International Journal for Computational Civil and Structural Engineering, 21(2) 43–61 (2025)

DOI:10.22337/2587-9618-2025-21-2-43-61

ACCOUNTING FOR THE ADDED MASS OF WATER OF RESERVOIR IN CALCULATION OF SEISMIC ACCELERATIONS FOR PRESSURE HYDRAULIC STRUCTURES

Alexandra S. Bestuzheva, Georgy P. Gotsiridze

National Research Moscow State University of Civil Engineering, Moscow, RUSSIA

Abstract: The analysis of domestic and foreign studies on the issue of hydrodynamic pressure of water on the pressure face of hydraulic structures under dynamic influences is carried out. The authors have developed an alternative method for numerically analytical solution of the problem of seismic water pressure on the inclined pressure face of the dam. Based on the grapho-analytical solution of the Laplace equation under specified boundary conditions, hydrodynamic grids and diagrams of the distribution of the coefficient of the attached mass of water for pressure faces with a vertical angle of 0...75 degrees are constructed, the obtained solution is compared with the reference solution of Westergaard, Zangar, as well as recommendations of relevant regulatory documentation. Based on the conducted research, a nomogram was constructed to determine the coefficient of the attached mass of water. The obtained results are approximated by a quadratic function, which allows us to obtain the values of the coefficients of the combined mass of water depending on the vertical angle of inclination of the pressure face and the relative depth of the element location on the pressure face, which is used in the algorithm of the calculation program for computers. A graph of the change in the hydrodynamic force acting on the structure, depending on the vertical angle of inclination of the pressure face, is given, the accuracy of plotting is investigated depending on the number of elements on the pressure face. On the basis of the linear spectral method, accelerations in the elements of the pressure face of dams of various designs are calculated, both taking into account the attached mass of water and without. The results obtained indicate a significant effect of the attached mass of water on seismic accelerations not only on the crest of a concrete gravity dam, but also in the zone of the upper slope of groundwater dams. The increase in accelerations in these zones is up to 30% for concrete dams and about 10% for ground dams.

Keywords: hydrodynamic pressure, Laplace equation, attached water mass, pressure structure, seismic acceleration, linear spectral method

УЧЕТ ПРИСОЕДИНЕННОЙ МАССЫ ВОДЫ ВОДОХРАНИЛИЩА В РАСЧЕТЕ СЕЙСМИЧЕСКИХ СИЛ ДЛЯ НАПОРНЫХ ГИДРОТЕХНИЧЕСКИХ СООРУЖЕНИЙ

А.С. Бестужева, Г.П. Гоцiридзе

Национальный исследовательский Московский государственный строительный университет, г. Москва, РОССИЯ

Аннотация: Произведен анализ отечественных и зарубежных исследований по вопросу гидродинамического давления воды на напорную грань гидротехнических сооружений при динамических воздействиях. Авторами разработан альтернативный метод численно-аналитического решения задачи о сейсмическом давлении воды на наклонную напорную грань плотины. На основе графоаналитического решения уравнения Лапласа в заданных граничных условиях построены гидродинамические сетки и эпюры распределения коэффициента присоединенной массы воды для напорных граней с вертикальным углом 0…75 градусов, проведено сопоставление полученного решения с эталонным решением Вестергарда, Зангара, а также рекомендациями актуальной нормативной документации. На основе проведенных исследований построена номограмма для определения коэффициента присоединенной массы воды. Полученные результаты аппроксимированы квадратичной функцией, позволяющей получить значения коэффициентов присоединенной массы воды в зависимости от вертикального угла наклона напорной

грани и относительной глубины расположения элемента на напорной грани, что использовано в алгоритме расчетной программы для ЭВМ. Приводится график изменения гидродинамической силы, действующей на сооружение, в зависимости от вертикального угла наклона напорной грани, исследована точность построения эпюры в зависимости от числа элементов на напорной грани. На основе линейно-спектрального метода вычислены ускорения в элементах напорной грани плотин различной конструкции как с учетом присоединенной массы воды, так и без. Полученные результаты говорят о существенном влиянии присоединенной массы воды на сейсмические ускорения не только на гребне бетонной гравитационной плотины, но и в зоне верхового откоса грунтовых плотин. Увеличение ускорений в этих зонах составляет до 30% для бетонных плотин и около 10% для грунтовых плотин.

Ключевые слова: гидродинамическое давление, уравнение Лапласа, присоединенная масса воды, напорное сооружение, сейсмическое ускорение, линейно-спектральный метод

INTRODUCTION

One of the most difficult and unresolved issues of designing ground dams in the high seismicity zone is the interaction of the structure with the reservoir's water environment during an earthquake. In the modern regulatory literature (SP 358.1325800.2017 " Hydraulic structures. Rules of design and construction in seismic areas", SP 14.13330.2018 "Construction in seismic areas", hereinafter-SP) regulates the determination of hydrodynamic loads at the contact of a structure with water as additional seismic loads during earthquakes. Despite a large amount of scientific research on the hydrodynamic pressure of water on the pressure face of structures under dynamic impacts, in the current standards for plotting the seismic water pressure, there are only provisions on the "attached mass of water" with the determination of the coefficients of the attached mass according to the solution of Westergaard [1], obtained for the vertical pressure face. The Westergaard solution obtained analytically is a reference one (hereinafter referred to as the Westergaard problem) and underlies the applied methods for calculating hydraulic structures. It is based on the following assumptions about: - an ideal, incompressible fluid; - a rigid, non-deformable, interacting with water body; - non-eddy motion of the liquid; - absence of the phenomenon of wave formation on the water surface; - horizontal translational vibrations with a small amplitude, which are performed by the "dam (with base)" system- reservoir"; - a flat problem in the assumption of an infinite length of the reservoir with a horizontal bottom line.

In further works of Russian [2-7] and foreign scientists [8-12], the problems of hydroelasticity were solved taking into account the viscous properties of a compressible liquid and the elastic properties of a solid, taking into account wave formation on the water surface, the special geometry of the reservoir bed, and other factors that specify the solutions.

Of particular importance in the practice of hydraulic engineering construction are structures with a slope profile, often with a broken pressure face, for which the definition of seismic pressure in the current regulatory documentation during an earthquake is limited to taking into account the angle of the upper slope (vertical angle) of 15°.

However, for the vast majority of structures of ground dams, many structures of concrete buttresses, dams made of rolled concrete, berthing and enclosing structures, the upper pressure faces have an angle significantly greater than 15 degrees, which must be taken into account when designing structures in the zone of high seismicity.

The problem of the distribution of the seismic water pressure on the inclined pressure face of the dam was solved in the 50s of the XX century in the works of Sh. G. Napetvaridze on the basis of the conformal mapping method [13], where he gave an approximate formula for determining the seismic water pressure on the inclined face in the form:

$$p_\alpha(z) = p_w(y)\cos^2(\alpha) \quad (1)$$

where α is the vertical angle of inclination of the pressure face.

