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MODELLING OF A THREE-DISPERSED SUSPENSION FILTRATION

Galina L. Safina

Moscow State University of Civil Engineering, Moscow, RUSSIA

Abstract. The injection method of soil stabilization is one of the methods to improve the soil base during the road construction. This method consists in the introduction of special compounds into the ground with the help of special equipment. Such compositions, as a rule, are solutions of polymers or cement, which harden, forming a solid base. The advantages of the injection method of soil stabilization for roads include rapid completion of work, minimal environmental impact, the possibility of application in difficult geological conditions, as well as strengthening the soil base at great depth. Injection solutions penetrate into microcracks and micropores of soils, forming a deposition. The study of liquid filtration in a porous soil system is of great practical importance. The paper considers the filtration of liquid in a porous medium with three types of particles. In the considered problem, each of the three types of suspension particles is characterized by its linear filtration function. It determines the size-exclusion particle capture mechanism, in which particles whose diameter exceeds the size of the pores get stuck in them, the rest pass through them unhindered. Numerical solutions are obtained for the concentrations of suspended particles of three types, as well as the total deposition. Depending on the initial parameters of the problem, the concentrations of suspended particles are either monotonic functions or non-monotonic, reaching the maximum value. In this paper, asymptotic solutions are constructed for the concentrations of suspended and retained particles near the concentration front, which are compared with numerical ones. The solution is obtained at infinity using a traveling wave.

Keywords: suspension filtration, suspended and retained particles, porous medium, filtration function, asymptotic solution, traveling wave

МОДЕЛИРОВАНИЕ ФИЛЬТРАЦИИ ТРЁХДИСПЕРСНОЙ СУСПЕНЗИИ

Г.Л. Сафина

Национальный исследовательский Московский государственный строительный университет, г. Москва, РОССИЯ

Аннотация: Инъекционный способ укрепления грунтов является одним из методов улучшения грунтового основания при строительстве автодорог. Этот метод заключается во введении специальных составов в грунт при помощи специального оборудования. Такие составы, как правило, являются растворами полимеров или цемента, которые затвердевают, образуя прочную основу. К преимуществам инъекционного способа укрепления грунтов для автодорог можно отнести быстрое выполнение работ, минимальное воздействие на окружающую среду, возможность применения в сложных геологических условиях, а также усиление грунтового основания на большой глубине. Инъекционные растворы проникают в микротрещины и микропоры грунтов, образуя осадок. Изучение фильтрации жидкости в пористой системе грунта имеет большое практическое значение. В работе рассматривается фильтрация жидкости в пористой среде с тремя типами частиц. В рассматриваемой задаче каждый из трех видов частиц суспензии характеризуется линейной функцией фильтрации. Это определяет размерный механизм захвата частиц, при котором частицы, диаметр которых превышает размер пор, застревают в них, остальные проходят через них беспрепятственно. Получены численные решения для концентраций взвешенных частиц трех типов, а также полного осадка. В зависимости от исходных параметров задачи концентрации взвешенных частиц являются либо монотонными функциями, либо немонотонными, достигающими максимального значения. В работе построено асимптотические решения для концентраций взвешенных и осажденных частиц вблизи фронта концентраций, которые сравниваются с численными. Получено решение на бесконечности с помощью бегущей волны.

Ключевые слова: фильтрация суспензии, взвешенные и осажденные частицы, пористая среда, функция фильтрации, асимптотическое решение, бегущая волна

INTRODUCTION

Many negative factors affect pavements throughout their service life. The mechanical effects of road transport cause bending stresses in concrete and abrasion of surface layers. Fluctuations in temperature and relative humidity during the day and calendar year, as well as cyclic processes of freezing and thawing, lead to the occurrence of tensile and compressive stresses in concrete. Hydrogeological conditions of the location of the road route, frosting of soil bases as a result of winter freezing and spring thawing lead to non-uniform deformations of concrete. Anti-icing chemical reagents and chemical compounds of the environment aggressively affect the concrete of the road bed. All this leads to the need to solve the problem of durability of road surfaces [1-5].

There are currently many ways to strengthen the ground bases in road construction [6-10], one of them is the soil stabilization by injection [11-15]. This method consists in introducing special compositions (injection solutions) into the soil to strengthen its properties and increase its bearing capacity. The process of injecting soil usually occurs as follows. First, the weak zones, zones with reduced bearing capacity, zones with subsidence, etc. on the ground surface are determined. Then drills are inserted into these zones and an injection fluid is injected through them, which fills the pores and cracks of the soil, strengthening its properties. Usually injection compositions consist of cement, sand and water, as well as additives to improve the properties of the composition.

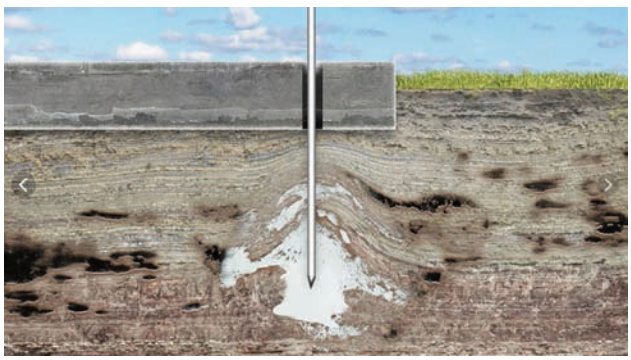


Figure 1. Soil stabilization by injection

Injecting soils during the road construction has a number of advantages. This is a fast and effective method of strengthening the soil foundations, which can significantly reduce the construction time of the road and reduce its cost. Injecting soils can significantly increase the bearing capacity of the soil and reduce the likelihood of subsidence and failures on the road. This method does not require a large amount of equipment and labor, which allows it to be used in the construction of roads in difficult geological conditions. Soil injection is an environmentally safe method of strengthening soil foundations, as it does not require the usage of chemicals or other hazardous substances.

Solving the filtration problem [16, 17] in a porous medium is an integral part of achieving the efficiency of the injection process. Currently, many papers are devoted to the problems of monodisperse [18-20] and polydisperse [21-23] suspension filtration. Suspension filtration in a porous medium with several types of particles is the process of passing a liquid with particles through a porous medium in which particles of various sizes and shapes may be present.

In the case of several types of particles, different filtration processes may occur for each type of particle. For example, larger particles may linger in the pores of the medium, and smaller particles may pass through the pores. It may also be possible for one type of particle to block the pores, which prevents the passage of other types of particles.

To optimize the filtration process in a porous medium with several types of particles, various methods can be used, for example, changing the parameters of the porous medium (pore size, particle packing density), changing the speed of liquid passage through the porous medium, etc. A combination of different methods can also be used to achieve the best result.

The paper is devoted to the numerical solution of the problem of bed deep filtration with three-dispersed suspension. It is assumed that three types of particles, with different sizes move in a

porous medium with a constant velocity. Linear functions for each type of particles differing only by a multiplier are considered as filtration functions. The usage of linear filtration functions trains the size-exlusion capture mechanism [24], in which particles freely pass through pores with a larger diameter than the pore diameter itself and plug pores with a smaller diameter.

THE MATHEMATICAL MODEL

In the domain $\Psi = \begin{cases} 0 \le x \le 1 \\ t \ge 0 \end{cases}$ consider a hyperbolic system of first-order equations defining a mathematical model of filtration of a three-dispersed liquid:

$$\frac{\partial C_i(x,t)}{\partial t} + \frac{\partial C_i(x,t)}{\partial x} + \frac{\partial S_i(x,t)}{\partial t} = 0, \quad (1)$$

$$\frac{\partial S_i(x,t)}{\partial t} = \Lambda_i\left(S_1, S_2, S_3\right) C_i(x,t), \quad (2)$$

$$i = 1, 2, 3,$$

where $C_i(x,t)$ is the concentration of suspended particles of the *i*-th type, $S_i(x,t)$ is the concentration of retained particles of the *i*-th type, $\Lambda_i\left(S_1, S_2, S_3\right)$ is the filtration function of the *i*-th type particles.

