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NUMERICAL MODELING TECHNIQUE FOR THE HYDRODYNAMIC STABILITY OF SWIRLING FLOWS

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Abstract. A numerical technique is proposed for investigating the hydrodynamic stability of axisymmetric swirling flows of an incompressible viscous fluid within the linear stability theory framework. The associated eigenvalue problem is governed by a system of sixth-order ordinary differential equations. Asymptotic expansions are constructed at both artificial and irregular singular points to facilitate the solution. The system is integrated numerically using a high-order accurate scheme, and the Gram–Schmidt orthonormalization procedure is employed to suppress the rapid growth of particular solutions, thereby ensuring their linear independence. A computational package based on the developed methodology has been implemented and applied to the stability analysis of the Batchelor vortex. Validation tests are conducted to compare the obtained results against available reference data. The existence of a viscous instability mode is established, for which the maximum growth rate is found to decrease monotonically with increasing Reynolds number.

Keywords: hydrodynamic stability, small-disturbance method, swirling flow, viscous mode, growth rate coefficients

МЕТОДИКА ЧИСЛЕННОГО МОДЕЛИРОВАНИЯ ГИДРОДИНАМИЧЕСКОЙ УСТОЙЧИВОСТИ ЗАКРУЧЕННЫХ ТЕЧЕНИЙ

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Аннотация. Представлена методика численного исследования задачи гидродинамической устойчивости осесимметричных закрученных течений вязкой несжимаемой жидкости в рамках линейной теории. Рассматриваемая спектральная задача описывается системой обыкновенных дифференциальных уравнений шестого порядка. Для ее решения в регулярной и иррегулярной особых точках строятся асимптотические разложения. Численное интегрирование системы проводится методом повышенной точности с использованием процедуры ортогонализации Грамма-Шмидта для подавления быстрого роста одного или нескольких решений с целью обеспечения их линейной независимости. Разработанный пакет программ по описанной методике реализован для исследования устойчивости вихря Бэтчелора. Проведены тестовые вычисления для сравнения результатов с имеющимися данными. Показано существование вязкой моды неустойчивости, для которой максимальный коэффициент усиления монотонно убывает с ростом числа Рейнольдса.

Ключевые слова: гидродинамическая устойчивость, метод малых возмущений, закрученный поток, вязкая мода, коэффициенты усиления

1. INTRODUCTION

Swirling flows of liquid and gas are frequently encountered in nature (tornadoes, cyclones, whirlpools) and are widely used in various technical applications. Contemporary areas of application for swirling flows include the following [1,2]: the creation of efficient pulverized coal combustion facilities for modern thermal power plants; optimization of gas turbine engine performance through the use of flow swirl in combustion chambers; the development and design of cyclone-type devices for gas dust removal, particle separation, and liquid mixture separation, for example, for treating produced oil, as well as vortex atomizers for liquid spraying;

and the use of the Ranque–Hilsch effect in vortex tubes for thermal flow separation.

In hydraulic engineering and special construction applications, swirling flows are employed in the design of draft tubes for hydraulic turbines, vortex spillways, antivortex energy dissipators, and antivortex aerators for ecological treatment of natural and artificial water bodies, settling ponds, and storage basins in biological and chemical industries [3].

Investigations into the hydrodynamic stability of swirling flows are extremely relevant. Loss of flow stability often leads to disruption of the designed operating regime, intensification of vibration effects, and damage (or destruction) of the structure.

There are several general stability criteria for swirling flows. For free-vortex type flows, they are formulated in [4, 5]. These criteria are predominantly sufficient in nature and do not allow obtaining precise stability characteristics of the flow.

The primary tool for investigating the hydrodynamic stability of vortex flows is numerical modeling using the perturbation method. For free swirling flows and internal channel flows (excluding the near-wall region), the profiles of the axial U and azimuthal W velocity components can be reasonably well described by expressions

$$U(r) = W_1 + W_2 \exp(-br^2),$$
$$W(r) = \frac{K}{r}[1 - W_2 \exp(-br^2)], \quad (1)$$

where W_1, W_2, K, b are empirically determined constants, and r is the dimensionless distance from the axis. By elementary transformations, profiles (1) can be reduced to the form

$$U = \exp(-r^2), \quad W = \frac{q}{r}(1 - \exp(-r^2)), \quad (2)$$

where q is the flow swirl parameter. Therefore, in recent years, considerable attention has been devoted to the stability investigation of the flow with velocity distribution (2), often referred to as the Q – vortex (or Batchelor vortex).

