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OPTIMIZATION OF THE MAIN RIGIDITY PLANES POSITION OF INHOMOGENEOUS ROD IN SPATIAL BENDING

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Abstract: The problem of finding the optimal rotation angle of the principal planes of stiffness for a spatially loaded, inhomogeneous rod of variable cross-section is considered. This angle serves as an additional regulator of the stress state for inhomogeneous rod designed in accordance with the adopted optimization criteria.
The rod rigidly connected to a movable coordinate system capable of rotation around rod's longitudinal axis is composed of quasi-homogeneous structural elements. The rod's structure is symmetrical with respect to one of the plane coordinates. The specified loads are applied to the rod's centerline and are located in two mutually perpendicular fixed planes.
Based on the minimum strain energy criterion, a formula for the optimal rotation angle of the rod's rigidity planes was derived. This formula contains six integral strain energy components. These components represent the actual and potential work of internal forces in the original rigidity planes and those rotated by 90 degrees. A numerical example shows strain energy graphs for cross-sections with different layouts. The relationship between the energy criterion and the strength criterion is highlighted.
Based on the minimum strain energy criterion, a formula for the optimal rotation angle of the rod 's rigidity planes was derived. This formula contains six integral strain energy components. These components represent the actual and possible work of internal forces initial rigidity planes and those rotated by 90 degrees. A numerical example shows strain energy graphs for cross sections with different layouts. The relationship between the energy criterion and the strength criterion is noted.
In the example discussed, using axial rotation of the rod, without requiring changes to its physical structure or geometry, allows for an increase in its load-bearing capacity by 20-30% and a reduction in maximum stress levels. This method can also be applied to prefabricated standard structural elements, including homogeneous ones.
In the example discussed, the use of axial rotation of the rod, without requiring changes to its physical structure or geometry, allowed for an increase in its load-bearing capacity by 20-30% and a reduction in maximum stress levels. This method can also be applied to prefabricated standard structural elements, including homogeneous ones.
The development of this method is its application in “controlled structures”, which acquire the ability to effectively adapt to changes in the parameters of external force impact.

Keywords: optimal design of inhomogeneous rods, layered rod, spatial bending, criterion of minimum strain energy, controlled structures

ОПТИМИЗАЦИЯ ПОЛОЖЕНИЯ ГЛАВНЫХ ПЛОСКОСТЕЙ ЖЕСТКОСТИ НЕОДНОРОДНОГО СТЕРЖНЯ ПРИ ПРОСТРАНСТВЕННОМ ИЗГИБЕ

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Аннотация: Для пространственно-нагруженного неоднородного стержня переменного поперечного сечения рассмотрена задача нахождения оптимального угла поворота главных плоскостей жёсткости. Данный угол является дополнительным регулятором напряженного состояния для неоднородного стержня, запроектированного в соответствии с принятыми критериями оптимизации.

Стержень составлен из квазиоднородных структурных элементов, жестко связан с подвижной системой координат, способной поворачиваться вокруг продольной оси. Структура стержня обладает симметрией относительно одной из координатных плоской. Заданные нагрузки приведены к осевой линии стержня, расположены в двух взаимно перпендикулярных неподвижных плоскостях.
На основе критерия минимума энергии деформации получена формула для оптимального угла поворота плоскостей жесткости стержня, содержащая шесть интегральных компонент энергии деформации. Они представляют собой действительные и возможные работы внутренних усилий в исходных и повернутых на угол 90 градусов плоскостях жесткости. На численном примере показаны графики энергии деформации для поперечных сечений с различной компоновкой. Отмечена связь энергетического критерия с критерием прочности.
Применение осевого поворота стержня, не требуя изменения физической структуры и геометрии стержня, в рассмотренном примере позволяет повысить его несущую способности на величину 20-30%, снизить уровень максимальных напряжений. Данный способ может применяться и к готовым типовым конструктивным элементам, в том числе, – однородным.
Развитием данного метода является его применение в «управляемых конструкциях», которые приобретают способность эффективно адаптироваться к изменениям параметров внешнего силового воздействия.

Ключевые слова: оптимальное проектирование неоднородных стержней, слоистый стержень, пространственный изгиб, критерий минимума энергии деформации, управляемые конструкции

INTRODUCTION

Settings of optimization problems for deformable systems and structures used in various branches of engineering are quite diverse and are determined by the adopted optimality criteria and the types of variable parameters and functions [1–4]. In rod systems, one of the key means of improving operational performance is the optimization of the geometry, physical structure, and topology of the system [5]. Significant effects are also achieved by selecting a rational arrangement of the cross-sections of the elements. Among the formulations of optimization problems in which the geometric shapes of structural elements subject to various types of impacts are varied, we note the following.