The boundary conditions of the problem at $x = 0$ are

$$C_i(0,t) = C_i^0, \ i = 1,2,3, \quad (3)$$

at that $\sum_{i=1}^{3} C_i^0 = 1$.

The initial conditions at $t = 0$ are

$$C_i(x,0) = 0, S_i(x,0) = 0, \ i = 1,2,3. \quad (4)$$

As filtration functions, we will consider functions of the form

$$\Lambda_i = \lambda_i \left(1 - \sum_{i=1}^{3} \beta_i C_i^0 S_i(x,t)\right), \ i = 1, 2, 3, \quad (5)$$

where λ_i and β_i are some positive constants. Assume that $\lambda_1 > \lambda_2 > \lambda_3$, it means that the particles of the first type are the largest, and particles of the third type are the smallest

Let us denote the linear combination in (5) as

$$P = \sum_{i=1}^{3} \beta_i C_i^0 S_i(x,t),$$

it takes into account the different probabilities of deposit for different type particles.

The straight line divides the domain Ψ into two subdomaines Ψ^1 and Ψ^2. In the subdomain $\Psi^1 = \begin{cases} 0 \le x \le 1 \\ 0 \le t < x \end{cases}$, i.e. under the line $t = x$, the solutions $C_i(x,t)$ and $S_i(x,t)$ for every type of particles are zero. Below this line, i.e. in the subdomain $\Psi^2 = \begin{cases} 0 \le x \le 1 \\ t > x \end{cases}$ the solutions $C_i(x,t)$ and $S_i(x,t)$, $i = 1, 2, 3$ are positive. On the line $t = x$ the solutions $C_i(x,t), i = 1, 2, 3$ are positive and the solutions $S_i(x,t), i = 1, 2, 3$ are equal to zero. This line is called the concentration front. So, along the concentration front the solutions $C_i(x,t)$, $i = 1, 2, 3$ are discontinuous and the solutions $S_i(x,t)$, $i = 1, 2, 3$ are continuous.

The solution at x=0

Taking into account the conditions (3) we have the heterogeneous system of equations (2) in the form

$$\begin{cases} \dfrac{\partial S_1}{\partial t} = \lambda_1 \left(1 - P\right) C_1^0, \\ \dfrac{\partial S_2}{\partial t} = \lambda_2 \left(1 - P\right) C_2^0, \\ \dfrac{\partial S_3}{\partial t} = \lambda_3 \left(1 - P\right) C_3^0. \end{cases}$$

Let's solve the corresponding homogeneous system of equations

$$\begin{cases} \dfrac{\partial S_1}{\partial t} = \lambda_1 C_1^0 P, \\ \dfrac{\partial S_2}{\partial t} = \lambda_2 C_2^0 P, \\ \dfrac{\partial S_3}{\partial t} = \lambda_3 C_3^0 P. \end{cases}$$

Write down the characteristic polynomial of the system

$$\begin{vmatrix} \lambda_1\beta_1\left(C_1^0\right)^2 + k & \lambda_1\beta_2 C_1^0 C_2^0 & \lambda_1\beta_3 C_1^0 C_3^0 \\ \lambda_2\beta_1 C_1^0 C_2^0 & \lambda_2\beta_2\left(C_2^0\right)^2 + k & \lambda_2\beta_3 C_2^0 C_3^0 \\ \lambda_3\beta_1 C_1^0 C_3^0 & \lambda_3\beta_2 C_2^0 C_3^0 & \lambda_3\beta_3\left(C_3^0\right)^2 + k \end{vmatrix}$$

and construct the characteristic equation

$$k^3 + \lambda_1\beta_1\left(C_1^0\right)^2 k^2 + \lambda_2\beta_2\left(C_2^0\right)^2 k^2 + \\ + \lambda_3\beta_3\left(C_3^0\right)^2 k^2 = 0,$$

whose solutions are numbers

$$\begin{bmatrix} k_{1,2} = 0, \\ k_3 = -\lambda_1\beta_1\left(C_1^0\right)^2 - \lambda_2\beta_2\left(C_2^0\right)^2 - \lambda_3\beta_3\left(C_3^0\right)^2. \end{bmatrix}$$

Omitting the calculations, we obtain the solution of homogeneous system of equations

$$\begin{cases} S_1 = \alpha_1 \dfrac{\beta_2 C_2^0}{\beta_1 C_1^0} + \alpha_2 \dfrac{\beta_3 C_3^0}{\beta_1 C_1^0} + \alpha_3 \dfrac{\lambda_1 C_1^0}{\lambda_3 C_3^0} e^{-\sum_{j=1}^{3}\lambda_j\beta_j\left(C_j^0\right)^2 t}, \\ S_2 = -\alpha_1 + \alpha_3 \dfrac{\lambda_2 C_2^0}{\lambda_3 C_3^0} e^{-\sum_{j=1}^{3}\lambda_j\beta_j\left(C_j^0\right)^2 t}, \\ S_3 = -\alpha_2 + \alpha_3 e^{-\sum_{j=1}^{3}\lambda_j\beta_j\left(C_j^0\right)^2 t}, \end{cases}$$

where $\alpha_1, \alpha_2, \alpha_3$ are arbitrary constants.

After that we solve the original inhomogeneous system of equations by the method of variation of constants. Taking into account the initial conditions (4) we get the solutions

$$S_i^0 = S_i(0,t) = \frac{\lambda_i C_i^0 \left(1 - e^{-\sum_{j=1}^{3}\lambda_j\beta_j\left(C_j^0\right)^2 t}\right)}{\sum_{j=1}^{3}\lambda_j\beta_j\left(C_j^0\right)^2}, \quad (6)$$

$$i = 1, 2, 3.$$

The solution at the concentration front

Taking into account that $S_i(x,t) = 0$, $i = 1, 2, 3$ at $t = x$ the filtration functions (5) are $\Lambda_i = \lambda_i$, $i = 1, 2, 3$, so equations (2) can be rewritten in the form

$$\frac{\partial S_i(x,t)}{\partial t} = \lambda_i C_i(x,t).$$

Turn in equation (1) to the characteristic variables $\tau = t - x$, $x = x$:

$$\frac{\partial C_i(x,t)}{\partial x} + \lambda_i C_i(x,t) = 0, \; i = 1, 2, 3.$$

Solving the equations with separable variables we obtain the solutions

$$C_i^f = C_i^0 e^{-\lambda_i x}, \; i = 1, 2, 3. \quad (7)$$

ASYMPTOTIC SOLUTION NEAR THE CONCENTRATION FRONT

Let us obtain solutions near the concentration front in the form of expansions

$$\tilde{C}_i(x,t) = \tilde{C}_i^0(x) + \tilde{C}_i^1(x)(t-x), \quad (8)$$

$$\tilde{S}_i(x,t) = \tilde{S}_i^1(x)(t-x) + \frac{1}{2}\tilde{S}_i^2(x)(t-x)^2, \quad (9)$$

$$i = 1, 2, 3.$$

Note, that in expansion (8) there are no terms $\tilde{S}_i^0(x)$, because $S_i(x,t) = 0$ at $t = x$.

