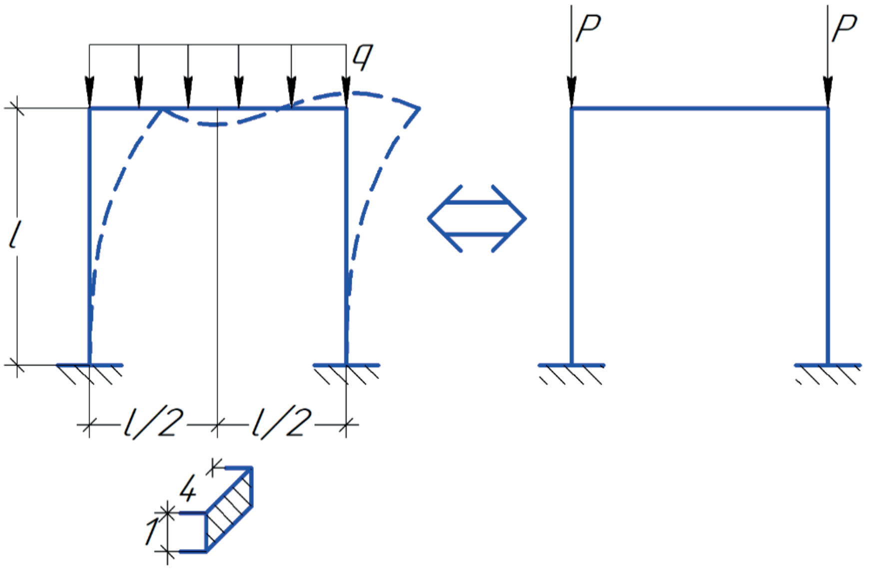


Figure 9. Π-shaped frame loaded with a distributed load

The solution to the characteristic equation gives an almost exact critical load (section 4×5 cm, $(ql)_{cr}\,J = 41{,}666\,cm, E = 2\cdot 10^6\,kg/\mathrm{cm}^2$)

$$\left(6+\frac{\nu}{tg\,\nu}\right)\frac{EJ}{l}=0, \nu_{cr\,1}=2{,}716,$$
$$P_{cr}=61472\,kg, P_{cr}^{\Sigma}=122944\,kg$$

When calculating for such a load, bifurcation into an asymmetric shape occurs at a slightly higher critical force ($P_{\Sigma}\sim 122600\,kg$) In this problem, all of the obtained results coincided within the first three digits.

Let us note that in this problem, the "forced" distribution of the values of the nodal load (and longitudinal forces in the vertical beams), due to the symmetry of the frame, coincides with the actual distribution of these forces from the action of the distribution load.

According to NASTRAN, $P_{cr}^{\Sigma}=121300\,kg$.

REFERENCES

1. **Aleksandrov A.V., Potapov V.D., Zylev V.B.** Stroitelnaya mekhanika: uchebnoye posobiye dlya studentov vuzov zheleznodorozhnogo transporta: v dvukh knigakh. – Moscow: Vyssh. shk., Kn. 2: Dinamika i ustoychivost uprugikh sistem. - 2008. - 383 p.
2. **Bleykh F.** Ustoychivost metallicheskikh konstruktsiy / M.: Fizmatgiz, 1959. — 544 p.
3. **Shimkovich D.G.** Raschet konstruktsiy v MSC/NASTRAN for Windows. M.: DMK Press, 2003, 448 p.
4. **Zhukov K.A., Manuylov G.A.** Dvustoronniye otsenki i priblizheniya k naimenshemu sobstvennomu znacheniyu ramnykh sistem metodom neosobennykh prodolzheniy. Tr. MIIT Chislennyye metody resheniya zadach stroitel'noy mekhaniki. – Moscow: MIIT. – 1990. – p. 88-99.
5. **Smirnov A.F.** Tablitsy funktsii dlya rascheta sterzhnevykh sistem na ustoychivost i kolebaniya. Moscow: MIIT. – 1961. – 93 p.
6. **Manuylov G.A.** O reshenii kharakteristicheskikh uravneniy metodom prodolzheniy s poiskom shaga. Trudy MIIT № 458 Voprosy stroitel'noy mekhaniki prostranstvennykh sistem na s transporte i v stroitelstve. Moscow. – 1974. – p. 104-115.
7. **Zhukov K.A., Manuylov G.A.** Dvustoronniye otsenki i priblizheniya k naimenshemu sobstvennomu znacheniyu ramnoy sistemy metodom neosobennykh prodolzheniy. Trudy MIIT № 827 Chislennyye metody resheniya zadach stroitelnoy mekhaniki transportnykh sooruzheniy. Moscow. – 1990. – p. 88-99.