Settings with using geometric shapes that profile the transverse dimensions of structural elements of rods (width, height, thickness, etc.), specified with an accuracy of one or more parameters [6]. A subclass of such task are optimization problems for prismatic rods, in which the geometric shape functions are of zero order. The use of local normative criteria for finding the required values of the sought-after parameters in such problems with fixed shapes leads to the need to identify dangerous stress localizations within the structure.

In more complex settings, the desired geometric functions of the structural elements of the rods are found under the condition of continuous satisfaction of specified normative local criteria. This is a class of equal-criteria problems, in particular, those concerning equal stress [2, 7, 8]. In general, such problems (of functional analysis) are characterized by high labor intensity and, as a rule, for real structures are solved by numerical methods using discretization of the calculation functions.

Normative criteria, reflecting the relationship between the desired design parameters and the standardized quantities, can be integral in nature, as is realized in problems of stability, vibration, deformability, and a number of others, solved based on the corresponding boundary value problems. In these problems, the defining differential and integral relations naturally define the essence of the quantities included in the normative criteria. Depending on the type of variable quantities, we have variational or finite-dimensional optimization [3].

In a number of studies, the solution is based on the use of integral criteria reflecting additional requirements for the optimal system, which have technical, economic, general physical, or other sense. One of these is the integral energy criterion, which allows for identifying the geometric shape, structure, and topology of the system that best satisfies the requirements of design standards [9, 10, 11, 12].

When solving optimization problems for inhomogeneous structures, the calculation hypothe-

ses and assumptions adopted during the formation of the mathematical model acquire particular significance. Along with refined non-classical models [13–18], models based on the Bernoulli and Timoshenko hypotheses can also be used in the calculations of elements of rod structures that satisfy certain geometric and physical constraints.

In this paper, we consider the problem of finding the optimal rotation angle of the rod's principal rigidity planes for a spatially loaded, inhomogeneous rod of variable cross-section. This angle acts as an additional regulator of the stress state. In subsequent or previous stages, the set of variable values can be expanded to expand the functionality of the rod's geometric and physical structure.

MATERIALS AND METHODS OF RESEARCH

1. Mathematical model of inhomogeneous rod.

An inhomogeneous rod composed of quasi-homogeneous structural elements (layers, phases, inclusions, reinforcing elements, etc.) is rigidly connected to the coordinate system xyz, and the rod's structure is symmetrical with respect to the plane yx. The specified loads are referenced to the rod's centerline and are located in two mutually perpendicular fixed planes y_0x, z_0x. We will assume the coordinate system xyz to be movable, capable of rotating around the axis through a certain angle α, together with the rod (Fig. 1).

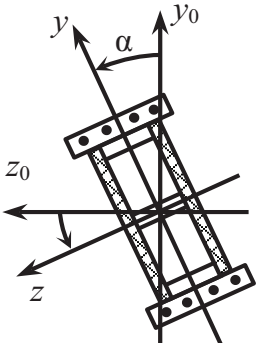


Figure 1. Rotation of the principal planes of rigidity xy, xz relative to the force planes y_0x, xz_0.

By solving the corresponding boundary value problems of spatial transverse bending with tension (compression), the functions of internal forces are found - bending moments, transverse and longitudinal forces, combined into a vector

$$\mathbf{S}_0(x) = [M_{z_0}, M_{y_0}, Q_{y_0}, Q_{z_0}, N]^{\mathrm{T}}(x).$$

Considering that the boundary conditions for the rod do not change when the transverse axes y, z are rotated relative to the centerline, we write down the forces in the coordinate system xyz

$$\mathbf{S}(x) = \mathbf{C}\,\mathbf{S}_0(x),$$
$$\mathbf{S}(x) = [M_z, M_y, Q_y, Q_z, N]^{\mathrm{T}}(x), \qquad (1)$$
$$\mathbf{C} = diag[\mathbf{\Phi}\ \ \mathbf{\Phi}\ \ 1],\ \mathbf{\Phi} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}.$$