Let us calculate partial derivatives of functions (8) and (9)

$$\frac{\partial \tilde{C}_i(x,t)}{\partial x} = \tilde{C}_i^{0\prime}(x) + \tilde{C}_i^{1\prime}(x)(t-x) - \tilde{C}_i^{1}(x),$$

$$\frac{\partial \tilde{C}_i(x,t)}{\partial t} = \tilde{C}_i^{1\prime}(x),$$

$$\frac{\partial \tilde{S}_i(x,t)}{\partial t} = \tilde{S}_i^{1}(x) + \tilde{S}_i^{2}(x)(t-x), \quad i = 1,2,3.$$

Substitute these expressions into (1)-(2) and group the terms at the same degrees of $(t-x)$:

$$\begin{cases} \lambda_i \tilde{C}_i^0 + \tilde{C}_i^{0\prime} + (t-x)\left(\tilde{C}_i^{1\prime} + \lambda_i \tilde{C}_i^1 - \lambda_i \tilde{C}_i^0 \left(\beta_1 C_1^0 \tilde{S}_1^1 + \beta_2 C_2^0 \tilde{S}_2^1 + \beta_3 C_3^0 \tilde{S}_3^1\right)\right) + \ldots = 0, \\ -\lambda_i \tilde{C}_i^0 + \tilde{S}_i^1 + (t-x)\left(\tilde{S}_i^2 - \lambda_i \tilde{C}_i^1 + \lambda_i \tilde{C}_i^0 \left(\beta_1 C_1^0 \tilde{S}_1^1 + \beta_2 C_2^0 \tilde{S}_2^1 + \beta_3 C_3^0 \tilde{S}_3^1\right)\right) + \ldots = 0. \end{cases}$$

Taking into account that $\tilde{C}_i^0 = C_i^0$, $i = 1,2,3$, the solution of the system of equations

$$\begin{cases} \lambda_i \tilde{C}_i^0 + \tilde{C}_i^{0\prime} = 0, \\ -\lambda_i \tilde{C}_i^0 + \tilde{S}_i^1 = 0, \quad i = 1,2,3 \end{cases}$$

$$\begin{cases} \tilde{C}_i^0 = C_i^0 e^{-\lambda_i x}, \\ \tilde{S}_i^1 = \lambda_i C_i^0 e^{-\lambda_i x}, \quad i = 1,2,3. \end{cases}$$

Obviously, solution $\tilde{C}_i^0$ is equal to C_i^f, $i = 1, 2, 3$, obtained in (6).

Solving the system for lower terms is

$$\begin{cases} \tilde{C}_i^{1\prime} + \lambda_i \tilde{C}_i^1 - \lambda_i \tilde{C}_i^0 \left(\beta_1 C_1^0 \tilde{S}_1^1 + \beta_2 C_2^0 \tilde{S}_2^1 + \beta_3 C_3^0 \tilde{S}_3^1\right) = 0, \\ \tilde{S}_i^2 - \lambda_i \tilde{C}_i^1 + \lambda_i \tilde{C}_i^0 \left(\beta_1 C_1^0 \tilde{S}_1^1 + \beta_2 C_2^0 \tilde{S}_2^1 + \beta_3 C_3^0 \tilde{S}_3^1\right) = 0, \end{cases}$$

we obtain

$$\begin{cases} \tilde{C}_i^1 = \lambda_i \tilde{C}_i^0 e^{-\lambda_i x}\left(\beta_1 \left(C_1^0\right)^2 \left(1 - e^{-\lambda_1 x}\right) + \beta_2 \left(C_2^0\right)^2 \left(1 - e^{-\lambda_2 x}\right) + \beta_3 \left(C_3^0\right)^2 \left(1 - e^{-\lambda_3 x}\right)\right), \\ \tilde{S}_i^2 = \lambda_i \tilde{C}_i^0 e^{-\lambda_i x}\left(\beta_1 \left(C_1^0\right)^2 \left(\lambda_i - (\lambda_i + \lambda_1) e^{-\lambda_1 x}\right) + \beta_2 \left(C_2^0\right)^2 \left(\lambda_i - (\lambda_i + \lambda_2) e^{-\lambda_2 x}\right) + \beta_3 \left(C_3^0\right)^2 \left(\lambda_i - (\lambda_i + \lambda_3) e^{-\lambda_3 x}\right)\right), \end{cases}$$
$$i = 1,2,3.$$

So, the asympotic solutions are

$$\tilde{C}_i(x,t) = C_i^0 e^{-\lambda_i x} + \lambda_i \tilde{C}_i^0 e^{-\lambda_i x}\left(\beta_1 \left(C_1^0\right)^2 \left(1 - e^{-\lambda_1 x}\right) + \beta_2 \left(C_2^0\right)^2 \left(1 - e^{-\lambda_2 x}\right) + \beta_3 \left(C_3^0\right)^2 \left(1 - e^{-\lambda_3 x}\right)\right)(t-x), \tag{10}$$

$$\tilde{S}_i(x,t) = \lambda_i C_i^0 e^{-\lambda_i x}(t-x) + \frac{1}{2}\lambda_i \tilde{C}_i^0 e^{-\lambda_i x}\left(\beta_1 \left(C_1^0\right)^2 \left(\lambda_i - (\lambda_i + \lambda_1) e^{-\lambda_1 x}\right) + \beta_2 \left(C_2^0\right)^2 \left(\lambda_i - (\lambda_i + \lambda_2) e^{-\lambda_2 x}\right) + \beta_3 \left(C_3^0\right)^2 \left(\lambda_i - (\lambda_i + \lambda_3) e^{-\lambda_3 x}\right)\right)(t-x)^2, \tag{11}$$

$$i = 1,2,3.$$

THE SOLUTION AT $t\to\infty$

Let us obtain the solution at $t\to\infty$ with usage of travelling wave. Consider $C_i = C_i(\theta)$ and $S_i = S_i(\theta)$, $i=1,2,3$, where $\theta = x - \nu t$ with unknown wave velocity ν. In this case the system (1)-(2) takes the form

$$(\nu-1)\frac{\partial C_i(\theta)}{\partial\theta} + \nu\frac{\partial S_i(\theta)}{\partial\theta} = 0, \qquad (12)$$

$$\nu\frac{\partial S_i(\theta)}{\partial\theta} = -\lambda_i(1-P)C_i(\theta), \qquad (13)$$

$$i = 1, 2, 3$$

with infinity boundary conditions

$$C_i(+\infty) = 0,\ S_i(+\infty) = 0, \qquad (14)$$

$$C_i(-\infty) = C_i^0. \qquad (15)$$

$$i = 1,2,3,$$

where C_i^0, $i=1,2,3$ are determined with conditions (3).
Substituting (13) into (12)

$$(\nu-1)\frac{\partial C_i(\theta)}{\partial\theta} - \lambda_i(1-P)C_i(\theta) = 0, \qquad (16)$$

$$i = 1,2,3,$$

we obtain

$$(1-P) = \frac{\nu-1}{\lambda_i C_i(\theta)}\frac{\partial C_i(\theta)}{\partial\theta},\ i=1,2,3. \qquad (17)$$

Using the expressions (17) we can construct the system of equations

$$\begin{cases} \dfrac{1}{\lambda_1 C_1(\theta)}\dfrac{\partial C_1(\theta)}{\partial\theta} = \dfrac{1}{\lambda_2 C_2(\theta)}\dfrac{\partial C_2(\theta)}{\partial\theta}, \\ \dfrac{1}{\lambda_1 C_1(\theta)}\dfrac{\partial C_1(\theta)}{\partial\theta} = \dfrac{1}{\lambda_3 C_3(\theta)}\dfrac{\partial C_3(\theta)}{\partial\theta}, \end{cases}$$

integrating it with conditions (15) we obtain

$$\begin{cases} C_2(\theta) = C_2^0\left(\dfrac{C_1(\theta)}{C_1^0}\right)^{\frac{\lambda_2}{\lambda_1}}, \\ C_3(\theta) = C_3^0\left(\dfrac{C_1(\theta)}{C_1^0}\right)^{\frac{\lambda_3}{\lambda_1}}. \end{cases} \qquad (18)$$