The inviscid stability of flow (2) with respect to non-axisymmetric perturbations of the form $\exp[i(\alpha z + n\varphi - \alpha ct)]$ was first numerically investigated in [6]. Primary attention was given to perturbations with $n \le -1$, since they proved more unstable than perturbations with positive n. The presented numerical results showed that instability of flow (2) is observed for $q < 1.5$.

The effect of viscosity on the stability of a free vortex was first numerically studied in [7]. The lowest critical Reynolds number corresponded to perturbations with $n = -1$. Two viscous instability modes of the Q-vortex at $n = 0$ and $n = 1$ were found in [8, 9]. In contrast to previously studied inviscid modes, their maximum amplification factor monotonically decreases with increasing Reynolds number $\mathrm{Re} \to \infty$.

The transition from absolute to convective instability in an inviscid formulation for the Batchelor vortex Q was investigated in [10]. In jet flows, absolute instability was observed for modes $n = -1, -2, -3$. For wake-type flows, absolute instability was found only for mode $n = -1$. Generalizations of these studies accounting for flow compressibility were carried out in [11, 12].

Similar investigations in an inviscid formulation for the Rankine vortex are presented in [13], and for shear swirling flows in [14].

Selective calculations of absolute instability for Q vortex accounting for viscosity, performed using direct numerical simulation, were conducted in [15], while systematic and detailed investigations over a wide range of Reynolds numbers are presented in [16].

Experimental results on the stability of swirling flows and the structure of the near-axis recirculation zone during vortex breakdown are presented in [17–19].

Issues of global instability following vortex breakdown are considered in [20]. The relationship between the vortex breakdown phenomenon and swirling flow stability is shown in [21].

With regard to problems of thermal flow separation in Ranque–Hilsch tubes, flow stability was studied in [22], and for stratified media in [23].

The conducted analysis of the scientific literature indicates a strong interest in the stability investigation of swirling flows. The Batchelor vortex has a velocity distribution to which a significant portion of such flows can be reduced. In this connection, the problem of Batchelor vortex stability is an important and pressing issue.

2. PROBLEM STATEMENT AND RESEARCH METHOD

Consider homogeneous axisymmetric flow of a viscous incompressible fluid with a velocity field in an unbounded medium

$$V_z = U(r), \quad V_r = 0, \quad V_\varphi = W(r), \tag{3}$$

where V_z, V_r, V_φ are the axial, radial, and azimuthal velocity components in a cylindrical coordinate system.

Consider small perturbations of the base flow (3) with velocity profiles (2) as solutions of the linearized Navier–Stokes equations in the form of travelling waves (normal modes):

$$\begin{aligned}\{V_z', V_r', V_\varphi', p'\} = \\ = \{F, iS, H, P\}\exp[i(\alpha z + n\varphi - \alpha ct)],\end{aligned} \tag{4}$$

where p is the pressure; α is the wavenumber; n is the perturbation mode $(n = 0; \pm 1; \pm 2; ...)$, with negative values corresponding to wave propagation in the direction opposite to the swirl; c is the wave speed; i is the imaginary unit. For the complex amplitude functions $F(r)$, $S(r)$, $H(r)$, $P(r)$ we obtain the system of equations:

$$\begin{aligned}r^2\gamma F + \alpha r^2 P + r^2 SU' = \\ = \frac{1}{i\,\mathrm{Re}}\left[r(rF')' - (\alpha^2 r^2 + n^2)F\right], \\ r^2\gamma S + 2rHW - r^2 P' = \\ = \frac{1}{i\,\mathrm{Re}}\left[r(rG')' - (\alpha^2 r^2 + n^2 + 1)S - 2nH\right], \\ r^2\gamma H + r^2 S\left(W' + \frac{W}{r}\right) + rnP = \\ = \frac{1}{i\,\mathrm{Re}}\left[r(rH')' - (\alpha^2 r^2 + n^2 + 1)H - 2nS\right], \\ \alpha rF + (rS)' + nH = 0,\end{aligned} \tag{5}$$

