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ANALYSIS OF VIBRATIONS OF AN ELASTIC PLATE ON A VISCOELASTIC BASE VIA THE FRACTIONAL DERIVATIVE KELVIN-VOIGT MODEL

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Abstract: In the present paper, free and forced vibrations of an elastic Kirchhoff-Love plate on a viscoelastic foundation of the Fuss-Winkler type are studied, the damping properties of which are described using the Kelvin-Voigt model with a fractional derivative. The integral Laplace transform method with further expansion of the sought functions into a series of eigenfunctions of the problem is used as the method for solving the problem of non-stationary vibrations of linear elastic plates on a viscoelastic foundation. The solution is obtained as the sum of two terms, one of which controls the drift of the equilibrium position of the system and is determined by the quasi-static creep processes occurring in the system, and the other term describes the damping of oscillations around the equilibrium position and is determined by the inertia and energy dissipation of the system.

Keywords: free and forced vibrations of an elastic plate, viscoelastic Fuss-Winkler-type foundation, Kelvin-Voigt model with fractional derivative, Riemann-Liouville fractional derivative, Laplace integral transform

АНАЛИЗ КОЛЕБАНИЙ УПРУГОЙ ПЛАСТИНКИ НА ВЯЗКОУПРУГОМ ОСНОВАНИИ С ПОМОЩЬЮ МОДЕЛИ КЕЛЬВИНА-ФОЙГТА С ДРОБНОЙ ПРОИЗВОДНОЙ

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Аннотация: Исследованы свободные и вынужденные колебания упругой пластины Кирхгофа-Лява на вязкоупругом основании типа Фусса-Винклера, демпфирующие свойства которого описываются при помощи модели Кельвина-Фойгта с дробной производной. В качестве метода решения использован метод интегрального преобразования Лапласа с дальнейшим разложением искомых функций в ряд по собственным функциям задачи. Решение получено в виде суммы двух слагаемых, одно из которых управляет дрейфом положения равновесия системы и определяется квазистатическими процессами ползучести, происходящими в системе, а другое слагаемое описывает затухание колебания вокруг положения равновесия и определяется инерцией и диссипацией энергии системы.

Ключевые слова: свободные и вынужденные колебания упругой пластинки, вязкоупругое основание типа Фусса-Винклера, модель Кельвина-Фойгта с дробной производной, дробная производная Римана-Лиувилля, интегральное преобразование Лапласа

INTRODUCTION

The analysis of plate vibrations on a viscoelastic base is an important area of structural mechanics, since plates are used as structural elements in the construction of buildings and structures. The effect of the base response on the dynamic behavior of the slabs is of common interest to structural engineers.

At present, there is an increased interest in the analysis of plates and beams resting on a viscoelastic base under the influence of various types of dynamic loads [1]. Viscoelastic base models of the Fuss-Winkler type with fractional derivatives have become widespread [2], since fractional calculus plays an important role in dynamic problems of structural mechanics [3-6].

Modeling the vibrations of elastic plates on a viscoelastic base is an important and urgent task not only from a theoretical point of view. Due to the wide application of elastic plates in various fields of science and technology, including mechanics, construction and acoustics, the study and modeling of vibrations of elastic plates on a viscoelastic base, damping features of which are described by the fractional derivative Kelvin-Voigt model [7,8], is of great practical importance, since it makes it possible to adequately describe the dynamic reaction of the "plate + viscoelastic base" system with the fewest parameters. At the same time, the order of the fractional derivative is an additional structural parameter, the variation of which from zero to one corresponds to the change in the level of viscoelasticity and allows for a more flexible description and analysis of the hereditary behavior of systems with long-term memory.

Thus, the study of the dynamics of plates on a viscoelastic base using models with fractional derivatives is a significant area in mechanics of materials and structural mechanics, covering both theoretical and practical aspects. In recent years, researchers have focused on the use of fractional derivatives in models of viscoelastic properties of different structural elements, what makes it possible to describe the complex behavior of plates under loads more correctly [3-5, 9].

For example, Rossikhin and Shitikova [10] were among the first to study the dynamic response of an elastic rectangular plate oscillating in a viscous medium, the damping properties of which are described by the Kelvin-Voigt model with fractional derivatives. Vibrations of the plate were described by three linear differential equations with respect to in-plane and out-of-plane displacements of the plate. Laplace method of integral transformations was used as a method of solution.