Let us itegrate the equation (12) over variable θ

$$(\nu-1)C_i(\theta) + \nu S_i(\theta) = 0,\ i=1,2,3$$

and express

$$S_i(\theta) = -\frac{\nu-1}{\nu}C_i(\theta),\ i=1,2,3. \qquad (19)$$

Substite expressions (18) and (19) into (12) for the first type of particles we obtain the equation

$$\frac{\partial C_1(\theta)}{\partial\theta} - \lambda_1\left(\frac{1}{\nu-1} + \frac{1}{\nu}\left(\beta_1 C_1^0 C_1(\theta) + \beta_2\left(C_2^0\right)^2\left(\frac{C_1(\theta)}{C_1^0}\right)^{\frac{\lambda_2}{\lambda_1}} + \beta_3\left(C_3^0\right)^2\left(\frac{C_1(\theta)}{C_1^0}\right)^{\frac{\lambda_3}{\lambda_1}}\right)\right)C_1(\theta) = 0 \qquad (20)$$

with initial condition $C_1(0) = \hat{C}_1$, where $0 < \hat{C}_1 < C_1^0$.
Integrating separable differential equation (20), we obtain

$$\int_{\hat{C}_1}^{C_1}\frac{dC_1}{C_1\left(\frac{1}{\nu-1} + \frac{1}{\nu}\left(\beta_1 C_1^0 C_1(\theta) + \beta_2\left(C_2^0\right)^2\left(\frac{C_1(\theta)}{C_1^0}\right)^{\frac{\lambda_2}{\lambda_1}} + \beta_3\left(C_3^0\right)^2\left(\frac{C_1(\theta)}{C_1^0}\right)^{\frac{\lambda_3}{\lambda_1}}\right)\right)} = \lambda_1\theta. \qquad (21)$$

Taking into account the equality (15) we obtain the formula for velocity ν from (21)

$$\nu = \frac{\beta_1 (C_1^0)^2 + \beta_2 (C_2^0)^2 + \beta_3 (C_3^0)^2}{1 + \beta_1 (C_1^0)^2 + \beta_2 (C_2^0)^2 + \beta_3 (C_3^0)^2}. \tag{22}$$

NUMERICAL SOLUTION

Since the problem (1)-(4) has no analytical solution, we will solve it numerically using explicit difference schemes. To approximate first-order derivatives, we use the values $h_t = h_x = 0.01$ as steps along the coordinate axes, in this case the Courant condition is fulfilled [25]. Let us set the coefficients of the filtration functions Λ_i, $i = 1, 2, 3$: $\lambda_1 = 10$, $\lambda_2 = 5$, $\lambda_3 = 2.5$. We solve the problem for three different sets of parameters C_i^0 and β_i^0, $i = 1, 2, 3$, presented in Table 1.

Table 1

№	C_1^0	C_2^0	C_3^0	β_1	β_2	β_3
1	0.3	0.2	0.5	4	6	12
2	0.3	0.5	0.2	4	6	8
3	0.4	0.5	0.1	8	5	7

The graphs of retained particle concentrations $S_1(x)$ (the black line), $S_2(x)$ (the red line), $S_3(x)$ (the blue line) and the total deposit concentration $S(x)$ (the green line) are shown in Fig. 1 at different times for the first parameter combination from the Table 1. The figure shows that the concentrations $S_1(x)$, $S_2(x)$ and $S(x)$ decrease on the total segment $x \in [0;1]$. The solution $S_3(x)$ decreases monotonically until about time $t \approx 0.2$, then the graph behaves non-monotonically: it first increases, reaching its maximum value, and then begins to decrease. Beginning at a point in time $t \approx 1.3$ the solution $S_3(x)$ is an increasing function for all $x \in [0;1]$. The movement of the maximum point of $S_3(x)$ is shown in Fig. 2.

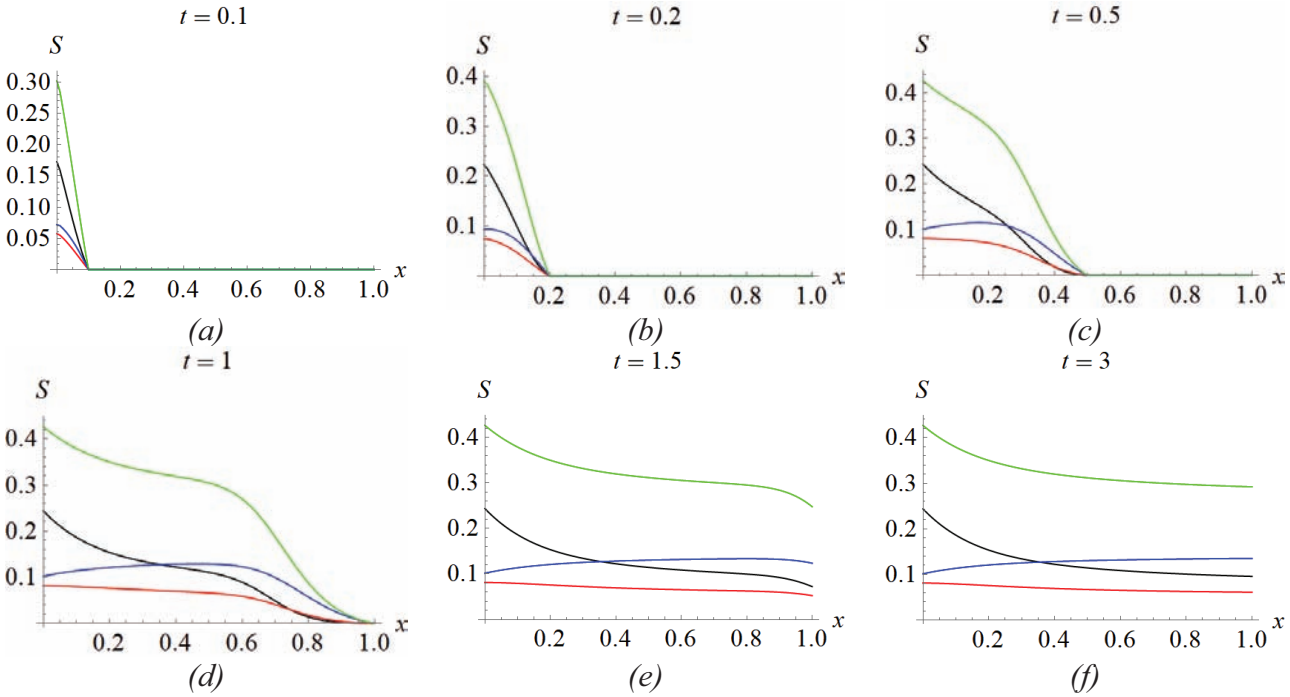


Figure 1. The solutions of retained particles concentrations for three types of particles and total deposit at different times (set 1)

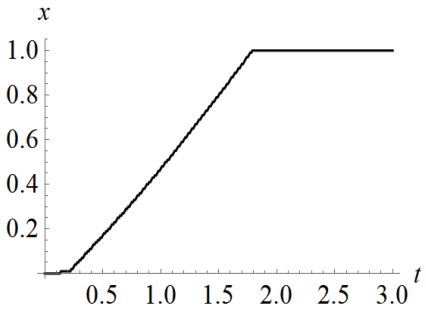


Figure 2. Movement of the maximum point of the concentration $S_3(x)$

The plots of $S_1(x)$ (the black line), $S_2(x)$ (the red line), $S_3(x)$ (the blue line) and $S(x)$ (the green line) for the second parameter combination from the Table 1 are shown in Fig. 3. The solutions $S_1(x)$ and $S(x)$ are monotonously decreasing functions for any time values. $S_2(x)$ is a monotonously decreasing curve until $t \approx 0.23$, after that it is non-monotonic. $S_3(x)$ always decreases until $t \approx 0.15$, after that it has the maximum point. In Fig. 4 the graphs of moving the maximum of the concentrations $S_2(x,t)$ (Fig. 4a) and $S_3(x,t)$ (Fig. 4b) are presented. From Fig. 3 it can be seen that starting from the moment of time $t \approx 0.2$ the solutions $S_2(x,t)$ (Fig. 4(a)) and $S_3(x,t)$ (Fig. 4(b)) have the maximum point which moves along the axis x until it reaches the filter outlet.