where $\gamma = \alpha(U - c) + nW/r$, and the prime denotes differentiation with respect to r. The boundary conditions for system (5) at $r = 0$ are derived from the requirements of regular behavior of the solution near the axis and decay of perturbations at infinity, and have the form:

$$\begin{aligned}S(0) = H(0) = 0, \\ F(0), P(0) \ \ are \ \ limited \ \ for \ \ n = 0; \\ S(0) \pm H(0) = 0, \\ F(0) = P(0) = 0 \ for \ n = \pm 1; \\ S(0) = H(0) = F(0) = P(0) = 0 \ \ for \ \ |n| > 1; \\ S(\infty) = H(\infty) = F(\infty) = P(\infty) = 0.\end{aligned} \tag{6}$$

Consider perturbations (4) that are periodic in z, whose amplitude changes with time. Then α is a real number ($\alpha = 2\pi/\lambda$, where λ is the perturbation wavelength), and $c = c_r + ic_i$ is complex one; c_r represents the speed of perturbation propagation in the direction z (phase velocity), c_i is the temporal growth rate of the perturbation, $\omega_i = \alpha c_i$ is the perturbation amplification factor, and $\omega_r = \alpha c_r$ is the oscillation frequency. If $c_i < 0$, the perturbation amplitudes (4) decay (the flow is stable). And if $c_i > 0$, they grow with time (the flow is unstable). Thus, here it is necessary to determine c_i as a function of α.

In another case, one studies perturbations (4) that are periodic in time with amplitude varying

in the direction z, but independent of time. Then the frequency $\omega = \alpha_r c$ should be considered real (c is the phase velocity), and $\alpha = \alpha_r + i\alpha_i$ is a complex one; α_i determines the spatial growth rate of the perturbation. If $\alpha_i > 0$ the perturbation decays (the flow is stable), and if $\alpha_i < 0$, it grows (the flow is unstable). In this case, it is important to determine the dependence of α_i on ω.

The eigenvalue computation method comprises several stages. Near the singular points $r = 0$ and $r = \infty$, asymptotic solutions are constructed using the Frobenius method [24], which allow the boundary conditions to be transferred to the points $r = r_a$ and $r = r_d$, respectively. Then numerical integration is performed from r_a and r_d into the computational domain towards the point r_c $(r_a < r_c < r_d)$, where the solutions are matched according to the condition

$$\sum_{j=1}^{3} a_j y_{i,j} = \sum_{j=1}^{3} a_{j+3} y_{i,j+3} \quad (i = 1, 2, \ldots, 6), \quad (7)$$

where $y_{i,j}$ are the solutions obtained by integration from r_d to r_c, and $y_{i,j+3}$ are those obtained by integration from r_a to r_c, a_j – произвольные постоянные. The vanishing of the determinant of the linear homogeneous system (7) is achieved by adjusting c_r, c_i (or α_r, α_i) using Newton's method.

In the calculations, $r_a = 0.01$, $r_c = 0.5$ were taken, and the value of r_d was chosen individually for each set of parameters based on the required accuracy of the eigenvalue determination, and varied from $r_d = 2.7$ to $r_d = 150$. The widest integration domain was required for long-wave perturbations, at large Reynolds numbers and swirl parameters. When computing the dependencies of the eigenvalues on the governing parameters, the step in the parameter was determined automatically by the number of iterations k_{it} in Newton's method for solving the characteristic equation. The value of k_{it} was maintained in the interval $3 \le k_{it} \le 7$. If $k_{it} > 7$ or $k_{it} < 3$ for three successively found eigenvalues, the parameter step was correspondingly halved or doubled. To accelerate convergence, the next initial guess for the eigenvalue in Newton's method was determined by linear extrapolation.

