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MODEL OF CAKE FILTRATION IN POROUS MEDIUM

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Abstract: Strengthening of loose soil and creation of water-resistant underground walls are associated with filtration of small particles in a porous medium. Liquid solution pumped into a well under pressure spreads through hollow channels and strengthens the soil upon hardening. Many porous filters retain particles near the entrance. The particles deposited on the filter surface form a crust, which does not allow suspended particles to penetrate deep into the porous medium. A model of cake filtration - the formation of a surface crust during filtration of a monodisperse suspension in a homogeneous porous medium is considered. This model is a modification of the standard mathematical description of deep bed filtration with linear accessible fractional flow decreasing to zero. An exact solution is obtained using the method of characteristics. Asymptotics is constructed for a long time. It is shown that the dynamics and profiles of suspended and deposited particles concentrations exponentially decrease.

Keywords: cake filtration, deep bed filtration, porous medium, exact solution, asymptotics

МОДЕЛЬ ПОВЕРХНОСТНОЙ ФИЛЬТРАЦИИ В ПОРИСТОЙ СРЕДЕ

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Аннотация: Укрепление рыхлого грунта и создание водонепроницаемых подземных стен связаны с фильтрацией мелких частиц в пористой среде. Жидкий раствор, закаченный в скважину под давлением, распространяется по полым каналам и при застывании укрепляет грунт. Многие пористые фильтры задерживают частицы вблизи входа пористой среды. Осажденные частицы на поверхности фильтра образуют корку, которая не пропускает частицы вглубь пористой среды. Рассматривается модель образования поверхностной корки при фильтрации монодисперсной суспензии в однородной пористой среде. Модель является модификацией стандартного математического описания глубинной фильтрации, включающей убывающую до нуля функцию допустимого потока. Методом характеристик найдено точное решение. Построена асимптотика при большом времени. Показано, что динамика и профили концентраций взвешенных и осажденных частиц экспоненциально убывают.

Ключевые слова: поверхностная фильтрация, глубинная фильтрация, пористая среда, точное решение, асимптотика

1. INTRODUCTION

Filtration of liquids with particles is used in construction technologies to strengthen soil during the construction of structures and reconstruction of roads. The injected grout penetrates far into the soil through hollow pores and closes them, strengthening the underground layer and preventing the penetration of groundwater [1-5].

Effective filters used in water wells and in the treatment of industrial and domestic wastewater collect particles on the surface of the filter and purify the water of impurities. The formation of a sediment on the surface of a porous medium, which does not allow particles to pass deeper, is called cake filtration. As a rule, the crust is formed by large particles that cannot penetrate into narrow pores [6-8]. The surface layer of deposited particles is also a porous medium that

can allow small particles to pass through. If small particles spread along the entire length of the porous medium, then cake filtration and deep bed filtration occur simultaneously [9, 10].
Standard deep bed filtration models assume that a porous medium has large pores through which a carrier fluid easily transfers particles from inlet to outlet [11-13]. During layer-by-layer deposition, small particles can block large pores on the surface and limit the transfer of suspended particles deep into the porous medium [14]. In [15] a model of surface deposition is considered, which relates the growth of retained concentration at the inlet with the decreasing velocity of suspended particles. To describe the cake filtration, this article proposes the deep bed filtration model with vanishing accessible fractional flow [16, 17]. The model includes the mass exchange equation of suspended and deposited particles and the kinetic equation of deposit growth with the Langmuir filtration function [18]. The accessible fractional flow is proportional to the linear filtration function and vanishes to zero when the deposit limit is reached. For the new model, an exact solution and asymptotics for large times are obtained.
As a rule, filtration equations do not allow an exact solution and are solved numerically [19-21]. The presence of an exact solution for the cake filtration model under consideration allows us to check the accuracy of numerical methods and adjust the program code.

2. CAKE FILTRATION MODEL

Consider the cake filtration model in an infinite homogeneous porous medium. In the domain $\Omega = \{x \geq 0,\, t \geq 0\}$, the model in a dimensionless form is specified by a system of first-order partial differential equations with unknown volumetric concentrations of suspended and deposited particles $C(x,t),\ S(x,t)$:

$$\frac{\partial C}{\partial t} + \frac{\partial}{\partial x}\left(\left(1 - \frac{S}{S_m}\right)C\right) + \frac{\partial S}{\partial t} = 0, \qquad (1)$$

$$\frac{\partial S}{\partial t} = \lambda\left(1 - \frac{S}{S_m}\right)C \qquad (2)$$

with the conditions

$$x = 0: C = 1, \quad t = 0: C = 0,\ S = 0. \qquad (3)$$

Here $\lambda > 0$ is the filtration coefficient and $S_m > 0$ is the maximum limiting deposit.
According to the condition (3), at the inlet of the porous medium x = 0, equation (2) takes the form

$$\frac{\partial S_0}{\partial t} = \lambda\left(1 - \frac{S_0}{S_m}\right). \qquad (4)$$