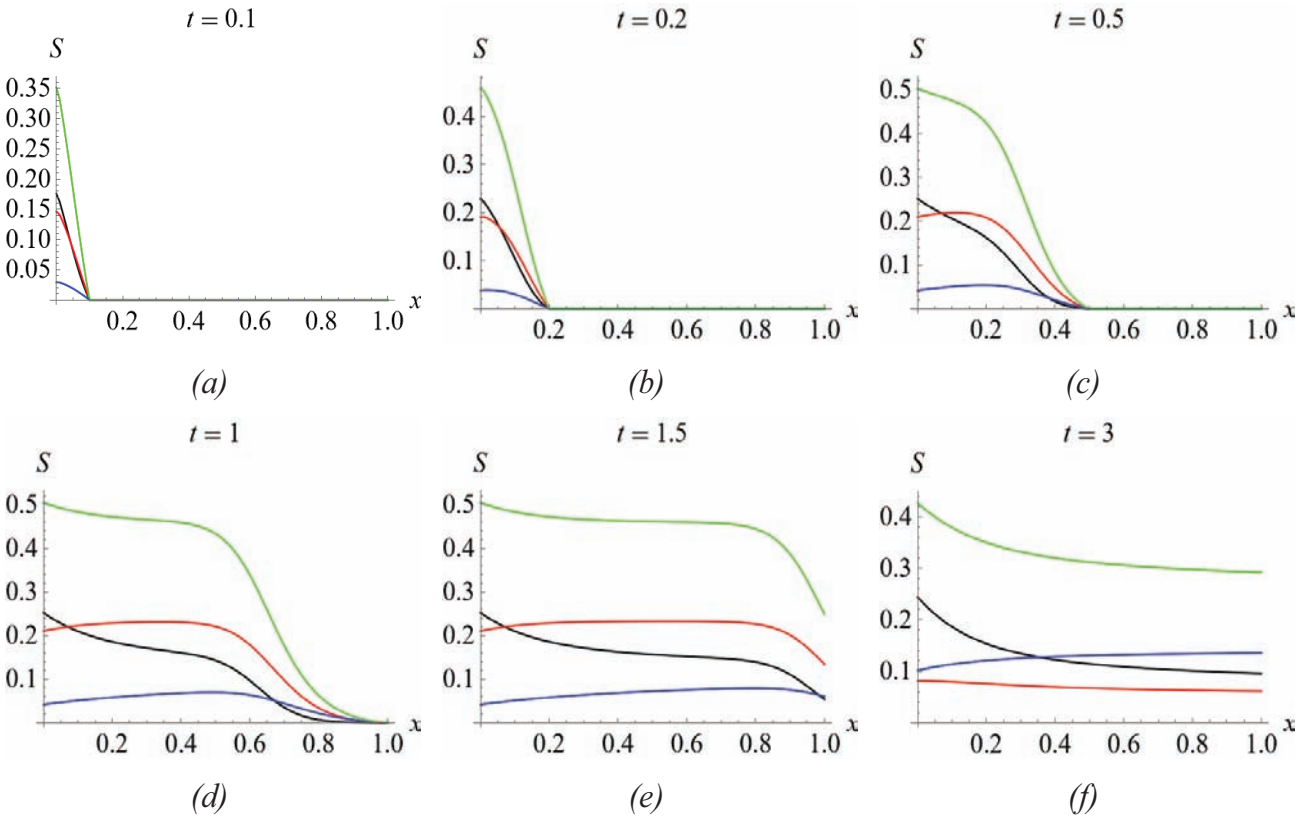


Figure 3. The solutions of retained particles concentrations for three types of particles and total deposit at different times (set 2)

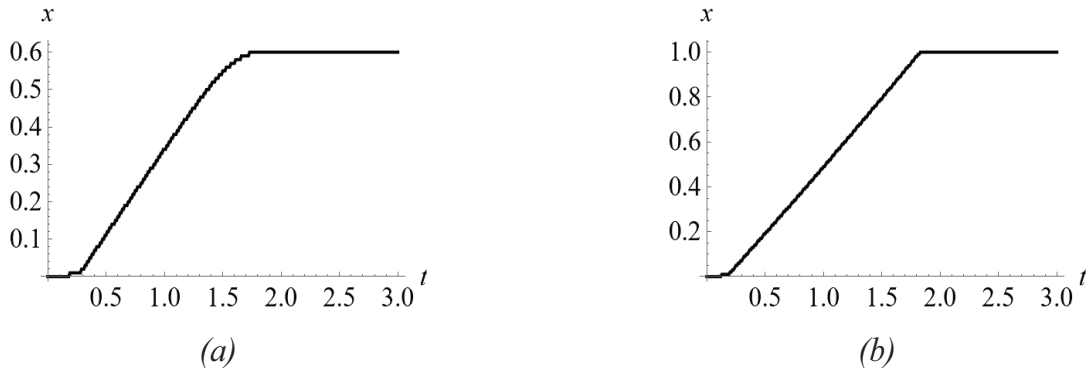


Figure 4. Movement of the maximum point of the concentrations $S_2(x)$ and $S_3(x)$

In Fig. 5 the graphs of functions $S_1(x)$ (the black line), $S_2(x)$ (the red line), $S_3(x)$ (the blue line) and $S(x)$ (the green line) for the third case from Table 1 are shown. The solution $S_1(x)$ always monotonously decreases. There is a decrease in functions $S_2(x)$, $S_3(x)$ and the total deposit $S(x)$: $S_2(x)$ and $S_3(x)$ decrease until $t \approx 0.1$, $S(x)$ decreases until $t \approx 0.15$. After that each function becomes non-monotonic: every maximum point moves from inlet to outlet of filter with increasing time. The graphs of movement of the maximum point of $S_2(x,t)$ (Fig. 6(a)), $S_3(x,t)$ (Fig. 6(b)) and $S_3(x,t)$ (Fig. 6(c)) are presented in Fig. 6.