The analysis of the dynamic behavior of viscoelastic thin Kirchhoff-Love plates under impact was carried out for circular and rectangular plates, respectively, in [11] and [12,13]. A modified Hertzian law was used, in which the modulus of elasticity was replaced by a viscoelastic operator of the fractional order.

Quasi-static analysis of an elastic rectangular plate lying on a viscoelastic Winkler base, the properties of which are described by the Kelvin-Voigt model or a standard linear rigid body with fractional derivatives, was carried out by the Laplace transform method in [14] and [15], respectively.

Free and forced linear oscillations of plates on a viscoelastic Winkler base under the action of a moving harmonic force and a harmonic force whose position does not change over time are considered in [16]. The damping properties of the foundation were described by a standard linear solid model with fractional derivatives. Numerical studies of the amplitude-frequency characteristics of the system were carried out. The results demonstrate how the oscillation parameters change under different types of loads and how the viscoelastic properties of the foundation affect the dynamic behavior of the system.

In this paper, the forced linear oscillations of the system "elastic plate + viscoelastic base of the Fuss-Winkler type" are investigated, the damping properties of which are described by the fractional derivative Kelvin–Voigt model, under the influence of a concentrated harmonic load, using the approach proposed in [10]. It is based on the assumption that each mode of oscillation possesses its own damping coefficient and retardation time, what makes it possible to obtain a solution in a form convenient for engineering applications.

PROBLEM FORMULATION

Let us consider the oscillations of a linearly elastic rectangular plate with sides a and b along the x- and y-axes, respectively, lying on the base (Fig. 1).

The dynamic behavior of a Kirchhoff-Love plate on a viscoelastic base under the action of an external transverse load $q = q(x, y, t)$ is described by the following differential equation:

$$D\nabla^2\nabla^2 w + \rho h\ddot{w} = q - F_1, \qquad (1)$$

where $w = w(x, y, t)$ is the transverse displacement (deflection of the plate), $D = Eh^3/12(1-\nu^2)$ is the cylindrical stiffness of the plate, E is the Young's modulus of the plate, h is its thickness, ρ is the density of the plate material, $\nabla^2\nabla^2$ is the biharmonic operator, $q = F\delta(x - x_0)(y - y_0)\cos\Omega_F t$ is the external harmonic load, F and Ω_F is its amplitude and frequency, $\delta(x - x_0)(y - y_0)$ is the Dirac delta function, and F_1 is the reaction of the viscoelastic base.

Let us assume that damping properties of the viscoelastic base are described by the Fuss-Winkler model:

$$F_1 = \tilde{\lambda} w, \qquad (2)$$

where $\tilde{\lambda}$ is the compliance operator of the viscoelastic base, the damping properties of which are modeled using the Kelvin-Voigt operator of the fractional order [2]:

$$\tilde{\lambda} = \lambda_0(1 + \tau^\gamma D^\gamma), \qquad (3)$$

and λ_0 is the coefficient of compliance of the base, τ is the retardation time,

$$D^\gamma u = \frac{d}{dt}\int_0^t \frac{u(t-t')dt'}{\Gamma(1-\gamma)t'^\gamma}, \quad 0 < \gamma \le 1 \qquad (4)$$

- fractional derivative of Riemann-Liouville [17], Γ(1-γ) is a gamma function, γ is the order of a fractional derivative, which is a parameter of the fractionality of the underlying viscoelastic medium.

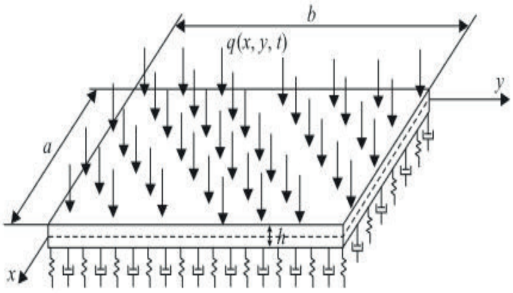

Figure 1. Elastic plate on a viscoelastic base

The boundary conditions for a plate hinged along the contour are as follows:

$$x = 0 \text{ and } a, \quad w = \frac{\partial^2 w}{\partial x^2} = 0$$

$$y = 0 \text{ and } b, \quad w = \frac{\partial^2 w}{\partial y^2} = 0.$$

Let us represent the dynamic deflection of the plate as an eigenfunction expansion of the problem: $W_{mn}(x, y)$:

$$w(x, y, t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} W_{mn}(x, y)x_{mn}(t), \qquad (5)$$

where $x_{mn}(t)$ are generalized displacements that depend only on time, and

$$W_{mn}(x, y) = \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b},$$

are eigenfunctions that satisfy boundary conditions.