In works [19, 20] for the Timoshenko rod, physical relations are given that connect the forces (1) with the generalized deformations of the axis: curvatures κ_z, κ_y, averaged shears γ_{y0}, γ_{z0}, and axial deformation ε_0. Let us represent them in the form

$$\mathbf{D}\cdot\mathbf{\Delta} = \mathbf{S}, \qquad (2)$$
$$\mathbf{\Delta}(x) = [\kappa_z, \kappa_y, \gamma_y, \gamma_z, \varepsilon_0]^{\mathrm{T}}(x) \qquad (3)$$

with a stiffness matrix

$$\mathbf{D}(x) = \begin{bmatrix} D_{zz} & -D_{yz} & 0 & 0 & -D_z \\ -D_{yz} & D_{yy} & 0 & 0 & D_y \\ 0 & 0 & D_{Qy} & 0 & 0 \\ 0 & 0 & 0 & D_{Qz} & 0 \\ -D_z & D_y & 0 & 0 & D_0 \end{bmatrix}, \qquad (4)$$

containing components

$$D_0 = \sum_{k=1}^{s} E_k \iint_{A_k} dA, \quad \left[D_z, D_y\right] = \sum_{k=1}^{s} E_k \iint_{A_k} [y,z]\,dA, \quad \left[D_{zz}, D_{zy}, D_{yy}\right] = \sum_{k=1}^{s} E_k \iint_{A_k} \left[y^2, yz, z^2\right] dA \tag{5}$$

by zero, first, and second geometric orders, respectively. The characteristic D_{yz} for the adopted calculation scheme in relations (2) is missing. The presence of stiffnesses D_z, D_y determines the coupling of bending and stretching deformations. The stiffnesses of cross sections under shear

$$D_{Qy} = \frac{1}{\sum_{k=1}^{s} \frac{1}{G_k} \int_{y_k}^{y_{k+1}} \left(\frac{D_0 D_z^{\text{sec}} - D_z D_0^{\text{sec}}}{D d_z} \right)^2 \frac{dy}{b_k}}, \quad D_{Qz} = \frac{1}{\sum_{k=1}^{s} \frac{1}{G_k} \int_{z_k}^{z_{k+1}} \left(\frac{D_0 D_y^{\text{sec}} - D_y D_0^{\text{sec}}}{D d_y} \right)^2 \frac{dy}{h_k}}, \quad D d_z = D_0 \cdot D_{zz} - D_z^2, \quad D d_y = D_0 \cdot D_{yy} - D_y^2 \tag{6}$$

are written using the rigidity characteristics of the cut-off parts of the inhomogeneous sections D^{sec} according to the corresponding expressions from (5) when integrating over a part of the section $[y, y_{\max}]$ or $[z, z_{\max}]$.

In addition, to find the shear stiffness characteristics, an approximate approach can be used based on the use of approximation functions for the relative shear forces $f_{y,\tau}(y)$, $f_{z,\tau}(z)$, satisfying the boundary conditions on the rod surfaces. As a result, we obtain the second version of the dependencies for shear stiffnesses

$$D_{Qy} = \frac{F_{y,\tau}^{\ 2}}{\sum_{k=1}^{s} \int_{y_k}^{y_{k+1}} \frac{f_{y,\tau}^{\ 2}}{b_k G_k} dy}, \quad D_{Qz} = \frac{F_{z,\tau}^{\ 2}}{\sum_{k=1}^{s} \int_{z_k}^{z_{k+1}} \frac{f_{z,\tau}^{\ 2}}{h_k G_k} dz}.$$

The symmetry of the rod relative to the plane yx ensures the fulfillment of the conditions $D_y = D_{yz} = 0$. In addition, we will assume that the structure of the variable cross-section rod is such that it allows the rod to be tied to the plane xz while ensuring the condition $D_z(x) \equiv 0$. Due to the dependence of characteristics (5), (6) on geometric and physical parameters, this condition can be satisfied even in the absence of geometric symmetry. Then from (2) we obtain

$$D_{zz}\kappa_z = M_z, \quad D_{yy}\kappa_y = M_y, \quad D_0\varepsilon_0 = N.$$

2. Optimality criterion

As a criterion for the optimality of the parameters of an inhomogeneous rod, we will adopt the condition of minimum deformation energy [6, 9, 11]

$$W(\mathbf{P}) = \frac{1}{2}\int_0^l \mathbf{S}^{\mathrm{T}} \boldsymbol{\Delta}\, dx \to \min. \tag{7}$$