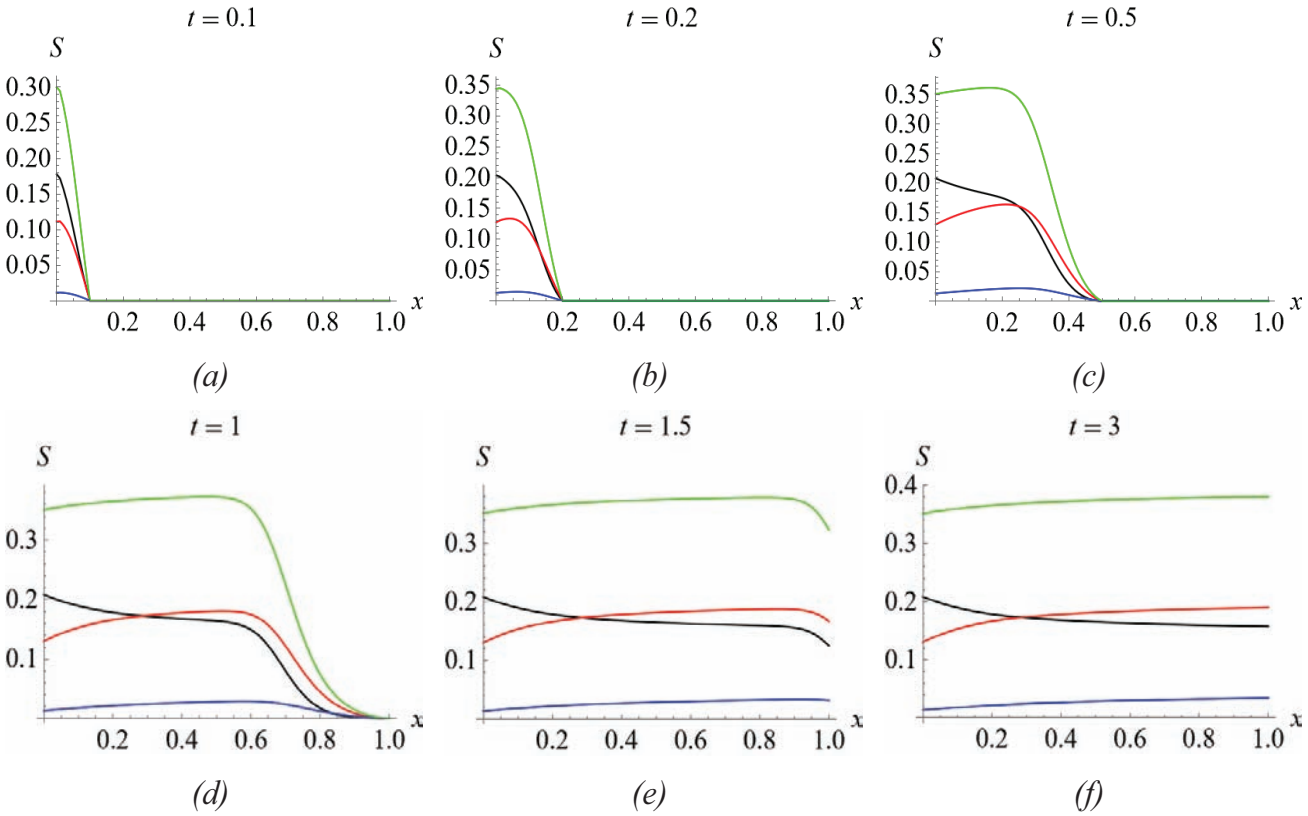


Figure 5. Solutions of retained particles concentrations for three types of particles and total deposit at different times (set 3)

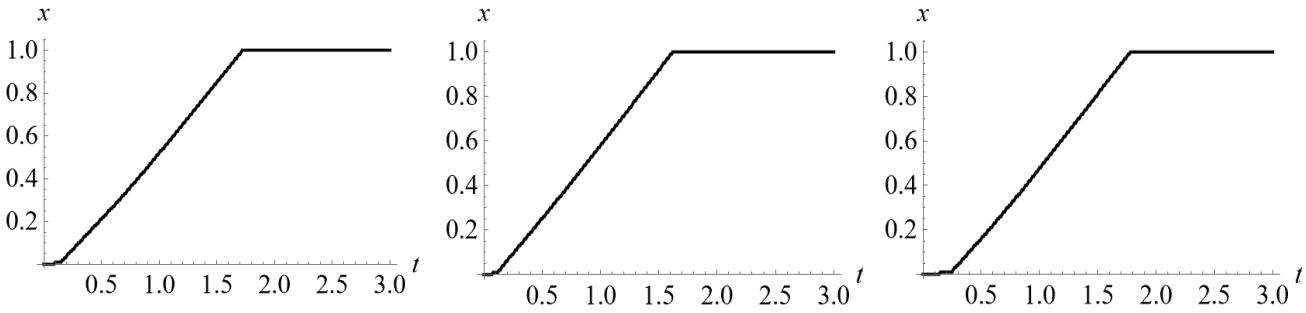


Figure 6. Movement of the maximum point of the concentrations $S_2(x)$, $S_3(x)$ and $S(x)$

The graphs of concentrations $S_1(t)$ (Fig. 7(a)), $S_2(t)$ (Fig. 7(b)) and $S_3(t)$ (Fig. 7(c)) and their asymptotic solutions $\tilde{S}_1(t)$, $\tilde{S}_2(t)$ and $\tilde{S}_3(t)$, obtained by formulas (11), at $x = 0.5$ are shown in Fig. 7. The black curves correspond to the numerical solutions and the red lines correspond to the asymptotic ones. Fig. 8 shows similar graphs at the filter outlet at $x = 1$. The following parameters are choosen: $\lambda_1 = 0.5$, $\lambda_2 = 0.25$, $\lambda_3 = 0.125$, $C_1^0 = 0.2$, $C_2^0 = 0.3$, $C_2^0 = 0.5$, $\beta_1 = 2$, $\beta_2 = 4$, $\beta_3 = 5$. It can be seen from the graphs that the concentration of suspended particles of the first type has the best approximation, while for all three types of suspended particles, the closer to the filter output, the closer the asymptotic solution is to the numerical one. The graphs of relative errors of suspended particles concentration asymptotics $\tilde{S}_1(t)$, $\tilde{S}_2(t)$ and $\tilde{S}_3(t)$ are shown in Fig. 9. The black line corresponds to $x = 0.5$ and the red line corresponds to $x = 1$. With increasing, i.e. with distance from the concentration front, the relative errors of the asymptotics for each type of particles increase. At the same time, it can be seen that the relative errors at $x = 0.5$ are much greater than at $x = 1$. Both at $x = 0.5$ and $x = 1$, the asymptotic $\tilde{S}_1(t)$ has the smallest error and $\tilde{S}_3(t)$ has the largest one. The maximum values of the relative errors are presented in Table 2.

Table 2

	$\tilde{S}_1(t)$	$\tilde{S}_2(t)$	$\tilde{S}_3(t)$
$x = 0.5$	10 %	19 %	23 %
$x = 1$	5 %	4 %	9 %