$p_w(y)$ - seismic water pressure on the vertical pressure face at a depth y from the water level (Westergaard's solution), $p_\alpha(z)$ - seismic water pressure on the inclined pressure face at the length *z* from the water edge along the face.

In the Reference manual of I. S. Sheinin [3], based on the solution of hydroelasticity problems, a similar formula is given for the coefficient of the attached mass of water μ_α taking into account the angle of inclination of the pressure face (vertical) in the form:

$$\mu_\alpha = \mu C(\alpha)\cos^2(\alpha) \quad (2)$$

where is the coefficient of attached mass for the vertical pressure face;

$C(\alpha)$ - coefficient depending on the angle of inclination of the pressure face (Figure 1)

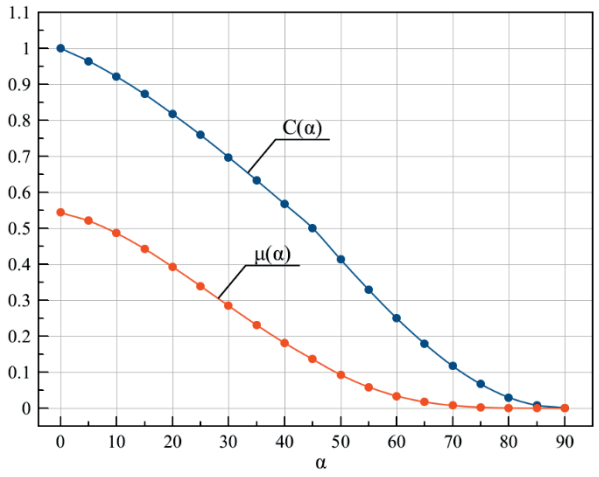


Figure 1. Coefficient of attached water mass

In [14], an experimental solution was obtained for determining the seismic pressure of water on the inclined face of a dam based on the electrodynamic analogy method (EGDA), given in the form of plots of coefficients of the attached mass of water. Comparison of the obtained values of the attached mass coefficients with the values obtained by the authors of the work is given below.

The authors have developed an alternative grapho-analytical method for solving the prob-lem of seismic water pressure on the inclined pressure face of an underground dam in the formulation of the "Westergaard problem" (hereinafter referred to as the GAM) [15,16], compared with known solutions, and compiled calculation programs for determining seismic forces in the body of a ground dam, taking into account the forms of natural vibrations, the seismic pressure of the reservoir, and the direction of approach of the seismic waves to the dam alignment [17].

METHODS

During an earthquake, the dam oscillates and shifts towards the upstream, while hydrodynamic pressure will act on it from the water side, depending on the speed of movement, the elastic properties of interacting bodies, and the geometry of the calculated area. The solution of the contact problem of hydroelasticity for a hydraulic structure is based on the provisions of continuum mechanics and the theory of small deformations, and the prerequisites are used to simplify the problem without losing the accuracy of the solution for a particular structure. For the massive body of a ground dam, small values of the amplitudes of seismic vibrations can be noted, which are measured in centimeters in the most severe earthquakes. Taking into account the fact that the dam dimensions are much larger than the amplitudes of its displacements during seismic vibrations, we can neglect the shape of the deformed axis of the pressure face of the dam during vibrations and take it as a displacement plot uniform in height, which corresponds to the model of "translational movement of a rigid wall". Due to the insignificant speed of water movement in the reservoir during an earthquake, the viscous properties of water can be ignored. Due to the minimal oxygen content in the deep

layers of the reservoir, the water medium can be considered as a homogeneous and incompressible liquid with a constant density $p = const$.

The solution of the problem of pressure distribution in a liquid under given boundary conditions and a known displacement on the contour (the pressure face of the dam) is based on dynamic equilibrium equations in the Euler form, one of which can be written as:

$$\rho \frac{dv}{dt} = \rho X - \frac{\partial p}{\partial x} \tag{3}$$

where $v(x,z,t)$ is a function of the velocity of the water medium particle, m/s

$p(x,z,t)$ – pressure function, kPa

X – projection of volumetric forces on the x-axis, kN

ρ - density of liquid (water), t/m3[3]

The Euler equation in the form (3) is written for volume forces assigned to a unit of volume.

Assuming that there is no rotation in the motion of particles, we can write down the following relative to the velocity components:

$$\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} = 0 \tag{4}$$

what happens if v_x both v_z are coordinate derivatives of the same function, i.e. we can write:

$$\begin{cases} v_x = \dfrac{\partial u_x}{\partial t} = \dfrac{\partial \varphi}{\partial x} \\ v_z = \dfrac{\partial u_z}{\partial t} = \dfrac{\partial \varphi}{\partial z} \end{cases} \tag{5}$$

The function is usually called the velocity potential, and the irrotational motion is called the potential.

The Euler equation in the absence of external forces, when the fluid motion is related to the pressure difference at points in a continuous medium caused only by seismic action, taking into account (5), in projections on the axes x and z is written as:

$$\begin{cases} \dfrac{\partial P_0}{\partial x} = -\rho \dfrac{\partial v_x}{\partial t} \\ \dfrac{\partial P_0}{\partial z} = -\rho \dfrac{\partial v_z}{\partial t} \end{cases} \tag{6}$$

where P_0 is the excess hydrodynamic pressure caused by dam vibrations, kPa.

v_x, v_z - horizontal and vertical velocity components.

The solution of problem (6) involves determining the vector velocity field of fluid particles and the scalar pressure field at each moment of time t according to the law of base motion during an earthquake.

The solution of the dynamic problem is constructed based on a series of quasi-stationary states, for which the velocity and pressure functions are determined. The harmonic function can be taken as a regularity of the change in acceleration over time in the given problem:

$$\ddot{u}(t) = \ddot{U}_0 \cos \omega t \tag{7}$$

where is the maximum acceleration of the base corresponding to the earthquake score and local ground conditions, m/s[2], according to the SP is set as a coefficient;

ω - the frequency of forced vibrations during an earthquake, in harmonic vibrations, T - the period closest to the frequency of natural vibrations of the dam, for local earthquakes the period does not exceed $T \le 1...1,2\,s$.