The set of variable parameters $\mathbf{P}$ in the case under consideration is represented by the rotation angle α. Substituting relations (2)–(4) into (7) gives the dependence of the deformation energy on the rotation angle of the axes

$$W(\alpha) = (W_{yy}^{\alpha=0} + W_{zz}^{\alpha=0})\cos^2\alpha + (W_{zy} - W_{yz})\sin 2\alpha + (W_{yy}^{\alpha=\pi/2} + W_{zz}^{\alpha=\pi/2})\sin^2\alpha + \frac{1}{2}\int_0^l \frac{N^2}{D_0} dx. \tag{8}$$

The energy components in (8) are determined by the expressions

$$W_{yy}^{\alpha=0} = \frac{1}{2}\int_0^l \frac{M_{z_0}^2}{D_{zz}} dx + \frac{1}{2}\int_0^l k_y \frac{Q_{y_0}^2}{D_{Q_y}} dx, \quad W_{zz}^{\alpha=0} = \frac{1}{2}\int_0^l \frac{M_{y_0}^2}{D_{yy}} dx + \frac{1}{2}\int_0^l k_z \frac{Q_{z_0}^2}{D_{Q_z}} dx,$$

$$W_{yy}^{\alpha=\pi/2}=\frac{1}{2}\int_0^l\frac{M_{z_0}^2}{D_{yy}}dx+\frac{1}{2}\int_0^l k_z\frac{Q_{y_0}^2}{D_{Q_z}}dx,$$
$$W_{zz}^{\alpha=\pi/2}=\frac{1}{2}\int_0^l\frac{M_{y_0}^2}{D_{zz}}dx+\frac{1}{2}\int_0^l k_y\frac{Q_{z_0}^2}{D_{Q_y}}dx, \quad (9)$$
$$W_{yz}=\int_0^l\frac{M_{z_0}M_{y_0}}{D_{yy}}dx+\int_0^l k_z\frac{Q_{y_0}Q_{z_0}}{D_{Q_z}}dx,$$
$$W_{zy}=\int_0^l\frac{M_{y_0}M_{z_0}}{D_{zz}}dx+\int_0^l k_y\frac{Q_{z_0}Q_{y_0}}{D_{Q_y}}dx$$

and have the following mechanical meaning: $W_{yy}^{\alpha=0}$, $W_{zz}^{\alpha=0}$ are the actual works of the forces acting in the force planes y_0x, z_0x respectively, in the absence of rotation of the rod ($\alpha=0$); $W_{yy}^{\alpha=\pi/2}$, $W_{zz}^{\alpha=\pi/2}$ are the actual works of the forces acting in the force planes y_0x, z_0x respectively, when the rod is rotated by an angle $\alpha=\pi/2$; W_{yz}, W_{yz} are the possible works of the forces acting in the force planes y_0x, y_0x respectively, on the deformations of the inhomogeneous rod in the orthogonal planes z_0x, y_0x, caused by other force factors.

The minimum energy (8) is achieved at a rotation angle α_{opt}, for which the condition is satisfied

$$tg2\alpha_{\text{opt}}=\frac{W_{zy}-W_{yz}}{W_{yy}^{\alpha=0}+W_{zz}^{\alpha=0}-W_{yy}^{\alpha=\pi/2}-W_{zz}^{\alpha=\pi/2}}. \quad (10)$$

In [8], a variational setting of the optimization problem for a homogeneous spatially loaded flat frame was considered with variations in the external dimensions of the box-shaped cross-section and the angular position of the principal axes of inertia. The volume of materials was fixed. The following criteria were adopted as continuous in the direction of the axial coordinate: the strength condition for normal stresses and the condition of coincidence of the principal axis of inertia with the vector of the resulting bending moment. As a result, it was shown that adopting local criteria leads to a natural (initial) twisting of the rods. The resulting solutions are of some theoretical interest, however, such designs are characterized by high labor intensity of production and sensitivity to load variations, which reduces the feasibility of their practical use.

In contrast, formula (10) reflects the integral average value of rational local transformations of forces on the main planes of rigidity of an inhomogeneous rod, in the form of rotation of the main planes of rigidity by a general angle α_{opt}.

A number of special cases follow from the considered setting.

1. When the conditions $D_{zz}=D_{yy}$, $D_{Q_y}=D_{Q_z}$ are simultaneously met the deformation energy (8) becomes stationary – $W(\alpha)=\text{const}$, independent of the rotation angle α.

2. The arrangement of loads in a single force plane. By combining it with, for example, plane y_0x, we obtain $M_{y_0}=Q_{z_0}=0$, which leads to the fulfillment of $W_{zz}^{\alpha=0}=W_{zz}^{\alpha=\pi/2}=W_{yz}=W_{zy}=0$ and optimal angle $\alpha_{\text{opt}}=0$ at $D_{zz}>D_{yy}$ and $\alpha_{\text{opt}}=\pi/2$ at $D_{zz}<D_{yy}$.