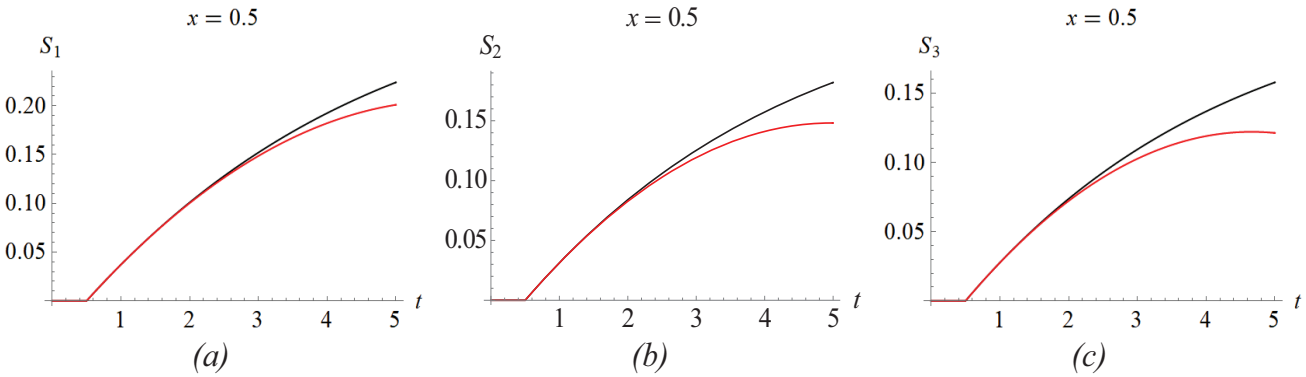
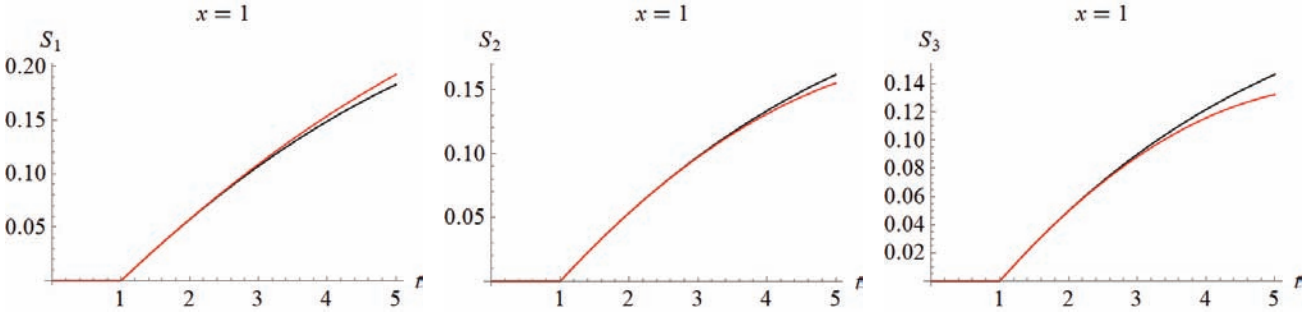
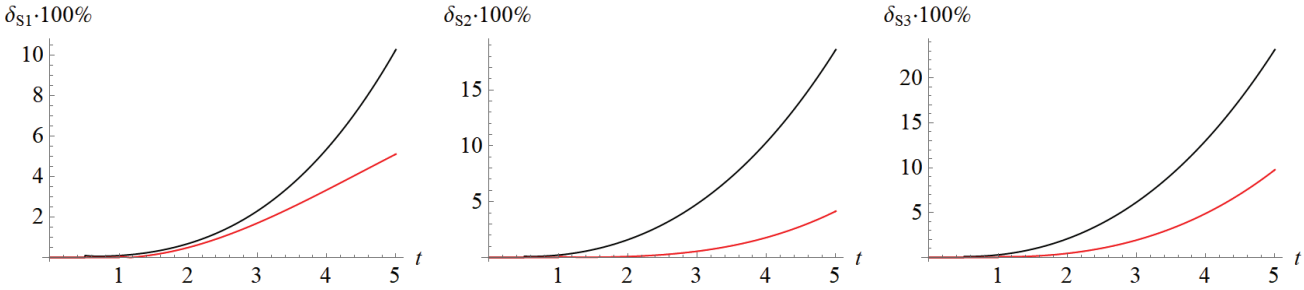


Figure 7. Numerical and asymptotic solutions for three types of retained particles at $x = 0.5$



(a) (b) (c)

Figure 8. Numerical and asymptotic solutions for three types of retained particles at $x = 1$



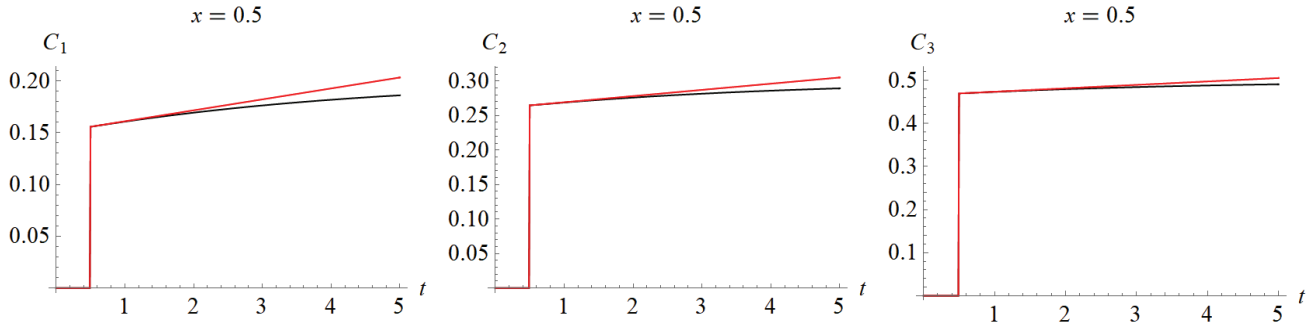
(a) (b) (c)

Figure 9. Relative errors of asymptotics for three types of retained particles for $x = 0.5$ *and* $x = 1$

The graphs of the numerical solutions $C_1(t)$ (Fig. 10(a)), $C_2(t)$ (Fig. 10(b)), $C_3(t)$ (Fig. 10(c)) and their asymptotics $\tilde{C}_1(t)$, $\tilde{C}_2(t)$ and $\tilde{C}_3(t)$, obtained by formulas (10) at $x = 0.5$, are presented in Fig. 10. Numerical solutions are shown with black lines, the asymptotic solutions are shown with red lines. Similar graphs at the filter inlet are shown in Fig. 11. In contrast to the asymptotics of retained particles concentrations, the deviations of asymptotics of suspended particles concentrations do not depend on x, they remain the same at $x = 0.5$ and $x = 1$. The asymptotics of the concentration of the third type of particles approximates the numerical solution in the best way. Fig. 12 shows plots of relative errors of asymptotics for three types of suspended particles at $x = 0.5$ and $x = 1$. The black lines correspond to the case of $x = 0.5$, the red ones correspond to $x = 1$. Analysis of these dependencies shows that the behavior of the relative errors of the asymptotics $\tilde{C}_2(t)$ and $\tilde{C}_3(t)$ are very similar: at first, the relative errors calculated for $x = 0.5$ and $x = 1$, practically coincide, and after reaching a certain point in time ($t \approx 3$ for $\tilde{C}_2(t)$ and $t \approx 2.6$ for $\tilde{C}_3(t)$), the relative error at the filter outlet begins to increase faster than at $x = 0.5$. The relative deviations of the asymptotics $\tilde{C}_1(t)$ at $x = 0.5$ and $x = 1$ by the time $t \approx 2.6$ maximum deviate from each other, then begin to converge again so that by the time $t \approx 9$ they reach the same value. Every function in this case is monotonically increasing. The values of the largest relative errors of asymptotics of suspended particles concentrations are presented in Table 3.

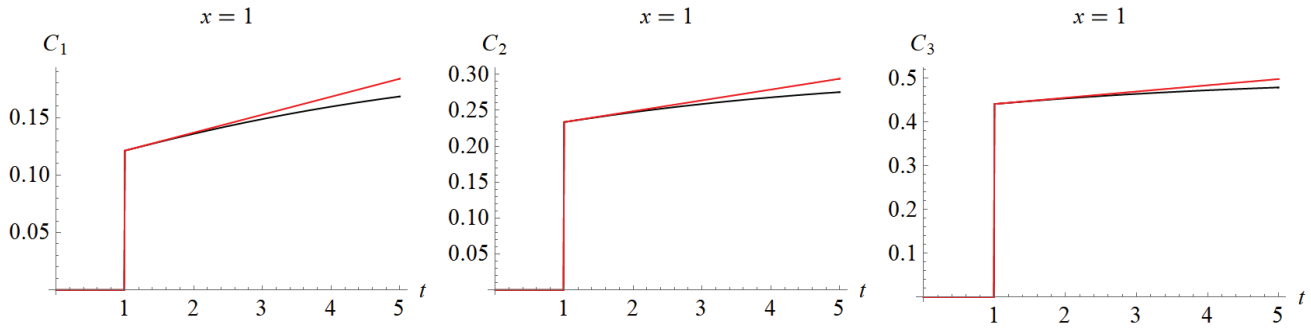
Table 3

	$\tilde{C}_1(t)$	$\tilde{C}_2(t)$	$\tilde{C}_3(t)$
$x = 0.5$	10 %	19 %	23 %
$x = 1$	5 %	4 %	9 %