When deriving differential equations, we will take into account the filtering property of the delta function:

$$\iint \delta(x - x_0)(y - y_0)\, f(x, y)dxdy = f(x_0, y_0) \qquad (6)$$

Substituting the proposed solution (5) into the system of equations (1) and (2), taking the relation (3) into account, multiplying each of the obtained equations by $W_{mn}(x, y)$, and integrating the length and width of the plate with due account for the orthogonality properties of eigenfunctions, we obtain the following system of differential equations with respect to generalized displacements $x_{mn}(t)$:

$$\ddot{x}_{mn} + \Omega_{mn}^2 x_{mn} + \frac{\lambda_0}{\rho h}(1 + \tau^\gamma D^\gamma)x_{mn} = P_{mn}(t) \qquad (7)$$

where $\Omega_{mn}^2 = \frac{Eh^2}{12\rho(1-\nu^2)}\nabla^2 W_{mn}(x, y)$ are the squares of the natural frequencies of oscillations of the elastic plate, and

$$P_{mn}(t)=\frac{\int_0^a\int_0^b q(x,y,t)W_{mn}(x,y)dxdy}{\rho h\int_0^a\int_0^b[W_{mn}(x,y)]^2dxdy}=\frac{\int_0^a\int_0^b(F\cos\Omega_F t)\delta(x-x_0)(y-y_0)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}dxdy}{\rho h\int_0^a\int_0^b\sin^2\frac{m\pi x}{a}\sin^2\frac{n\pi y}{b}dxdy}=$$
$$=\frac{4F\cos\Omega_F t\sin\frac{m\pi x_0}{a}\sin\frac{n\pi y_0}{b}}{\rho hab}.$$

Let us introduce dimensionless values:

$x^*_{mn}=\frac{(ab)^{1/2}}{h^2}x_{mn},\ t^*=\frac{\pi^2 h}{ab}\sqrt{\frac{E}{12\rho(1-\nu^2)}}\,t,\quad x^*=\frac{x}{a},\ y^*=\frac{y}{b}.$

Equation (7) could then be rewritten in a dimensionless form as

$$\ddot{x}^*_{mn}+{\omega^*}^2_{mn}x^*_{mn}+{\lambda_0}^*\tau^\gamma{}_{mn}\ D^\gamma x^*_{mn}-$$
$$-4F^*\sin\pi m{x_0}^*\sin\pi n{y_0}^*\cos{\Omega_F}^*t^*=0,\qquad(8)$$

where is $\Omega^*_{mn}=\frac{\zeta^2m^2+n^2}{\zeta},\ \zeta=\frac{b}{a},$

${\lambda_0}^*=\frac{12a^2b^2(1-\nu^2)}{h^3\pi^4E}\lambda_0,\ F^*=\frac{12(ab)^{\frac{3}{2}}(1-\nu^2)}{E\cdot h^5\pi^4}F,$

${\Omega_F}^*=\frac{ab}{\pi^2h}\sqrt{\frac{12\rho(1-\nu^2)}{E}}\Omega_F,\ {\omega^*}^2_{mn}={\Omega^*}^2_{mn}+$
$+{\lambda_0}^*$ are squares of the dimensionless natural frequencies of oscillations of the "plate + base" system.

Equations (8) for each fixed pair of values of m and n in the Laplace domain are reduced to equation (9) (here and below * for dimensionless quantities are omitted for ease of presentation):

$$p^2\overline{x}_{mn}+\omega^2_{mn}\overline{x}_{mn}+\lambda_0p^\gamma\tau^\gamma\overline{x}_{mn}=\overline{P}_{mn},\qquad(9)$$

where p is the Laplace variable, $\overline{P}_{mn}=\beta\frac{p}{p^2+\Omega_F}$, $\beta=4F\sin\pi m x_0\sin\pi n y_0$.

Equation (9) coincides with an accuracy of coefficients with the equation of forced oscillations of a single-mass mechanical system in a viscoelastic medium, the damping properties of which are described by the fractional derivative Kelvin-Voigt model. The solution of a similar equation for the case of free oscillations of the Kelvin-Voigt oscillator was obtained in [18]. Using the solution method proposed in [18] to equation (9) and then applying the inverse Laplace transformation, it is possible to find the dynamic deflection of the plate in the form of the expansion (5).