Substituting expressions (5) into (6) and integrating the full differentials of the functions: $dP_0(x,z)$ and $d\varphi(x,z)$ [15], we obtain that the hydrodynamic pressure function is related to the velocity potential by the relation:

$$P_0 = -\rho \frac{\partial \varphi}{\partial t} + C \tag{8}$$

The resulting expression indicates that the pressure function is also a potential function, therefore, its distribution in space obeys the Laplace equation and can be written:

$$\frac{\partial^2 P_0}{\partial x^2}+\frac{\partial^2 P_0}{\partial z^2}=0 \text{ or } \Delta P_0=0 \qquad (9)$$

The solution of the Laplace equation under given boundary conditions is called the Dirichlet problem and is completely determined by the boundary conditions, i.e. it is a boundary value problem. The desired function in equation (9) is the overpressure function, the distribution of which in space can be represented as a family of equipotentials, lines of equal pressure

Often, when solving hydraulic (filtration) problems, the solution of the Laplace equation is presented as a family of two potential functions – the velocity (pressure) potential function and the current line function. Potential functions of current and pressure (velocity potential) they are complex conjugate harmonic functions connected by the Cauchy-Riemann relations:

$$\begin{cases}\dfrac{\partial\varphi}{\partial x}=\dfrac{\partial\psi}{\partial z}\\ \dfrac{\partial\varphi}{\partial z}=-\dfrac{\partial\psi}{\partial x}\end{cases} \qquad (10)$$

In this case, the system of equipotentials $\varphi=const$ and current lines $\psi=const$ form an orthogonal hydrodynamic grid, in which the position of one of the functions uniquely determines the position of the other. Thus, the solution of the Laplace equation for the current function, represented as a family of streamlines, allows graphoanalytically constructing a hydrodynamic grid with determining the position of equipotentials for the velocity potential function (hydrodynamic pressure). This property of orthogonality of two functions allows us to use the boundary conditions of one of the functions when integrating the other (9).

Based on physical representations of the pressure distribution in a liquid during translational (horizontal) motion of a solid wall located at an angle α to the vertical, the following boundary conditions can be determined for the design scheme shown in Figure 2:

1) the boundary of the reservoir bed is non-deformable, impenetrable and stationary, which means that there is no gradient of the velocity potential function along the normal to the boundary ($\frac{\partial\varphi}{\partial n}=0$) and the boundary position for the current line function ($\psi=1$);

2) the vertical boundary of the design area at a great distance from the moving wall is stationary, watertight and does not experience hydrodynamic pressure from wall vibrations, along it it can be assumed $\varphi=0$, however, in the case of consider-ing the problem with a limited reservoir length, or if the slope of the reservoir bottom line is taken into account, the boundary conditions should be set differently: $\psi=1$ and $\frac{\partial\varphi}{\partial n}=0$;

3) there are no wave formation phenomena on the reservoir surface, the pressure is constant: $\varphi=const$ therefore, the current lines are orthogonal to the water surface;

4) on the mobile impenetrable pressure face of the dam, the points of which move at the same horizontal speed, the boundary values of the hydrodynamic pressure function are not determined and are the desired ones. At the same time, for the function of streamlines, the boundary conditions can be defined as follows. At the lower point of the pressure face (A), the intersection with the boundary current line is

set, where $\psi = 1$. At the upper point of the pressure face (D) , at the upstream water cut, the value for the current line function is: $\psi = 0$. The assumption of a uniform "penetrating-reflecting" capacity of the movable pressure face of the structure determines a uniform distribution of flow rates of "current tubes" over the height of the dam. Based on the latter assumption, you can specify a linear distribution of current lines along the length of the pressure face with a step $\Delta\psi$ that corresponds to the accuracy of the problem solution. The constant gradient for the current functions over the height of the pressure face $\Delta\psi$ was assumed to be equal 5...10% h to, where h is the height of the pressure face of the dam.

Determining the boundary conditions for the potential current function allows us to solve the problem of its distribution in the domain (Figure 2) by integrating the Laplace equation for this function:

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial z^2} = 0 \qquad (11)$$

Determination of the hydrodynamic pressure function from the obtained solution for the function with the condition of orthogonality of these functions can be obtained graphoanalytically based on the construction of hydrodynamic grids.

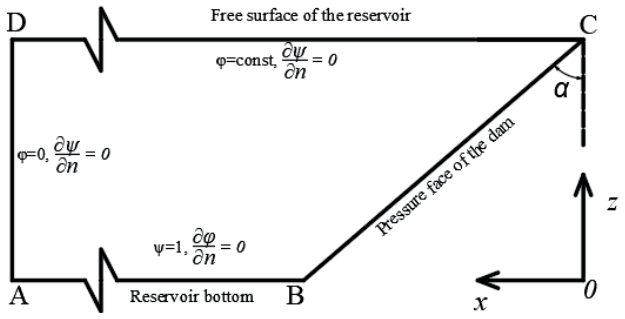


Figure 2. Design scheme with boundary conditions

Based on the principle of quasi-stationary states, it is possible to determine the maximum value of the hydrodynamic pressure function that will occur at the points of the pressure face of the dam, provided that the external impact is given by the maximum acceleration amplitude according to the earthquake score. For $\cos\omega t = 1$, according to (7), $\ddot{u}(t) = \ddot{U}_0 = A_0 g$ The distribution of the maximum pressure along the pressure face will be determined by the nature of the change in the function, which, according to the accepted boundary conditions, varies from 0 to 1 and determines the normalized values of hydrodynamic pressure.

When determining hydrodynamic loads on a structure due to the influence of reservoir water, the concept of "attached mass" of water is used as a body that performs joint vibrations with the structure, changing the inertial properties of the structure.

The force of hydrodynamic water pressure on the pressure face of the dam (P_w) can be represented on the basis of Newton's Second law as the maximum acceleration caused by wall vibrations ($\ddot{U}_0$), multiplied by the mass of water" attached" to the pressure face in the form:

$$P_w = m_w \ddot{U}_0 \qquad (12)$$

where m_w is the "attached mass" involved in the joint vibrations of the dam, which can be used to describe the hydrodynamic pressure on the structure during an earthquake as inertial.

The volume of the attached mass of water per 1 sq. m. of the width of the dam can be described by the area of the plot of the attached mass of water:

$$V_w = H \cdot \int_0^H \mu(Z)dZ \qquad (13)$$

where H is the height of the pressure face, m.
$\mu(Z)$ - function of the coefficient of the attached mass of water;
Z - the height of the element on the pressure face from the bottom of the reservoir, m
Thus, the attached mass of water for the entire structure can be defined by the expression:

$$m_w = \rho H \cdot \int_0^H \mu(Z) dZ \qquad (14)$$

In the solution of Westergaard for the vertical pressure face, it was assumed that the area of the plot of the attached mass of water in the conditions of the problem is determined by the expression:

$$\int_0^H \mu(Z) dZ = \mu_{eq} H \qquad (15)$$

where $\mu_{eq} = 0,544$ is the value of the coefficient of the attached mass of water obtained by the total area of the plot of the attached mass of water according to the accepted calculation scheme of Westergaard.