3. If the following conditions are met for the calculation model: a) the presence of symmetry in the stiffness characteristics relative to the midsection ($x=l/2$), b) the functions of the forces acting in one of the force planes have the properties of direct symmetry, and in the other (orthogonal to the first) – inverse symmetry, then we obtain a zero value of the possible works W_{yz}, W_{zy} and angles $\alpha_{\text{opt}}=0$, $\alpha_{\text{opt}}=\pi/2$. The plane with large stiffness characteristics should be combined with the force plane in which forces acting with direct symmetry.

4. In rods with a constant cross-section (in geometry and structure), with a simplified consideration of only one force factor (*M* or *Q*) in the energy components (9), formula (10) yields two fixed values of the angle α_{opt}, depending on a) the considered force functions (*M* or *Q*) and b) the results of comparing the corresponding stiffness characteristics ($D_{zz}<D_{yy}$ or $D_{zz}>D_{yy}$,

when taking M into account; $D_{Q_y} > D_{Q_z}$ or $D_{Q_y} < D_{Q_z}$, when taking Q into account). The absolute values of the stiffnesses in this case do not affect the value of the optimal angle. This is shown in the numerical example discussed below in Fig. 4.

5. The strain energy components in (9) have an integrand structure similar to Mohr integrals and can be calculated using known graphoanalytic methods. This is particularly applies to rods with constant stiffness characteristics along their length. These integrals can also be found numerically.

RESULTS AND DISCUSSION

Let us consider a inhomogeneous rod subjected to a force action in two orthogonal planes y_0x and z_0x (Fig. 2). The loads are specified with an accuracy of parameters q, l. Fig. 3 shows a scheme of a composite cross-section including four structural elements with numbers k = 1, 2, 3, 4.

Table 1 shows the geometric parameters, relative elastic moduli $e_k = E_k / E_0$ and strength characteristics $r_k = R_k / R_0$ of the materials for the section elements.

Table 1.

Parameters of structural elements of the section

k	**Width**	**Height**	e_k	r_k
1	2δ	b	0,9	1,4
2	h	δ	0,6	1,2
3	h	δ	0,6	1,2
4	2δ	0,8·b	1	1

Table 2.

Cross-section layout options

Option №	h	b	A	z_0	D_{yy}	D_{zz}
1	30·δ	10·δ	70·δ^2	16,54·δ	11400,7·δ^4	679,3·δ^4
2	20·δ	13,53·δ	70·δ^2	11,58·δ	6368,7·δ^4	1163,8·δ^4
3	10·δ	17,06·δ	70·δ^2	6,71·δ	2201,4·δ^4	1649,1·δ^4
4	8,48·δ	17,60·δ	70·δ^2	5,98·δ	1717,9·δ^4	1717,9·δ^4
5	7·δ	18,12·δ	70·δ^2	5,27·δ	1298,4·δ^4	1782,6·δ^4
6	5·δ	18,82·δ	70·δ^2	4,31·δ	815,4·δ^4	1866,6·δ^4

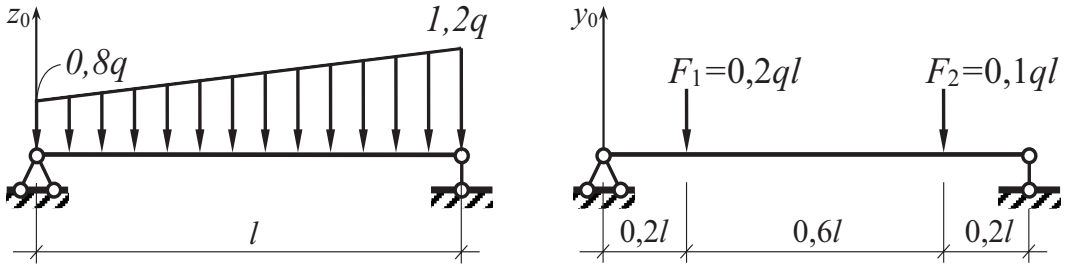


Figure 2. Given action: in plane z_0x, (a), in plane y_0x, (b)

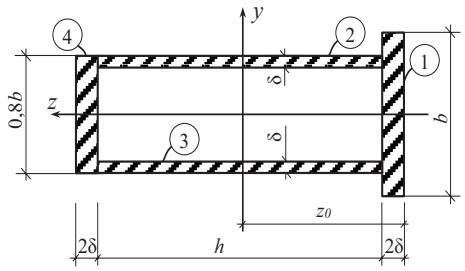


Figure 3. Cross-section of the rod in the initial position (α = 0)

We will do the research by specifying several cross-sectional layout options with varying dimensions *b* and *h*. We will also impose a constraint in the form of a constant cost of materials for the inhomogeneous rod.

$$C = l \cdot \sum_{k=1}^{4} c_k \gamma_k A_k = \text{const}, \qquad (11)$$

where A_k, c_k, γ_k is the cross-sectional area, specific cost and weight of the material of the *k*-th element.