METHOD OF SOLUTION

Analysis of the characteristic equation

The characteristic equation for the system of equations (8) has the form:

$$f_{mn}(p)=p^2+\lambda_0\tau^\gamma{}_{mn}\,p^\gamma+\omega^2_{mn}.\qquad(10)$$

Let us introduce the substitution of variables in the characteristic equation (10) using the formulas:

$$p=p^*\,\omega_{mn},\quad \tau_{mn}=\tau^*\,\omega^{-1}{}_{mn}.$$

As a result, we will get

$$f^*(p^*)=p^{*2}+\frac{\lambda_0}{\omega^2_{mn}}p^{*\gamma}\tau^{*\gamma}+1=0\,,\qquad(11)$$

whence at $\tau^*=0$ it follows that $p_0=\pm i$.

Substituting $p^*=r^*e^{i\psi^*}$ in equation (11) and separating the real and imaginary parts yield

$$r^{*2}\cos2\psi^*+\frac{\lambda_0}{\omega^2_{mn}}\tau^{*\gamma}r^{*\gamma}\cos\gamma\psi^*+1=0,$$
$$r^{*2}\sin2\psi^*+\frac{\lambda_0}{\omega^2_{mn}}\tau^{*\gamma}r^{*\gamma}\sin\gamma\psi^*=0.\qquad(12)$$

To calculate the roots of the system of equations (12) at $\frac{\pi}{2}<\psi^*<\pi$, following [18], let us introduce new variables $x_1=r^{*2}$ and $x_2=\frac{\lambda_0}{\omega^2_{mn}}\tau^{*\gamma}r^{*\gamma}$. Then, for each fixed angle $\frac{\pi}{2}<\psi^*<\pi\gamma$ at given λ_0 and ω^2_{mn}, we arrive at a set of two linear equations with two unknowns x_1 and x_2:

$$x_1\cos2\psi^*+x_2\cos\gamma\psi^*+1=0,$$
$$x_1\sin2\psi^*+x_2\sin\gamma\psi^*=0,\qquad(13)$$

the solution of which has the form

$$x_1 = \frac{\sin\gamma\psi^*}{\sin(2-\gamma)\,\psi^*},$$
$$x_2 = -\frac{\sin 2\psi^*}{\sin(2-\gamma)\,\psi^*}. \quad (14)$$

This allows one to calculate the values $r^* = {x_1}^{1/2}$, $\frac{\lambda_0}{\omega_{mn}^2}\tau^{*\gamma} = x_2\, r^{*-\gamma}$, which together with one selected value of ψ^*, governs the root of the characteristic equation. Substituting $-\psi^*$ instead of ψ^* results in its complex-conjugate root.

Thus, equation (11) for each fixed value $\frac{\lambda_0}{\omega_{mn}^2}\tau^{*\gamma}$ has two complex-conjugate roots $p^* = -\delta^* \pm \omega^*$ in the plane with the negative real semi-axis cut out $-\pi < \psi^* \leq \pi$.

The behavior of roots in the complex plane $-\pi < \psi^* \leq \frac{\pi}{2}$ as function of the parameter $\tau^{*\gamma}$ at $\omega_{mn}^2 = 1$ and given λ_0 is shown in Figure 2 (dashed lines indicate asymptotes for selected γ). Figure 2 shows that when the parameter $\tau^{*\gamma}$ changes from 0 to ∞, the root curves p^* leave the points $\pm i$ and grow monotonically and unlimitedly, while rapidly approaching the corresponding asymptotes coming out of the origin at the angle $\psi^* = \frac{\pi}{2-\gamma}$.

Having determined the roots of the basic equation (11), we could find all roots of the characteristic equation (10) using the formulas:

$$p_{mn} = \omega_{mn}\, r^* e^{i\psi^*}, \quad \tau_{mn} = \tau^*\, \omega^{-1}{}_{mn} \quad (15)$$

The values of the roots of the characteristic equation (10) are presented in Table 1 at $\gamma = 0.25$, $\zeta = 1$, $1 \leq m, n \leq 6$, for the initial retardation time $\tau^* = 1$, which corresponds to the main roots $p^* = -0.4013 \pm 2.2368i$.