If in expression (15) we assume that $z = \frac{Z}{H}$ is the relative coordinate of the node on the pressure face ($0 \le z \le h$), and replace the integration variable, then the integral for the function of the coefficient of the attached mass (15) can be rewritten as:

$$\int_0^H \mu(Z) dZ = H \cdot \int_0^h \mu(z) dz = \mu_{eq} H \qquad (16)$$

To determine the coefficient of the attached mass of water μ_{eq} in the works of I. S. Sheinin [3], graphs and tables are given depending on the problem statement, their calculation schemes, and the conditions of movement of the structure. For the calculation schemes considered in this paper with different angles of inclination of the pressure face, the coefficient μ_{eq} can be determined from the plot in the form:

$$\mu_{eq} = \int_0^h \mu(z) dz \qquad (17)$$

Using the concept of the coefficient of attached mass, the value of the attached mass of water for the entire structure can be determined, considering it as an average mass of water evenly distributed over the pressure face of the structure, attributed to the unit area of the pressure face. Based on (14) and (15), for the attached mass of water, we can write:

$$m_{eq} = \frac{\rho \mu_{eq} H^2 \cdot 1 r.m.}{H \cdot 1 r.m.} = \rho \mu_{eq} H \qquad [\frac{kg}{m^2}] \qquad (18)$$

Accordingly, the hydrodynamic force of water pressure, which falls on the entire area of the pressure face of the dam (horizontal component), can be defined as:

$$P_{\Sigma} = \rho H \ddot{U}_0 \int_0^H \mu(Z) dZ = \rho H^2 \ddot{U}_0 \mu_{экв} \qquad (19)$$

Since the distribution of the attached mass of water over the height of the pressure face is uneven, the hydrodynamic pressure acting in the k-element of the pressure face of the dam can be written as:

$$\begin{aligned} p_k &= \rho H \ddot{U}_0 \cdot \mu_k(z) \\ p_k &= \gamma H \cdot \mu_k(z) \cdot A_0 \end{aligned} \qquad (20)$$

where is the ordinate on the plot of coefficients of the attached mass of water in the k-element of the pressure face of the dam, located at a relative height $z = \frac{Z}{H}$ from the bottom of the reservoir.

In the modern regulatory literature, the definition of hydrodynamic pressure is differentiated for different elements by the

height of the pressure face using the table for the coefficient of attached mass $\mu = f(\frac{y}{H})$, where it determines the depth of the calculated element under the water level. However, there are no such reference data for structures of hydraulic structures with an inclined pressure face, which led to the need for research in this area.

To determine the coefficient of attached mass μ_k, it is necessary to solve the problem of the distribution of hydrodynamic pressure over the height of the pressure face of the dam. Taking the seismic acceleration of the base, the water density, and the reduced reservoir depth as unit values, we can conclude that the plot of the distribution of the coefficient of the attached water mass over the height of the pressure face will be identical to the plot of the hydrodynamic water pressure, which is determined by the expression:

$$\mu_k(Z) = \frac{p_k(Z)}{\gamma^* h^* A_0^*} \qquad (21)$$

Where A_0^* is the unit seismic acceleration of the base in fractions of g, γ^* is the unit volume weight of water, h^* is the relative depth of the reservoir, reduced to unity in the calculation scheme ($h = 1$).

The character of the distribution of the hydrodynamic pressure function in the calculated area with an inclined pressure face of the dam is determined by integrating the Laplace equation (9).

RESULTS AND DISCUSSING

For the numerical solution of (11), we used the apparatus of numerical methods in the MATLAB software package [18,19]. Design schemes for dams with different angles of inclination of the pressure face are constructed in relative coordinates XOZ, in which the height of the pressure face of the dam is taken as $h = 1$. In relative coordinates, the ordinate of an element on the pressure face z_k is defined as

$$z_k = \frac{Z}{H}$$

The boundary conditions for solving the problem of the distribution of the potential function in the computational domain "dam-reservoir" are set by the interface of the MATLAB software package on each of the sections of the figure ABCD, for the segments AB, BC, CD, DA (Figure 2). According to [MATLAB], two types of boundary conditions are given at the boundaries of the domain: Dirichlet conditions and Neumann conditions:

- the Dirichlet condition (or a condition of the 1st kind) is a method for setting a boundary condition when the value of the desired function is set explicitly, for example;
- the Neumann condition (or condition of the 2nd kind) is a method for setting a boundary condition when the desired function is given by its derivative, for example, $\frac{\partial V(x)}{\partial n} = 0$ which indicates that the equipotentials of the desired function are orthogonal to the selected boundary.

In the MATLAB PC interface, when solving elliptic equations, the boundary conditions for a potential function are set using the equation:

$$r(x) = V(x) \cdot a \qquad (22)$$

where a is a variable that can take any valid values, including a null value.

$r(x)$ - a function defined analytically by an equation.

At the same time, the potential function can take any values.

Within the scope of this task, the following conditions will be met at the boundaries of the calculation scheme (Figure 2):

1. AB – reservoir bed, set for all of them

$$a = 1;\ r(x) = 1$$

what creates the conditions for a potential function:

$$V(x,z)\big|_{z=0} = 1 \text{ (Dirichlet condition)}$$

2. BC – conditional watertight boundary of the reservoir, which is a current line where the condition can be set:

$$\frac{\partial V}{\partial n} = 0 \text{ (Neumann condition)}$$

3. CD is the free surface of the reservoir, which is a line of equal pressure, which determines the orthogonality of the function of streamlines at this boundary:

$$\frac{\partial V}{\partial n} = 0 \text{ (Neumann condition)}$$

4. DA is the pressure face of the dam. The pressure at each point of the boundary is defined as a function of the height and angle α between the pressure face and the vertical.

$$a = 1; r(x) = \frac{1}{h \cdot tg(\alpha)} x$$

what defines a potential function:

$$V(x) = r(x) \text{ (Dirichlet condition)}$$

5. To exclude the influence of the length of the reservoir on the solution of the Laplace equation, the following condition was fulfilled in the calculations:

$$L_{res} \geq 3h \qquad (23)$$

Numerical solutions of the Laplace equation for the potential function $\psi(x,z)$ in design schemes with different angles of inclination of the pressure face, in the range of vertical angles from 0° to 75° degrees, are presented in the form of a family of equipotentials for some design schemes in Figure 3, where the blue equipotentials are a function of current lines , the red equipotentials are a function of the hydrodynamic pressure level.

The family of equipotentials representing a solution for the current line distribution function allows, in compliance with the orthogonality principle, to construct a hydrodynamic grid with equipotentials corresponding to the distribution law of the hydrodynamic pressure function in the calculated domain (Figure 4). The plots of the hydrodynamic pressure distribution along the inclined pressure face of the dam are given in the form of function values $P_0 = \varphi(x,z)$, which allows them to be considered as plots of the coefficients of the attached water mass according to (17).