It is convenient to represent system (5) for numerical integration as six first-order differential equations $z' = f(r, z)$. To find the solution, we employ the Kutta–Merson method with automatic step-size selection. Denote

$$f_{n+\beta}^{[k]} = f(r_n + \beta h, z_{n+\beta}^{[k]}),$$

where h is the integration step, $z_{n+\beta}^{[k]}$ is the approximate solution found at the point $r_n + \beta h$ with local error h^k. Then the set of calculation formulas can be written as:

$$z_{n+1/3}^{[2]} = z_n^{[5]} + \frac{h}{3} f_n^{[5]},$$

$$z_{n+1/3}^{[3]} = z_n^{[5]} + \frac{h}{6}(f_n^{[5]} + f_{n+1/3}^{[5]}),$$

$$z_{n+1/2}^{[4]} = z_n^{[5]} + \frac{h}{8}(f_n^{[5]} + 3 f_{n+1/3}^{[3]}),$$

$$z_{n+1}^{[4]} = z_n^{[5]} + \frac{h}{2}(f_n^{[5]} - 3 f_{n+1/3}^{[3]} + 4 f_{n+1/2}^{[4]}),$$

$$z_{n+1}^{[5]} = z_n^{[5]} + \frac{h}{6}(f_n^{[5]} + 4 f_{n+1/2}^{[4]} + f_{n+1}^{[4]}).$$

Thus, approximate values $z(r_{n+1})$ are calculated with local errors h^4 and h^5. Comparison of the values $z_{n+1}^{[4]}$ and $z_{n+1}^{[5]}$ provides a practical means to assess the accuracy of the approximate solution $z(r_{n+1})$ during the computation. The integration step h for each computational point is determined from the condition

$$\left| (z_n^{[5]} - z_n^{[4]}) / z_n^{[4]} \right| \le \varepsilon,$$

in which it was assumed that $\varepsilon = 10^{-2}$.

A fundamental difficulty in implementing this method when solving hydrodynamic stability problems is the rapid (parasitic) growth of one or several solutions during integration, as a result of which their linear independence is lost. One of the most effective methods for overcoming these difficulties is the orthogonalization procedure of S.K. Godunov [25].

Such an orthogonalization procedure for the vectors can be performed either at each integration step, or after a certain number of steps when a predefined non-orthogonality criterion is violated. Such a criterion, for example, can be the condition $\max(y^j, y^j)^{1/2} > M$, where the constant M is usually of the order of $10^2 - 10^3$. In the present calculations, orthogonalization was carried out at every computational point.

To compute the complex determinant of system (7), the Gauss elimination method with column pivoting was used. The solution of an equation of the form $F(c) = 0$ was determined by Newton's method.

3. RESULTS AND DISCUSSION

Nontrivial solutions of the homogeneous boundary-value problem (5), (6) with distribution (2) determine the spectrum of complex eigenvalues and eigenfunctions for a given set of parameters Re, q, α, n. The first two parameters characterize the flow, and the latter two describe the perturbation wave.

The results of test calculations of the eigenvalues of the original problem for the viscous mode found in [8] and studied in detail in [9] are presented in Table 1. It can be seen that in all cases there is good agreement. This indicates that the numerical method employed allows obtaining a solution with high accuracy. At the same time, it is simpler in numerical implementation than the spectral collocation method with Chebyshev expansions [8].

In [8], two viscous instability modes with positive azimuthal wavenumbers were identified numerically, for which the maximum amplification factor monotonically decreases as $\mathrm{Re} \to \infty$. One of them corresponds to axisymmetric perturbations $n = 0$, and the other to non-axisymmetric ones with $n = 1$. In both cases, the instability turns out to be weaker compared to the inviscid mode at $n = -1$, and the corresponding critical Reynolds numbers calculated in [9] are higher: $\mathrm{Re}_c = 17.275$; 322.42 for the modes $n = 0; 1$, respectively. These two modes are of significant importance because they confirm the destabilizing effect of viscosity on flow stability.

The present numerical investigations have established that for there also exists an unstable mode of viscous nature (mode 2).

Provide analysis its properties in detail. The critical Reynolds number for it is $\mathrm{Re}_c = 43.934$ at $\alpha_c = 0.5012$, $q_c = 0.6603$. In this sense, it occupies an intermediate position between the viscous modes $n = 0$ and $n = 1$.