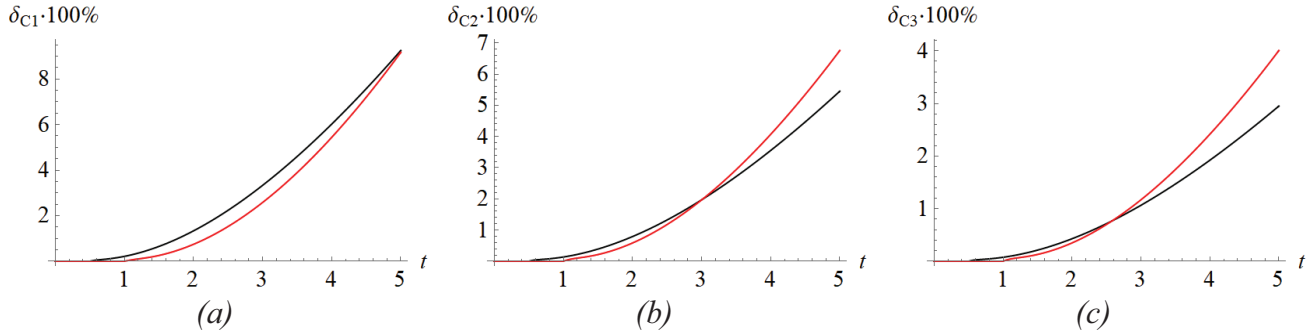
(a) (b) (c)

Figure 10. Numerical and asymptotic solutions for three types of suspended particles at $x = 0.5$



(a) (b) (c)

Figure 11. Numerical and asymptotic solutions for three types of suspended particles at $x = 1$



(a) (b) (c)

Figure 12. Relative errors of asymptotics for three types of suspended particles for $x = 0.5$ *and* $x = 1$

Fig. 13 shows the graphs of travelling waves, that determine the concentrations of suspended particles at $t \to \infty$, obtained using formulas (21) and (18). The values of parameters are: $\lambda_1 = 0.5$, $\lambda_2 = 0.25$, $\lambda_3 = 0.125$, $C_1^0 = 0.4$, $C_2^0 = 0.5$, $C_2^0 = 0.1$, $\beta_1 = 8$, $\beta_2 = 5$, $\beta_3 = 7$. The black curves correspond to the value of $\hat{C}_1 = 0.4$, the red curves correspond to $\hat{C}_1 = 0.3$. Fig. 14 shows similar graphs of retained particles concentrations obtained with equations (19).

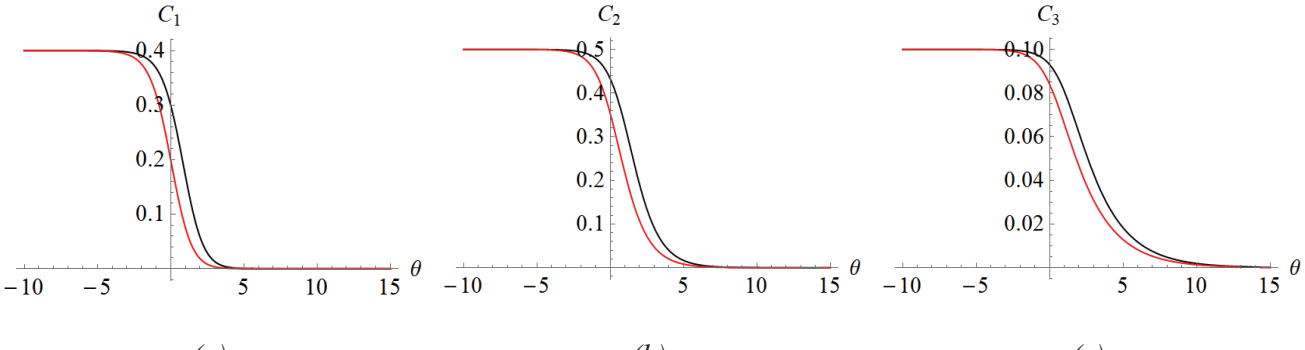


Figure 13. Solutions for three types of suspended particles at $t \to \infty$ with usage of travelling wave

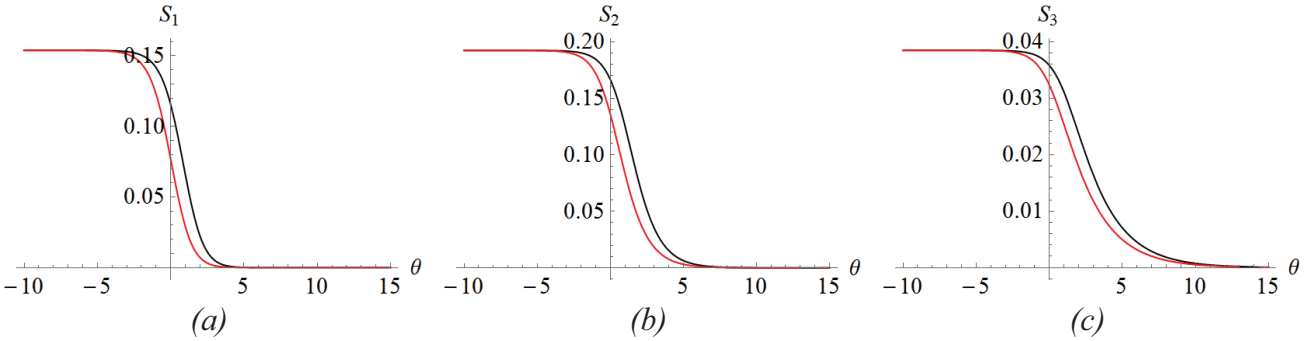


Figure 14. Solutions for three types of retained particles at $t \to \infty$ with usage of travelling wave

CONCLUSION

The problem with 3 types of particles is considered for the first time.

The asymptotics of the solutions near the concentration front are constructed.

The concentrations of partial deposits and total deposit are investigated. Similarly [22], the concentration of large retained particles always decreases monotonically, the concentration of small retained particles is non-monotonic. The concentration of total deposit remains monotonic or non-monotonic depending on the problem parameters. The concentration of medium retained particles can also be monotonic or non-monotonic.

Nonmonotonic particles concentrations have a maximum point. The movements of the maximum points in dependence of time were investigated. It was found that for the chosen parameters the points of maximum of the concentration of total deposit and deposit of small particles grow indefinitely with increasing time, while the point of maximum of the medium retained particles concentration is limited.

The travelling waves, determining the concentrations of retained and suspended particles at $t \to \infty$, are obtained.

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Galina L. Safina, Ph. D, Head of the Department of Fundamental Education, branch of Moscow State University of Civil Engineering in Mytishchi, Associate Professor of the Department of Computer Science and Applied Mathematics, Moscow State University of Civil Engineering; 129337, Russia, Moscow, Yaroslavskoe Shosse, 26, tel. +7(499)1835994, e-mail: minkinag@mail.ru.

Сафина Галина Леонидовна, кандидат технических наук, заведующий кафедрой фундаментального образования филиала НИУ МГСУ в г. Мытищи, доцент кафедры информатики и прикладной математики Национального исследовательского Московского государственного строительного университета; 129337, Россия, г. Москва, Ярославское шоссе, д. 26, тел. +7(499)1835994, e-mail: minkinag@mail.ru.