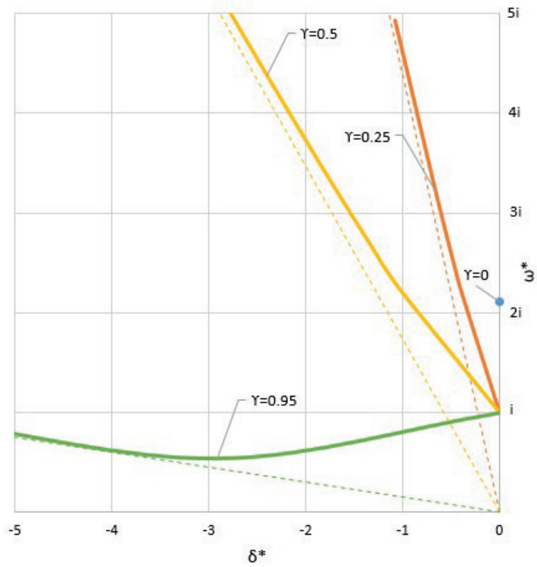

Figure 2. Behavior of the roots of the basic equation (11) of the plate+base system based on model (3)

Construction of the solution

Let us reduce equation (9) to the form:

$$(p^2 + \lambda_0 p^\gamma \tau^\gamma + \omega_{mn}^2)\,\overline{x}_{mn} = \overline{P}_{mn}. \quad (16)$$

or

$$\overline{x}_{mn} = \overline{P}_{mn}\ f^{-1}{}_{mn}(p), \quad (17)$$

where $f_{mn}(p)$ is defined by formula (10).

Equation (17) shows that functions $\overline{x}_{mn}$ on the complex plane are multivalued functions with branch points $p = 0$ и $p = -\infty$, which have poles at values $p = p_k$ that turn the denominator (17) into zero, that is, they are the roots of the characteristic equation (10).

For multivalued functions that have branch points, the Mellin-Fourier inversion formula

$$x_{mn}(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \overline{x}_{mn}(p)\ e^{pt} dp \quad (18)$$

is valid only for the first sheet of the Riemann surface, that is, when $-\pi < \arg p < \pi$. Therefore, the integration path should be selected as shown in Figure 3.

Table 1. Values of the roots of the characteristic equation (10)

m	n	Ω_{mn}	ω_{mn}	τ_{mn}	r_{mn}
1	1	2	2.730	0.366	6.204
1	2	5	5.334	0.187	12.122
2	1	5	5.334	0.187	12.122
2	2	8	8.213	0.122	18.664
1	3	10	10.171	0.098	23.114
3	1	10	10.171	0.098	23.114
2	3	13	13.132	0.076	29.843
3	2	13	13.132	0.076	29.843
3	3	18	18.096	0.055	41.123
1	4	17	17.101	0.058	38.863
4	1	17	17.101	0.058	38.863
2	4	20	20.086	0.050	45.646
4	2	20	20.086	0.050	45.646
3	4	25	25.069	0.040	56.970
4	3	25	25.069	0.040	56.970
4	4	32	32.054	0.031	72.843
1	5	26	26.066	0.038	59.236
5	1	26	26.066	0.038	59.236
2	5	29	29.059	0.034	66.038
5	2	29	29.059	0.034	66.038
3	5	34	34.051	0.029	77.381
5	3	34	34.051	0.029	77.381
4	5	41	41.042	0.024	93.269
5	4	41	41.042	0.024	93.269
5	5	50	50.035	0.020	113.704
1	6	37	37.047	0.027	84.189
6	1	37	37.047	0.027	84.189
2	6	40	40.043	0.025	90.999
6	2	40	40.043	0.025	90.999
3	6	45	45.038	0.022	102.350
6	3	45	45.038	0.022	102.350
4	6	52	52.033	0.019	118.246
6	4	52	52.033	0.019	118.246
5	6	61	61.028	0.016	138.688
6	5	61	61.028	0.016	138.688
6	6	72	72.024	0.014	163.675

According to Jordan's lemma, the curvilinear integral taken by arcs C_R tends to zero at $R \to \infty$, and the integral calculated by c_ρ, also tends to zero at $\rho \to 0$.
Using the main theorem of the residue theory, the solution of equation (18) can be written as

$$x_{mn}(t) = x_{mn}^{drift}(t) + x_{mn}^{vibr}(t), \quad (19)$$

$$x_{mn}^{drift}(t) = = \frac{1}{2\pi i}\int_0^\infty [\bar{x}_{mn}(s\, e^{-i\pi}) - \bar{x}_{mn}(s\, e^{i\pi})] \times \times e^{-st} ds, \quad (20)$$

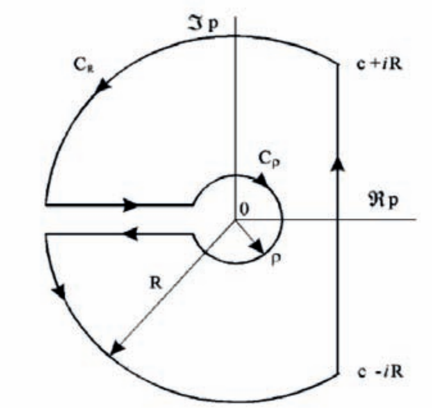

Figure 3. A contour used to compute the complex integral of inversion in the Laplace method

$$x_{mn}^{vibr}(t) = \sum_{k} res[\bar{x}_{mn}(p_k)\, e^{p_k t}], \quad (21)$$

where the summation is carried out over all isolated special points (poles).