Fig. 1 shows the relations $\omega_i = \omega_i(\alpha)$, $\omega_r = \omega_r(\alpha)$ for mode 2 by solid and dashed lines, respectively, calculated at $\mathrm{Re} = 100$, $q = 0.3,\ 0.4,\ 0.5,\ 0.6$ (curves 1–4). It is seen that the amplification factor $\omega_i = \omega_i(\alpha)$ has a peak-shaped distribution. The same character was noted for similar dependencies in [8] when studying viscous modes with $n = 0$ and $n = 1$. The phase velocities of perturbations with wavenumbers corresponding to the maximum amplification factor are negative, i.e., unstable perturbations in this case propagate upstream.

Confirmation that the found mode is viscous in nature is provided by the dependence of the amplification factor on the swirl $\omega_i = \omega_i(q)$ at fixed wavenumber $\alpha = 0.3$ and various $\mathrm{Re} = 300,\ 1000,\ 5000,\ 10000$ (Fig. 2, curves 1–4). It is evident that as the Reynolds number increases, the maximum amplification factor monotonically decreases.

Table 1

Re	α	*q*	$\omega_r \cdot 10^2$	$\omega_i \cdot 10^3$	$\omega_r \cdot 10^2$ [9]	$\omega_i \cdot 10^3$ [9]
25	0.05	0.5	–0.22387	0.98872	–0.22387	0.98851
25	0.05	0.5	0.00374	0.26682	0.00370	0.26618
100	0.3	0.4	1.72806	8.99650	1.72799	8.99607
1000	0.3	0.4	2.34621	0.97904	2.34621	0,97904

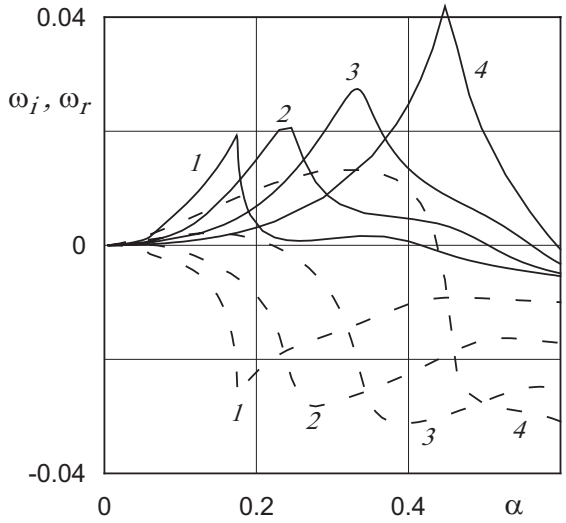


Figure 1. Relation of the amplification factor on the twist angle at Re=100, q=0.3, 0.4, 0.5, 0.6 (curves 1-4)

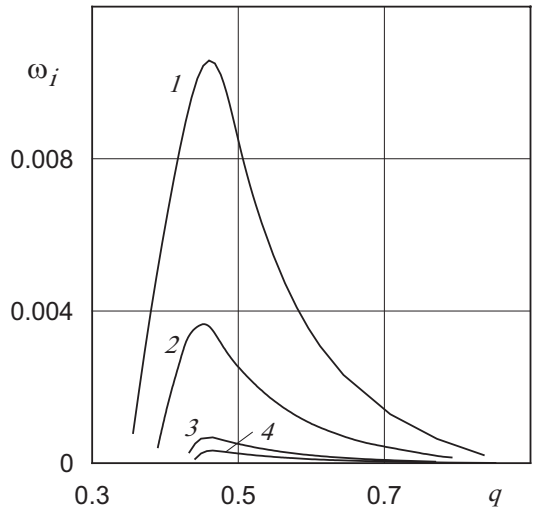


Figure 2. Relation of the amplification factor on the twist angle at α=0.3, Re=300, 1000, 5000, 10000 (curves 1-4)

4. CONCLUSION

An efficient numerical procedure has been developed for investigating the hydrodynamic stability of axisymmetric swirling flows with respect to non-axisymmetric travelling-wave perturbations. Calculations of temporal instability have been performed for the self-similar Batchelor vortex. A viscous instability mode has been found whose amplification factors exceed the values for previously known viscous modes. The application of the considered method is possible for a wide class of flow stability problems with arbitrary velocity profile distributions in various technical applications.

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