СПИСОК ЛИТЕРАТУРЫ

1. **Александров А.В., Потапов В.Д., Зылев В.Б.** Строительная механика: учебное пособие для студентов вузов железнодорожного транспорта: в двух книгах. - Москва: Высш. шк., Кн. 2: Динамика и устойчивость упругих систем. – 2008. – 383 с.
2. **Блейх Ф.** Устойчивость металлических конструкций / М.: Физматгиз, 1959. — 544 с.
3. **Шимкович Д.Г.** Расчет конструкций в MSC/NASTRAN for Windows. М.: ДМК Пресс. – 2003. – 448 с.
4. **Жуков К.А., Мануйлов Г.А.** Двусторонние оценки и приближения к наименьшему собственному значению рамных систем методом неособенных продолжений. Тр. МИИТ Численные методы решения задач строительной механики. М. МИИТ. – 1990 г. – с. 88-99.
5. **Смирнов А.Ф.** Таблицы функции для расчета стержневых систем на устойчивость и колебания. М. МИИТ. – 1961 г. – 93 с.
6. **Мануйлов Г.А.** О решении характеристических уравнений методом продолжений с поиском шага // Труды МИИТ № 458 Вопросы строительной механики пространственных систем на с

транспорте и в строительстве. Москва 1974. – с. 104-115.

7. **Жуков К.А., Мануйлов Г.А.** Двусторонние оценки и приближения к наименьшему собственному значению рамной системы методом неособенных продолжений // Труды МИИТ № 827 Численные методы решения задач строительной механики транспортных сооружений. – М. 1990. – с. 88-99.

Gaik A. Manuylov, Ph.D., Associate Professor, Department of Structural Mechanics, Russian University of Transport; 127994, Russia, Moscow, 9b9 Obrazcova Street; phone/fax +7(499)972-49-81.

Sergey B. Kosytsyn, Dr.Sc., Professor, Head of Department of Theoretical Mechanics, Russian University of Transport; 127994, Russia, Moscow, 9b9 Obrazcova Street; phone/fax: +7(499) 978-16- 73; E-mail: kositsyn-s@yandex.ru, kositsyn-s@mail.ru

Maxim M. Begichev, Ph.D., Associate Professor, Department of Theoretical Mechanics, Russian University of Transport; 127994, Russia, Moscow, 9b9 Obrazcova Street; phone/fax: +7(499) 978-16-73; E-mail: noxonius@mail.ru

Мануйлов Гайк Александрович, кандидат технических наук, доцент, доцент кафедры «Строительная механика» Российского университета транспорта; 127994, г. Москва, ул. Образцова, 9, стр. 9; тел./факс +7(499) 972-49-81

Косицын Сергей Борисович, доктор технических наук, профессор, заведующий кафедрой «Теоретическая механика» Российского университета транспорта; 127994, г. Москва, ул. Образцова, 9, стр. 9; тел./факс +7(499) 978-16-73; E-mail: kositsyn-s@yandex.ru, kositsyn-s@mail.ru

Бегичев Максим Михайлович, кандидат технических наук, доцент кафедры «Теоретическая механика» Российского университета транспорта; 127994, г. Москва, ул. Образцова, 9, стр. 9; тел./факс +7(499) 978-16-73; E-mail: noxonius@mail.ru