In other words, the solution (19) is represented as the sum of two terms, the first of which (20) determines the displacement of the equilibrium position of the system and is determined by the quasi-static creep processes occurring in the system, and the second term (21) describes the damped oscillations around the equilibrium position and is determined by the inertia of the system and the dissipation of energy [10].

Analysis of elastic plate vibrations

A) *in the case of free oscillations*

To obtain a solution in the case of free oscillations of the plate, it is sufficient to put $\bar{P}_{mn} = 0$ for all *m* and *n* in (17).

Knowing the roots of the characteristic equation (10) and substituting the relation (17) in (19) – (21), we get $x_{mn}^{drift}(t)$, $x_{mn}^{vibr}(t)$ and $x_{mn}(t)$:

$$x_{mn}^{drift}(t) = \frac{1}{\pi}\int_0^{\infty} \chi_{mn}(s)\, e^{-st} ds \qquad (22)$$

$$x_{mn}^{vibr}(t) = d_{mn} e^{-\delta_{mn}} \sin(\omega_{mn} t - \varphi_{mn}), (23)$$

where

$$\chi_{mn}(s) = \frac{\lambda_0 \tau_{mn}^{\gamma} s^{\gamma} \sin\gamma\pi}{s^4 + \omega_{mn}^4 + 2s^2\omega_{mn}^2 + {\lambda_0}^2 \tau_{mn}^{2\gamma} s^{2\gamma} + 2\lambda_0 \tau_{mn}^{\gamma} s^{\gamma}(s^2 + \omega_{mn}^2)\cos\gamma\pi},$$

$$d_{mn} = \frac{2}{\sqrt{4r^2{}_{mn} + {\lambda_0}^2 \gamma^2 \tau^{2\gamma}{}_{mn} r_{mn}{}^{2(\gamma-1)} + 4\lambda_0 \gamma \tau^{\gamma}{}_{mn} r^{\gamma}{}_{mn} \cos(2-\gamma)\, \psi_{mn}}},$$

$$\tan\varphi_{mn} = -\frac{2r_{mn}\cos\psi_{mn} + \lambda_0 \gamma \tau^{\gamma}{}_{mn} r_{mn}{}^{\gamma-1} \cos(1-\gamma)\psi_{mn}}{2r_{mn}\sin\psi_{mn} - \lambda_0 \gamma \tau^{\gamma}{}_{mn} r_{mn}{}^{\gamma-1} \sin(1-\gamma)\psi_{mn}}.$$

Then the solution for generalized movements will take the form:

$$x_{mn}(t) = d_{mn} e^{-\delta_{mn} t} \sin(\omega_{mn} t - \varphi_{mn}) + \\ + \frac{1}{\pi}\int_0^{\infty} \chi_{mn}(s)\, e^{-st} ds. \quad (24)$$

B) *in the case of forced oscillations*

To obtain a solution for the forced oscillations of the plate, let us reduce equation (9) to the form:

$$(p^2 + \lambda_0 p^{\gamma} \tau^{\gamma} + \omega_{mn}^2)\bar{x}_{mn} = \\ \frac{p}{p^2 + \Omega_F} 4F \sin\pi m x_0 \sin\pi n y_0 \qquad (25)$$