Without diminishing the significance of the obtained results, we will adopt in this problem $c_k = \gamma_k = 1$, which transforms condition (11) into a condition of constancy of the total area $\sum_{k=1}^{4} A_k = \text{const}$. Table 2 presents six variants of similar cross-sectional arrangements with the area fixed $A = 70\delta^2$. For each of them, the coordinate of the center of rigidity z_0, ensuring the fulfillment of $D_y = 0$, and the bending rigidities D_{yy}, D_{zz}, were found. In the first three variants, the condition $D_{yy} > D_{zz}$ is satisfied, in the 4th – $D_{yy} = D_{zz}$, and in the 5th–6th – $D_{yy} < D_{zz}$.

Let us consider the problem of determining the optimal angle α_{opt} of rotation of the principal planes of rigidity, ensuring the minimum deformation energy of a spatially loaded inhomogeneous rod (Fig. 2) with six variants of cross sections (Tables 1, 2). We introduce dimensionless quantities

$$\bar{M} = \frac{M}{ql^2}, \quad \bar{Q} = \frac{Q}{ql},$$

$$\left[\bar{D}_{yy}, \bar{D}_{zz}\right] = \frac{1}{E_0\delta^4}\left[D_{yy}, D_{zz}\right],$$

$$\bar{W} = \frac{W \cdot E_0\delta^4}{q^2 l^5}, \quad \bar{\sigma}_x = \frac{\sigma_x \delta^3}{ql^2}, \quad \bar{\delta} = \delta \sqrt[3]{\frac{R_0}{ql^2}}.$$

After solving the boundary value problem and identifying the function of forces acting in the force planes y_0x, z_0x, the dimensionless deformation energy function $W(\alpha)$ (8) was formed. Its graph in the interval $\alpha \in [-\pi, \pi]$ is shown in Fig. 4. The lines numbers correspond to the numbers of the section variants in Table 2.

For variants 1–3, the optimal angles are contained in the interval $\alpha_{\text{opt}} \in [-\pi/8, -\pi/16]$, while for variants 5–6, they are $\alpha_{\text{opt}} \in [3\pi/8, 7\pi/16]$. In the first case, the cross-sections have $D_{yy} > D_{zz}$, which requires a negative (clockwise) rotation of the rod; in the second case, for variants with $D_{yy} < D_{zz}$, a rotation in the opposite direction is required. Since the specified materials do not possess the properties of different resistance under tension and compression, periodicity of the function $W(\alpha)$ is observed over an interval equal to π.

If the stiffnesses are equal $D_{yy} = D_{zz}$ (option 4), the rod becomes insensitive to the position of the principal stiffness planes $W(\alpha) = \text{const}$. Rotating them does not change the energy level. The best of the six section options is option 1; it corresponds to the lowest value of the deformation energy at the minimum point $\alpha_{\text{opt}} = -0{,}0864\pi = -15{,}6°$. Option 6 requires a rotation angle $\alpha_{\text{opt}} = 0{,}4136\pi = 74{,}5°$. The same angle values are obtained using the formula (10).

To illustrate the practical effect of rotating the rod by an angle, we will evaluate the level of normal stresses in cross section 1 (Table 2). When

conditions $D_z = D_y = 0$ are met, they are determined by the formula [19, 20]

$$\sigma_{x,k}^{(M)}(x,y,z) = \\ = E_k \left(-\frac{M_z(x)\cdot y}{D_{zz}} - \frac{M_y(x)\cdot z}{D_{yy}} \right). \qquad (12)$$

Taking into account the stress limitation by the value of the calculated resistances r_k (Table 1), it is convenient to monitor the level of relative dimensionless stresses $s_{k,j}(x) = \overline{\sigma}_{x,k}(x, y_j, z_j)/r_k$ in an inhomogeneous rod. For this purpose, Fig. 5 shows 16 graphs $s_{k,j}(x)$ $(k = 1,...,4,$ $j = 1,...,4)$ for four corner points taken in each of the four structural elements.