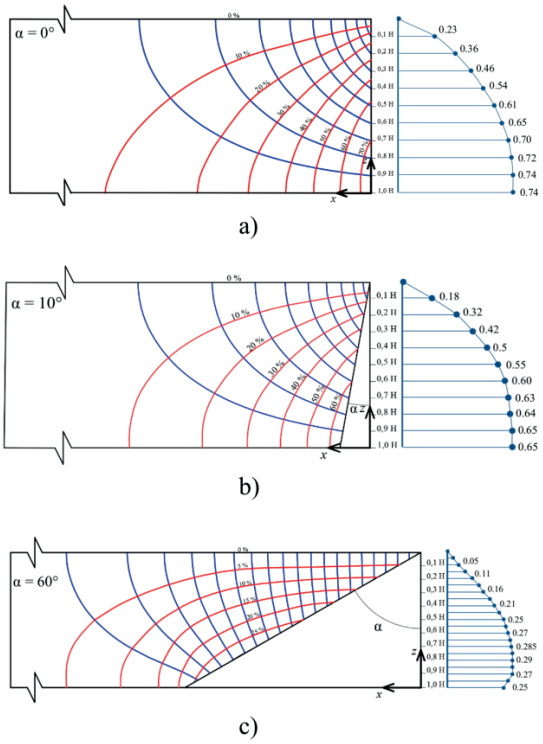


Figure 3. Hydrodynamic grid and plot for the coefficient of the attached mass of water at the angle of inclination of the pressure face: a) $\alpha = 0°$ b) $\alpha = 10°$ c) $\alpha = 60°$

Comparison of the obtained results for the coefficients of the added mass of water was carried out in 3 directions:

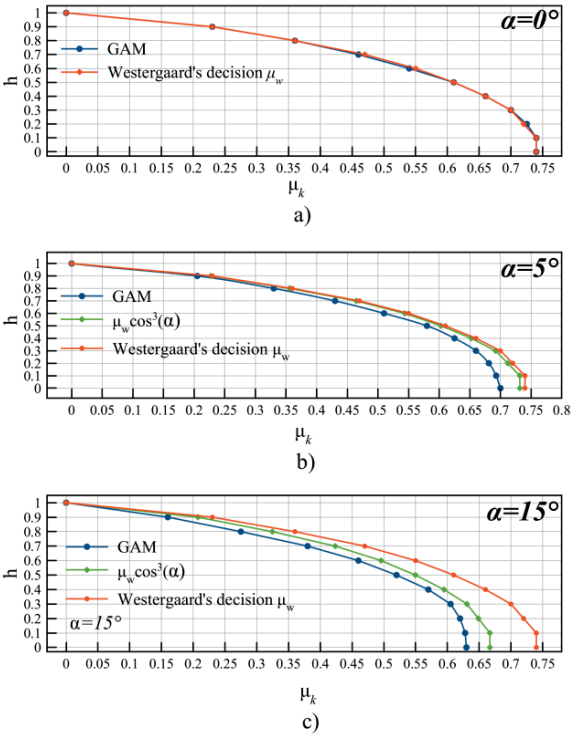
1. Comparison of the obtained functions $\mu(\alpha, z)$ with the Westergaard solution for the vertical pressure face (Figure 4a).
2. Comparison of the obtained functions $\mu(\alpha, z)$ with the recommendations of the SP for taking into account the angle of inclination of the pressure face (15° do) in the form:

$$\mu = \mu_w \cos^3(\alpha) \qquad (24)$$

where α is the angle of inclination of the pressure face relative to the vertical, deg.

μ_w - coefficient of the attached mass of water for the vertical face according to the solution of Westergaard.

3. Comparison of the obtained functions $\mu(\alpha, z)$ with the results of the Zangar experiment (Figure 4e)



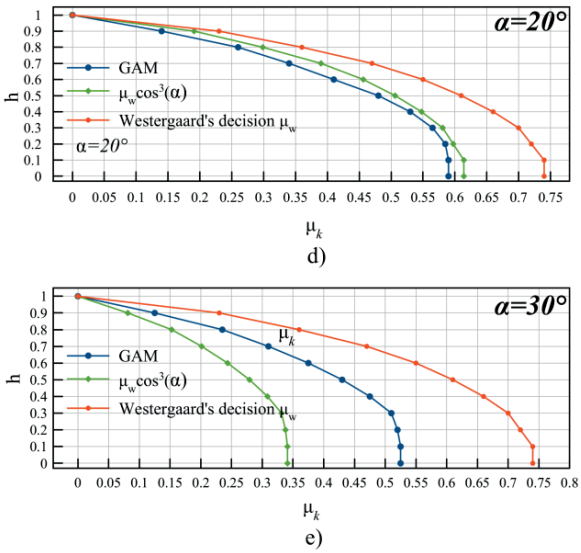


Figure 4. Comparison of results:
a) with the decision of Westergaard,
b) - e) with the recommendations of the SP for the corresponding angles

As can be seen from the comparison, for the angle, there is a high accuracy of matching the analytical solution of Westergaard. At the same time, the difference between the received decision and the JV recommendation reaches 10%, and for, obviously, the recommendations are no longer applicable.

The results of computational studies on the values of the coefficient of attached water mass obtained on the basis of constructed hydrodynamic grids for different angles of inclination of the pressure face of the dam, in the range, are summarized in a nomogram (Figure 5).

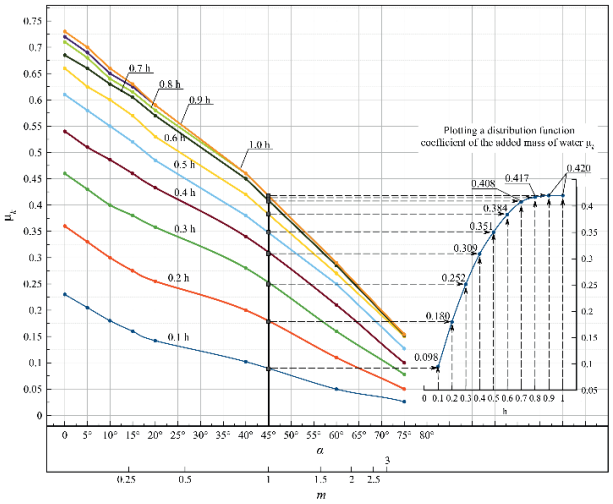


Figure 5. Nomogram

However, for convenient use in calculations, these curves are best represented as mathematical functions that depend on the corresponding arguments. Using the approximation method, we represent each experimental dependence $\mu = f(\alpha)$ as a linear function of the form:

$$\mu(z,\alpha) = a_{zi}\alpha + b_{zi} \tag{25}$$

The approximation functions are found by the least squares method using the Levenberg-Marquardt algorithm [20-22].