When an external harmonic load is applied to the plate, the expressions for $x_{mn}^{drift}(t)$, $x_{mn}^{vibr}(t)$ and $x_{mn}(t)$ take the form:

$$x_{mn}^{drift}(t) = \frac{\beta}{\pi}\int_0^{\infty} \frac{\lambda_0 \tau_{mn}^{\gamma} s^{\gamma+1} (\sin(\gamma+1)\pi + \Omega_F \sin(\gamma-1)\pi)}{(s+\Omega_F)^2 \left((s+\omega_{mn}^2)^2 + {\lambda_0}^2 s^{2\gamma} \tau_{mn}^{2\gamma} + 2(s+\omega_{mn}^2)\lambda_0 \tau_{mn}^{\gamma} s^{\gamma} \cos\gamma\pi\right.} e^{-st} ds, \qquad (26)$$

$$x_{mn}^{vibr}(t) = 2d_{mn}\beta e^{-\delta_{mn}t}\sin(\omega_{mn}t - \varphi_{mn}), \tag{27}$$

$$x_{mn}(t) = 2d_{mn}\beta e^{-\delta_{mn}t}\sin(\omega_{mn}t - \varphi_{mn}) + \frac{\beta}{\pi}\int_0^\infty \chi_{mn}(s)\, e^{-st}ds, \tag{28}$$

where $$\tan\varphi_{mn} = -\frac{4{r^4}_{mn}\cos 2\psi_{mn} + 2\gamma\chi {r_{mn}}^{2+\gamma}\cos\gamma\psi_{mn} + 4{r^2}_{mn}\Omega_F + 2\Omega_F\gamma\chi {r^\gamma}_{mn}\cos(2-\gamma)\psi_{mn}}{4{r^4}_{mn}\sin 2\psi_{mn} + 2\gamma\chi {r_{mn}}^{2+\gamma}\sin\gamma\psi_{mn} + 2\Omega_F\gamma\chi {r^\gamma}_{mn}\sin(2-\gamma)\psi_{mn}},$$

$$d_{mn} = \frac{\sqrt{\begin{gathered}4{r^8}_{mn} + \gamma^2\chi^2 {r_{mn}}^{2(\gamma+1)} + 4\gamma\chi {r_{mn}}^{2+\gamma}\left({r^4}_{mn} + {\Omega_F}^2\right)\cos(2-\gamma)\psi_{mn} + \\ +{\Omega_F}^2(4{r^4}_{mn} + \gamma^2\chi^2 {r_{mn}}^{2\gamma}) + \Omega_F(8{r^6}_{mn}\cos 2\psi_{mn} + 2\gamma^2\chi^2 {r_{mn}}^{2(\gamma+1)} \times \\ \times \cos(1-\gamma)2\psi_{mn} + 4\gamma\chi {r_{mn}}^{4+\gamma}\cos\gamma\psi_{mn}\end{gathered}}}{\begin{gathered}({r^4}_{mn} + {\Omega_F}^2 + 2\Omega_F {r^2}_{mn}\cos 2\psi_{mn})(4{r^2}_{mn} + \gamma^2\chi^2 {r_{mn}}^{2(\gamma-1)} + \\ +4\gamma\chi {r_{mn}}^\gamma\cos(2-\gamma)\psi_{mn})\end{gathered}},$$

$$\chi_{mn}(s) = \int_0^\infty \frac{\lambda_0\tau_{mn}^\gamma s^{\gamma+1}(\sin(\gamma+1)\pi + \Omega_F\cdot\sin(\gamma-1)\pi)}{(s+\Omega_F)^2\left((s+\omega_{mn}^2)^2 + {\lambda_0}^2 s^{2\gamma}\tau_{mn}^{2\gamma} + 2(s+\omega_{mn}^2)\lambda_0\tau_{mn}^\gamma s^\gamma\cdot\cos\gamma\pi\right)} e^{-st}ds.$$

Substituting the generalized displacements $x_{mn}(t)$ determined by equations (24) and (28) in relation (5), it is possible to obtain the desired displacements of an elastic plate lying on a viscoelastic base using the Kelvin-Voigt model with a fractional derivative under free oscillations and under the influence of harmonic loading, respectively.

NUMERICAL EXAMPLE

All calculations were performed in the MathCad program for mathematical and engineering calculations.

For numerical calculations, the following characteristics were taken for the plate and the base: $E = 32500$ MPa for concrete B30, $\lambda_0^* = 3.454$ for the base from hard loam, the material of the plate is concrete B30, $\rho = 2430\ \frac{kg}{m^3}$, dimensions in the plan $a = 6$ m, $b = 6$ m, and height $h = 0.2$ m, Poisson's ratio $\nu = 0.25$; $\gamma = 0,\ 0.25,\ 0.5,\ 0.95$.

The external load varies according to the law $q = 10\delta(x - x_0)(y - y_0)\cos 10t$ and is applied in the center of the plate, where $F = 10$, $\Omega_F = 10$, $x_0 = \frac{1}{2}, y_0 = \frac{1}{2}$.