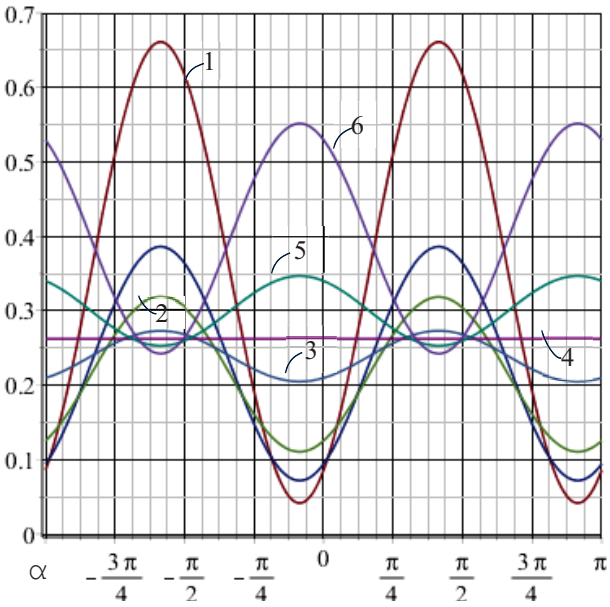


Figure 5. Dependence of the potential energy of deformation of a rod $\overline{W} \cdot 10^5$ on the angle of rotation of the rigidity planes for different cross-sectional options (Table 2)

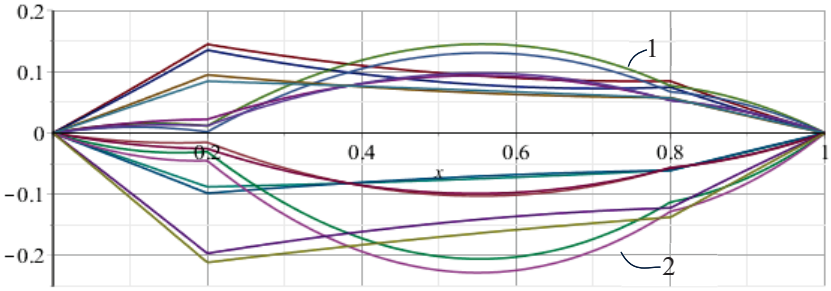


Figure 4. Graphs of relative stresses $10^3 \cdot s_{k,j}(x)$ at corner points of 4 structural elements of section 1, table 2 at $\alpha = \alpha_{opt}$.

The maximum relative stress is realized in graph 1, corresponding to the stress at point K of the first layer (Fig. 6).

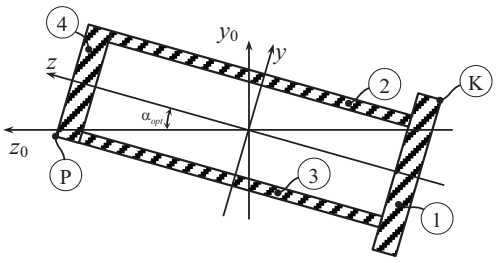


Figure 6. Optimal rotate of section 1 (Table 2) at the minimum value of deformation energy

The minimum is in graph 2, constructed for the stress at point P of the fourth layer. Taking point P as the design point – the point with the highest modulus of stress, from the strength condition

$$\delta \geq \delta_{\min} = \sqrt[3]{\frac{ql^2}{R_0} \max_{x,k,j} \left| s_{k,j}(x) \right|}$$

we find the required value of the geometric parameter $\overline{\delta}_{\min} = 0{,}0611$.

In this example, the settlement points are located at the corners of the outer contour of the section. Due to the discontinuity of stress functions (12) at the layer boundaries, with different physical characteristics, the settlement points may shift to the corners of the internal boundaries of the structural elements.

For comparison, we present the results corresponding to the initial position of the rod without rotation of the axes. Fig. 7 shows graphs of the relative stresses $s_{k,j}(x)$ acting in the same section 1 at the same 16 calculation points, at $\alpha = 0°$.