Further, representing families of angular coefficients $a_{zi}(z)$ and free terms $b_{zi}(z)$ as square parabolas:

$$\begin{aligned} a_{zi} &= f_\alpha(z) = m_1 z^2 + n_1 z + k_1 \\ b_{zi} &= f_b(z) = m_2 z^2 + n_2 z + k_2 \end{aligned} \tag{26}$$

you can express expression (25) in the following form:

$$\begin{aligned} \mu(z,\alpha) = (0.0073z^2 - 0.0132z - 0.0015)\alpha \\ -0.8448z^2 + 1.4894z + 0.0747 \end{aligned} \tag{27}$$

The value of the coefficient of the attached mass of water in the k-element of the pressure face of the dam can be calculated using expression (26) when $z = z_k$:

$$z_k = \frac{Z_k}{H} \tag{28}$$

where Z_k is the absolute height mark of the k-element, H is the absolute depth of the reservoir.

In accordance with the obtained approximation solution, expression (27) gives the values of the coefficients of the attached mass of water μ only over the interval $z_k = 0{,}1...1$. The function $\mu(z,\alpha)$ on the interval $z_k = 0...0{,}1$ can be approximated in a similar way:

$$\begin{cases} \mu(0,\alpha) \equiv 0 \\ \mu(0.1,\alpha) = a\alpha + b \end{cases} \tag{29}$$

In general, the function for finding the coefficient of the attached mass of water as a function of the relative depth of the element and the vertical angle of the pressure face is the following piecewise linear function with variable coefficients:

$$\mu(z,\alpha) = \begin{cases} 0,\ z \in \{0\} \\ -0.0026\alpha + 0.2103,\ z \in (0,0.1) \\ (0.0073z^2 - 0.0132z - 0.0015)\alpha - \\ -0.8448z^2 + 1.4894z + 0.0747, \\ z \in [0.1,1] \end{cases} \tag{30}$$

The equation of the values of the coefficients of the attached mass of water obtained using the approximation function (30) with the author's nomogram data and the results of experiments [14] for some angles α is shown in Figure 6.

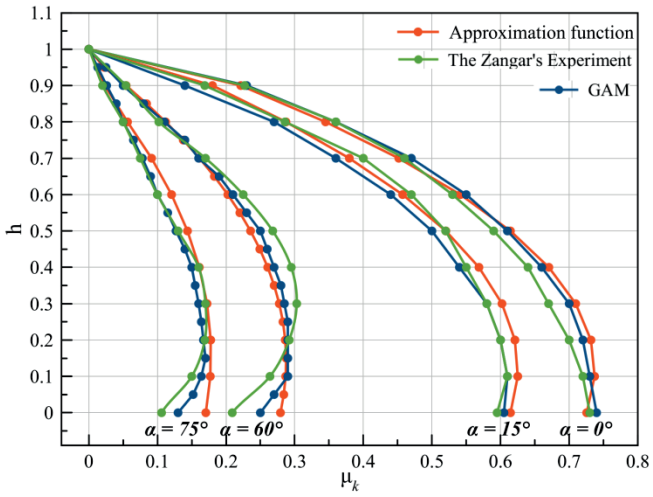


Figure 6. Comparison of results

The hydrodynamic pressure force according to (19) can be calculated by the formula:

$$P_{\Sigma} = \gamma_0 H^2 A_0 \int_0^h \mu(z,\alpha)dz \qquad (31)$$

where-the maximum relative acceleration of the base, - the volume weight of water, h- the relative depth of the reservoir, $\mu(z,\alpha)$– the function of the coefficient of the attached mass of water, constructed for the relative height of the pressure face z, varying from 0 to h

Calculation of hydrodynamic forces based on the obtained solutions for functions $\mu(z,\alpha)$ at different angles of inclination of the pressure face of the dam made it possible to plot changes in the hydrodynamic force $P_{hd} = f(\alpha)$. For this purpose, the integrals $\int_0^h \mu(z,\alpha)$ of the obtained plots $\mu(z,\alpha)$ were calculated. By analogy with (17), the obtained integrals determine the value of the coefficient of the equivalent attached mass of water for a dam with an inclined pressure face, and all other things being equal, the solutions of the problem depend only on the angle of inclination of the face.

For the resulting function, we introduce the following notation:

$$M(\alpha) = \int_0^h \mu(z,\alpha)dz \qquad (32)$$

The function graph $M(\alpha)$ is constructed based on the results of graphoanalytical plotting of the attached mass coefficients (Figure 7) and the approximating function constructed on their basis $\mu(z,\alpha)$.

The integral of the approximating function (30) can be obtained analytically in the form:

$$M(\alpha) = \int_0^h \mu(z,\alpha)dz = \int_0^{0.1} (-0.0026\alpha + 0.2103)dz + \int_{0.1}^1 ((0.0073z^2 - 0.0132z - 0.0015)\alpha - 0.8448z^2 + 1.4894z + 0.0747)dz$$

Performing direct integration, we get:

$$M(\alpha) = (0.021 - 0.00026\alpha) + (0.523 - 0.00545\alpha)$$
$$M(\alpha) = 0.544 - 0.00571\alpha \qquad (32)$$

Traditionally, it is interesting to represent the hydrodynamic pressure force as a fraction of the hydrostatic pressure force W (per 1 nm):

$$W = \frac{\gamma_0 H}{2}$$

Taking into account (30) and (31), we can write:

$$P_{\Sigma}(\alpha) = 2M(\alpha)WA_0 \qquad (33)$$

The obtained expression shows that for a strong earthquake ($A_0 = 0.4g$ for a 9-point earthquake) and a vertical pressure face ($M(0°) = 0.544$ - Westergaard's solution), the seismic pressure of

water on the pressure face of the dam will be $P_{\sum}(0°) = 0.435W$. When the pressure face is laid 1:1 ($\alpha = 45°$), which can occur for dams with reinforced concrete screens, the seismic water pressure will be $P_{\sum}(45°) = 0.23W$ almost a quarter of the hydrostatic pressure. The results obtained indicate a significant seismic pressure of water on the pressure face of the dam, including the slope profile.

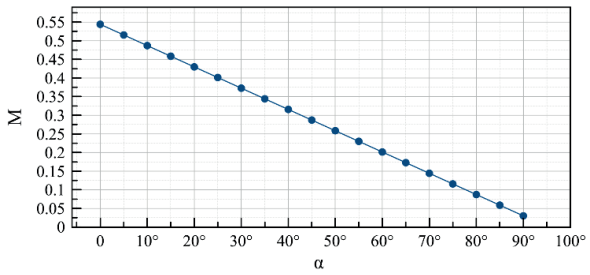


Figure 7. Graph of changes in hydrodynamic pressure from the slope of the pressure face

The obtained quadratic polynomials for the cefficients of the function depend on the height position of the element on the pressure face (23). Therefore, the number of elements forming the hydrodynamic pressure plot will affect the accuracy of plotting the coefficients of the attached water mass and the magnitude of seismic loads.

To assess the effect of the density of the FEM design grid on the hydrodynamic pressure force, calculations were performed with different numbers of elements on the pressure face. A model of a 100-meter dam with a 1:2 upper slope (vertical angle $\alpha = 63.43°$) was considered in 8 variants of breakdown: with 5, 10...70 elements on the pressure face.