Dynamic displacements $w(x, y, t)$ are shown in Figure 4 for the center of the plate $x = \frac{1}{2}, y = \frac{1}{2}$, considering 9 terms of the series, since for the middle of the plate with even modes, the eigenfunctions $W_{mn}(x, y)$ are zeroed. Figure 5 shows the deflection $w(x, y, t)$ at free vibrations for the point of the plate with coordinates $x = \frac{1}{3}, y = \frac{1}{3}$, with due account for 36 terms of the series.

Displacements $w(x, y, t)$ under force driven vibration s are shown in Figure 6 for the point of the plate with coordinates $x = \frac{1}{2}, y = \frac{1}{2}$ considering 9 terms of the series.

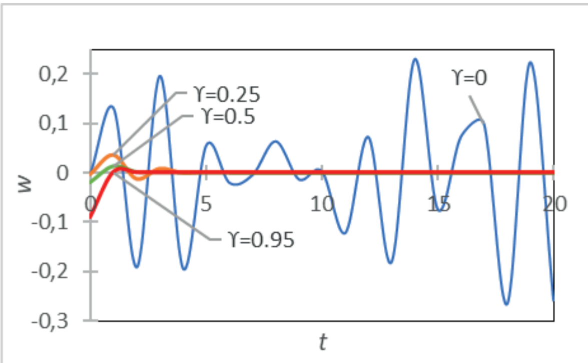

Figure 4. The time dependence of dimensionless displacements at free oscillations for a point of a plate with coordinates $x = \frac{1}{2}, y = \frac{1}{2}$

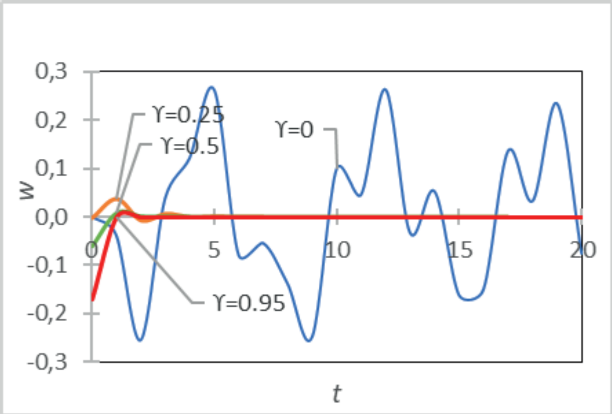

Figure 5. The time dependence of dimensionless displacements for free oscillations for a point of a plate with coordinates $x = \frac{1}{3}, y = \frac{1}{3}$

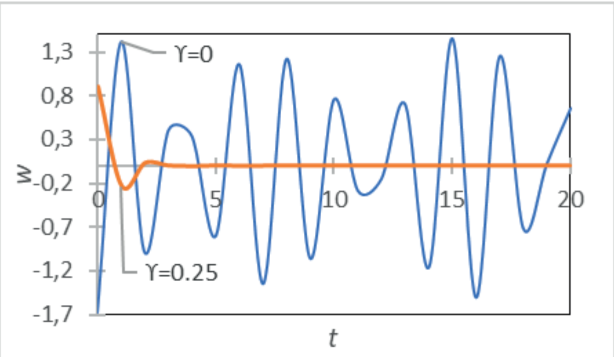

Figure 6. The time dependence of dimensionless displacements for forced oscillations for a point of a plate with coordinates $x = \frac{1}{2}, y = \frac{1}{2}$

CONCLUSION

This article presents an original method for solving the problem of non-stationary oscillations of linear elastic plates on a viscoelastic base, the properties of which are described by fractional derivatives.

The solution algorithm is based on the use of the integral Laplace transform with further decomposition of the desired functions into a series according to the eigenfunctions of the problem. However, in contrast to the traditional approach, when the rationalization of a characteristic equation with fractional powers is carried out during the transition from the frequency domain to the time domain, here the non-rationalized characteristic equation is solved by the method proposed in [10].

As a result of this approach, the solution is obtained as a sum of two terms, one of which controls the drift of the equilibrium position of the system and is determined by quasi-static creep processes occurring in the system, and the other term describes the attenuation of oscillation around the equilibrium position and is determined by the inertia and dissipation of the energy of the system [10, 18].

Modeling the vibrations of an elastic plate lying on a viscoelastic base using the fractional derivative Kelvin-Voigt model allows one to take the viscoelastic characteristics of the base into account and to obtain more accurate results. The findings of the research show that the oscillations depend on the parameters of the model and the foundation, which can be used to optimize the design of various devices and systems.

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