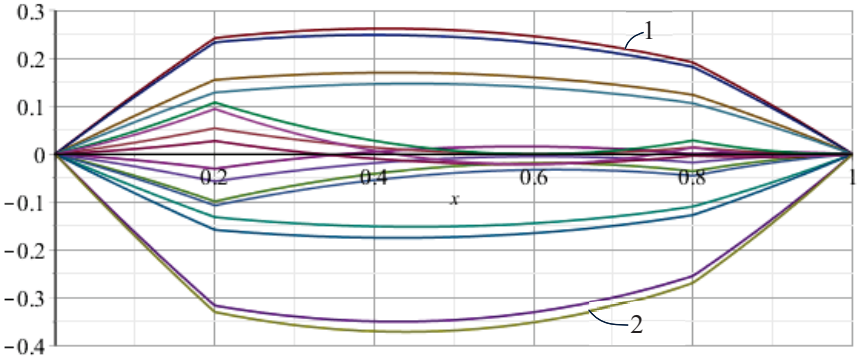


Figure 7. Graphs of relative stresses $10^3 \cdot s_{k,j}(x)$ at settlement points of section 1 at $\alpha = 0$

The required parameter value in this case was $\overline{\delta} = 0{,}0719$. Thus, setting the optimal rotate allowed us to reduce the size $\overline{\delta}$ by 15% and the volume of material by 38%.

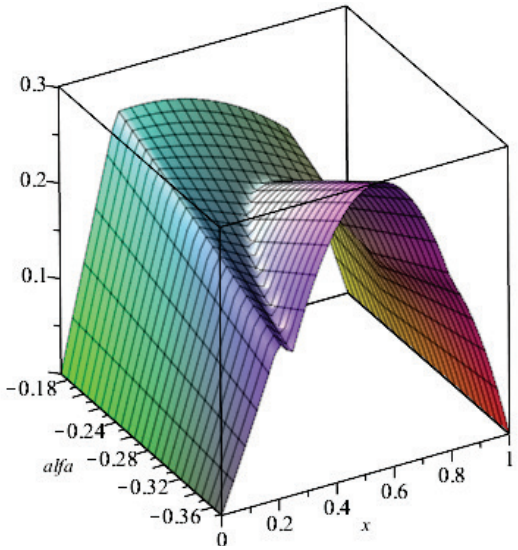


Figure 8. Dependency $10^3 \cdot \max\limits_{k,j} \left| s_{k,j}(\alpha, x) \right|$

In a number of studies, in particular in [21], it has been shown that as a result of the variational setting of optimization problems based on the energy criterion, when varying the geometric functions, equalization of stresses on the rod surface (at the layer boundaries) occurs. In this problem, solved without varying the geometry, continuous equalization of stresses does not occur (Fig. 5). However, here too, similar effects can be seen, manifested discretely in the form of equalization of local stress maxima. In Fig. 5, this is practically achieved for the greatest stresses in the cross-sections $x = 0{,}2$ and $x = 0{,}56$.

Fig. 8 shows a three-dimensional graph of relative stresses $\max\limits_{k,j} \left| s_{k,j}(\alpha, x) \right|$ as a function of x and angle α, which is the outer envelope of 16 surfaces – the moduli of stress constructed for the calculated points. Analysis of this surface allows us to formulate a criterion for minimizing local stress maxima.

$$\min_{\alpha} \left[\max_{k,j,x} \left| s_{k,j}(\alpha, x) \right| \right] \to \alpha_{\text{opt}}.$$

As a result of application it application, the optimal rod rotation angle was $\alpha_{\text{opt}} = -13{,}94°$ with a required strength dimension of $\overline{\delta}_{\sigma} = 0{,}0591$, resulting in a material savings of 32,4% compared to the design at $\overline{\delta} = 0{,}0719$ for the initial state at $\alpha = 0°$. With the determined rotation angle,

stresses are equalized at the two most heavily loaded points in the x-axis direction. One of these is located at the support $x = 0,2$, and the other is at the local span extremum at $x = 0,56$.

CONCLUSIONS

The use of rigid axial rotation of the rod as an external stress regulator is an effective additional optimization tool and allows for, in the example considered, allowed: a) increasing the load-bearing capacity of a inhomogeneous rod by 20–30% at a low cost, consisting of ensuring a specified fixed design position of the rod on the supporting devices; b) being an additional tool, it does not require changes to the physical structure and geometry of the rod, designed according to the optimization criteria. For this reason, this technique can also be applied to prefabricated standard structural elements, including homogeneous ones.

A significant advancement of this method is its application to "controlled structures," which, thanks to the presence of neural network systems, acquire the ability to effectively adapt to changes in external forces. To achieve this, the rod must be equipped with support devices capable of continuously adjusting the angular position of the rod's principal rigidity planes during operation. For the object in question – a inhomogeneous rod – implementing control systems is quite simple and feasible.

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