As a result of analytical integration of the function of the coefficient of the added mass of water (32), an expression was obtained that can be taken as an exact solution:

$$M(63.43°) = 0.544 - 0.00571 \cdot 63.43 = 0.182$$

In each of the 8 calculation variants, for the center of gravity of the element located on the pressure face, the values of the coefficients of the attached mass of water were determined, according to the approximating function $\mu(z, \alpha)$, and a plot was constructed from them. Determining the area of the plot (using Autocad tools) allowed us to compare the obtained values for the $\alpha = 63.43°$ following parameters. The results of the comparison are shown at Figure 8.

The obtained results allow us to conclude that when forming the FEM grid, it is necessary to provide for at least 20 elements in height on the pressure face of the dam. When dividing the pressure face into 20÷30 elements, the error will be less than 5% of the value obtained by direct integration, which is a satisfactory result for engineering calculations.

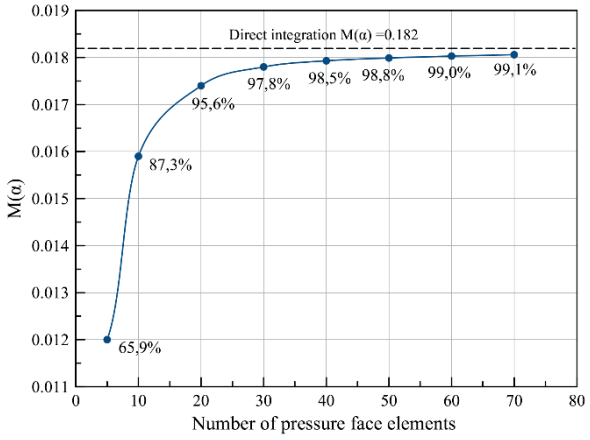


Figure 8. Comparison of numerical and direct integration results

In this problem, a method is developed for determining the hydrodynamic water pressure for a dam with an inclined pressure face, as a reaction to the horizontal component of dam vibrations during an earthquake. Taking into account

the horizontal hydrodynamic pressure of water in the attached mass of water involved in the horizontal vibrations of the dam, it is important to assess this effect in the magnitude of seismic forces that occur in dams during an earthquake, at different angles of inclination of the pressure face of the dam.

The linear-spectral method (hereinafter referred to as LSM), regulated by SP for determining seismic loads in hydraulic structures, involves solving the problem of natural dam vibrations and using calculation forms in determining seismic forces. When solving the problem of plane deformation for a momentless system, there are two components in the components of displacements according to the forms of natural vibrations-horizontal and vertical displacements, which will participate in the formation of both components of seismic forces – both horizontal $S_{k,x}$ and vertical $S_{k,z}$, while the attached mass of water affects only the inertial properties of the dam in the horizontal direction.

The study determined the seismic forces that occur in the dam elements in the horizontal direction, taking into account the horizontal hydrodynamic water pressure from the upstream side and the horizontal direction of propagation of the seismic wave. The considered problem statement determines the greatest seismic loads in the dam along the horizontal component. Studies were carried out for different dam structures: concrete, ground, combined, having different outlines of the pressure face.

The author's program "Seismic_Force_2D" was used to calculate and visualize the distribution of seismic forces in the dam [17]. As part of the calculation, the FEM grid is set, the properties of materials in the calculated area are calculated, the forms of natural vibrations are calculated, and seismic forces and accelerations are determined within the LSM with the calculated earthquake score.

Seismic forces in the dam elements were determined according to the linear-spectral method, taking into account the forms of natural dam vibrations by the formula:

$$S_{k,x} = k_f k_2 Q_{k\mu} \frac{A_0}{g} \sqrt{\sum_{i=1}^{12} ((k_\psi \beta)_i \eta_{ikx}))^2} \quad (34)$$

where k_f, k_2, k_ψ, β - coefficients are determined in accordance with SP, η_{ikx} - coefficient of the oscillation form in the k-element of the dam, according to the i-form of natural vibrations, in the horizontal direction:

$$\eta_{ikx} = \Gamma_{i\beta} x_{ik,x} \quad (35)$$

$$\Gamma_{i\beta} = \frac{\sum_{k=1}^{K} (Q_{k\mu} x_{ik} \cos\beta + Q_k z_{ik} \sin\beta)}{\sum_{k=1}^{K} (Q_{k\mu} x_{ik}^2 + Q_k z_{ik}^2)} \quad (36)$$

where $\Gamma_{i\beta}$ is the coefficient of participation of the i-form of vibrations in the oscillatory process (free vibrations) in the direction of propagation of the seismic wave, β is the angle of approach of the seismic wave with respect to the x-axis.

The change in the participation coefficient $\Gamma_{i\beta}$ when taking into account the attached mass of water for the model of the dam considered earlier is shown in Figure 9. As can be seen from the figure, for waveforms other than horizontal, in the horizontal direction, the coefficient $\Gamma_{i\beta}$ value is close to 0. Thus, there are not so many forms contributing to the seismic vibrations of the dam along the direction of wave propagation, they include 1,3,5 FGC, and the coefficient value of 5,6.9 FGC is not more than 10% $\Gamma_{i\beta}$ of 1 form, at high frequencies (more than 3.5 Hz) significant fluctuations of the dam in the direction of the seismic wave it is not observed, there are no significant fluctuations of the dam in the direction of the seismic wave, and the effect of the attached mass of water on the coefficient $\Gamma_{i\beta}$ for higher forms is minimal.

The weight of elements on the pressure face of the dam taking into account the attached weight of water, is determined by: $Q_{k\mu} = Q_k + Q_\mu$,

where Q_k - the weight of the element without taking into account the attached mass of water;

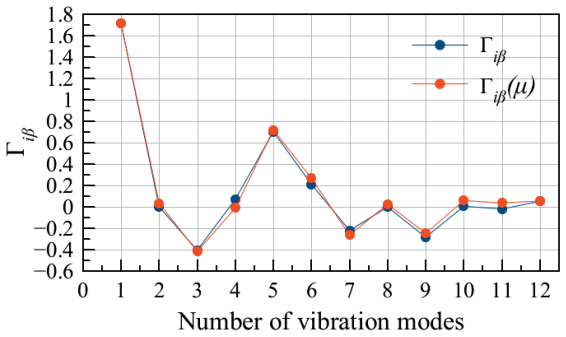


Figure 9. Effect of accounting for the added mass of water on the coefficient $\Gamma_{i\beta}$

x_{ik} and z_{ik} -normalized values of displacements of the k-nodes of the calculation scheme according to the i-form of natural vibrations (hereinafter referred to as FGC) in the horizontal and vertical directions, respectively.

The weight of the attached mass of water on the section of the pressure face of the dam height is determined as follows:

$$Q_\mu = \mu_k(z_k, \alpha_k)\gamma_w H \psi \Delta h_k \qquad (37)$$

where γ_w is the specific gravity of water, H is the depth of the reservoir, ψ and is a coefficient that takes into account the influence of the reservoir length.

The coefficient of the attached mass of water $\mu_k(z_k, \alpha_k)$ is determined by the formula (29), taking into account the change in angle α_k for each element separately.

The study examined three 100-meter-high dams located on the same base, experiencing seismic effects of the same intensity with an estimated base acceleration of 0.4g. The following dams are considered: a gravity concrete dam with a vertical pressure face, a ground dam with a 1:2 pressure face and an asphalt-concrete diaphragm, a combined dam with a concrete pedestal in the lower part of the pressure face and a reinforced concrete screen on the slope with a 1:1.5 foundation (Figure 10). For the correctness of the study, the number of elements on the pressure face in all calculated variants was assumed to be the same (n=30).

For each of the dam variants, according to the obtained function $\mu_k(z_k, \alpha_k)$, plots of the distribution of the coefficients of the attached mass along the pressure face were constructed (Figure 11).

For the design structures of dams, seismic accelerations were calculated using the linear-spectral method [23], taking into account the attached mass of water and without it. For a clear comparison of the effect of the added mass of water on acceleration, the set of coefficients is assumed to be the same.

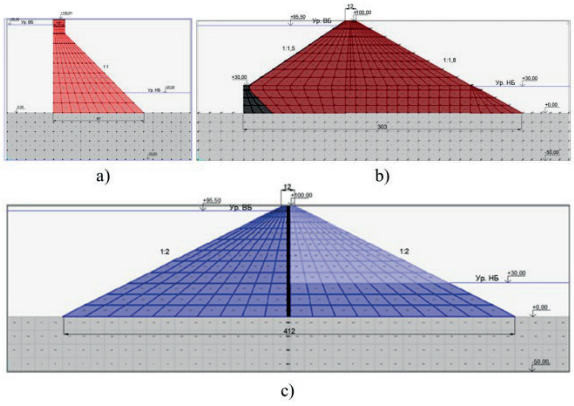


Figure 10. Structures of the considered dams (FEM grids).
a) concrete gravity dam
b) combined dam
c) ground dam

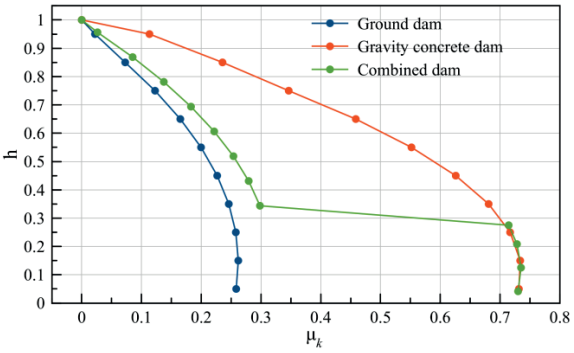


Figure 11. Plots of the distribution of the coefficient of attached water mass

As can be seen in Figure 12, the seismic accelerations of the ground and combined dams are comparable and do not exceed 0.3g in a 9-point earthquake. Accelerations on the crest of a concrete gravity dam are almost 0.7g.

Taking into account the added mass of water increases the acceleration of the elements of the pressure face of the dam over the entire height of the dam, reaching a difference of up to 10 % on the ridge. For a vertical pressure face, taking into account the added mass of water leads to an increase in acceleration on the ridge by almost 30%.

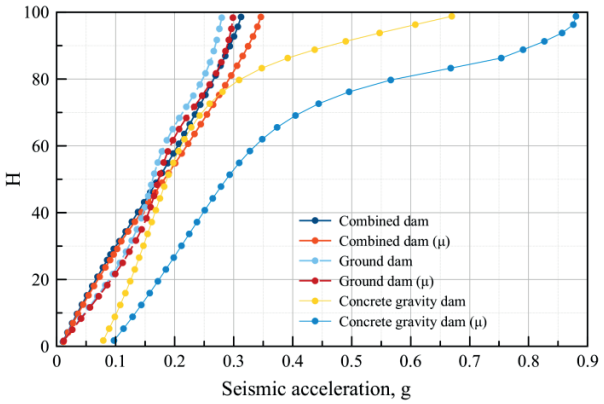


Figure 12. Seismic accelerations on the dam crest

CONCLUSIONS

1. A graph-analytical method for plotting plots of the attached mass of water for water structures with an inclined pressure face are developed for all possible angles of inclination in the range $\alpha = 0°...75°$, including when the slope is variable in height and when there are berms at ground dams.
2. The results of calculations are compared with the results of experimental studies, as well as with the well-known results of domestic and Western scientists, which showed good accuracy of the results.
3. For practical use, a nomogram is constructed with calculated values of the coefficients of the attached water mass, which allows determining the additional seismic pressure of the reservoir water on the structure during an earthquake.
4. An analysis of the accuracy of plotting epigrams of the attached water mass for different density of FEM grids is performed, based on which it is recommended that the maximum size of the element on the pressure face (in height) should not exceed 5% of the height of the dam.
5. The effect of the attached water mass on seismic accelerations for various dam designs is analyzed. The effect of the added mass of water increases the seismic acceleration at the crest of dams from 10% for ground dams to 30% for concrete dams with a vertical pressure face, which must be taken into account when designing hydraulic structures in seismically active areas.

ACKNOWLEDGEMENTS

The research was founded by the National Research Moscow State University of Civil Engineering (grant for fundamental and applied scientific research, project No. 34-392/130)

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Bestuzheva Alexandra Stanislavovna, Associate Professor, Candidate of Technical Sciences, Associate Professor of the Institute of Hydrotechnical and Power Engineering (IGES) of the National Research Moscow State University of Civil Engineering (NRU MGSU) 129337, Moscow, Yaroslavskoye sh., 26, phone +7 (495) 287-49-14, email: alex_bestu@mail.ru

Gotsiridze Georgy Paatovich, postgraduate student of the Institute of Hydrotechnical and Power Engineering (IGES) of the National Research Moscow State University of Civil Engineering (NRU MGSU) 129337, Moscow, Yaroslavskoye sh., 26, phone +7 (495) 287-49-14, email: gia19952@gmail.com

Гоциридзе Георгий Паатович, аспирант кафедры «Гидравлики и Гидротехнического строительства (ИГЭС) Национального исследовательского Московского государственного строительного университета (НИУ МГСУ) 129337, г. Москва, Ярославское ш., д. 26, тел. +7 (495) 287-49-14, email: gia19952@gmail.com

Бестужева Александра Станиславовна, доцент, кандидат технических наук, доцент кафедры «Гидравлики и Гидротехнического строительства (ИГЭС) Национального исследовательского Московского государственного строительного университета (НИУ МГСУ) 129337, г. Москва, Ярославское ш., д. 26, тел. +7 (495) 287-49-14, email: alex_bestu@mail.ru