Solution to the equation (4) with the condition (3) is

$$S_0(t) = S_m\left(1 - e^{-\lambda t / S_m}\right). \qquad (5)$$

Let's reduce problem (1)-(3) to one equation with unknown S. Substitute the equation (2) into (1)

$$\frac{\partial C}{\partial t} + \frac{1}{\lambda}\frac{\partial}{\partial x}\frac{\partial S}{\partial t} + \frac{\partial S}{\partial t} = 0$$

and change the order of differentiation in the mixed derivative

$$\frac{\partial C}{\partial t} + \frac{1}{\lambda}\frac{\partial}{\partial t}\frac{\partial S}{\partial x} + \frac{\partial S}{\partial t} = 0.$$

Integration over time gives

$$C + \frac{1}{\lambda}\frac{\partial S}{\partial x} + S = K(x)\,. \qquad (6)$$

The integration constant is obtained from the conditions (3): $K(x) = 0$.
Express C from the equation (2):

$$C = \frac{\partial S / \partial t}{\lambda (1 - S / S_m)} \tag{7}$$

and substitute in (6)

$$\frac{\partial S / \partial t}{\lambda (1 - S / S_m)} + \frac{1}{\lambda} \frac{\partial S}{\partial x} + S = 0 .$$

Transform the equation:

$$\frac{\partial S}{\partial t} + \left(1 - \frac{S}{S_m}\right) \frac{\partial S}{\partial x} + \lambda S \left(1 - \frac{S}{S_m}\right) = 0 . \tag{8}$$

The solution S is zero at $t \le x$ and is positive at $t > x$. The boundary $t = x$ is the front of deposit concentration.
To solve equation (8) in the domain $\Omega^+ = \{x > 0,\, t > x\}$, we will use the method of characteristics [22, 23] and make a change of variables: $(x,t) \to (\tau, t_0)$. Here τ is a variable along the characteristic, and t_0 is the initial value on the time axis. The characteristics are given by the equations

$$\dot{t} = 1,\;\; \dot{x} = 1 - \frac{S}{S_m},\;\; t(0) = t_0,\;\; x(0) = 0,$$

where $\dot{(\;)}$ is the derivative with respect to τ. In the characteristic variables, equation (8) takes the form

$$\dot{S} + \lambda S \left(1 - \frac{S}{S_m}\right) = 0,\;\; S(0) = S_0(t_0) . \tag{9}$$

The solution to problem (9) is obtained by the method of separation of variables:

$$S = S_m \frac{e^{\lambda t_0 / S_m} - 1}{e^{\lambda \tau} + e^{\lambda t_0 / S_m} - 1} . \tag{10}$$

Using solution (10), the characteristics are obtained:

$$t = \tau + t_0,\;\; x = \frac{1}{\lambda} \ln \frac{e^{\lambda \tau} + e^{\lambda t_0 / S_m} - 1}{e^{\lambda t_0 / S_m}} . \tag{11}$$

Express τ from the first equation (11) and substitute it into the second one:

$$x = \frac{1}{\lambda} \ln\left(e^{\lambda (t - t_0 (1 + 1/S_m))} - e^{-\lambda t_0 / S_m} + 1 \right).$$

Transform the equation to the form

$$F(z) = e^{\lambda t} z^{1 + S_m} - z + 1 - e^{\lambda x} = 0,\;\; z = e^{-\lambda t_0 / S_m} .$$

In the domain Ω^+ on the interval (0;1) the function $F(z)$ is differentiable and $F''(z) > 0$, at the ends of the interval $F(0) < 0,\, F(1) > 0$. Consequently, there is a single root $z^+(x,t),\; 0 < z^+ < 1$ of the transcendental equation $F(z) = 0$.
Now the characteristic variables τ, t_0 can be expressed in terms of Cartesian *x*, *t*:

$$t_0(x,t) = -\frac{S_m}{\lambda} \ln z^+(x,t),$$
$$\tau(x,t) = t + \frac{S_m}{\lambda} \ln z^+(x,t).$$

In variables *x*, *t,* solution (10) takes the form

$$S = S_m \frac{1 - z^+}{(z^+)^{1 + S_m} e^{\lambda t} - z^+ + 1} . \tag{12}$$

Transform formula (12) using (11)

$$S(x,t) = S_m e^{-\lambda x} (1 - z^+) . \tag{13}$$

The solution $C(x,t)$ is determined by the formulae (7) and (13):

$$C(x,t) = \frac{S_m z^+}{(1 + S_m)(e^{\lambda x} - 1) + S_m z^+} . \tag{14}$$

For large time *t,* the asymptotic expansion of the solution [24] is given by the formulae

$$S(x,t) = S_m e^{-\lambda x} \cdot \\ \cdot\left(1-(e^{\lambda x}-1)^{1/(1+S_m)} e^{-\lambda t/(1+S_m)}\right)+O\left(e^{-2\lambda t/(1+S_m)}\right), \quad (15)$$

$$C(x,t) = \frac{S_m (e^{\lambda x}-1)^{-S_m/(1+S_m)}}{(1+S_m)} e^{-\lambda t/(1+S_m)} + \\ +O\left(e^{-2\lambda t/(1+S_m)}\right). \quad (16)$$

Example 1. Let $S_m = 1$. Then (11) is a quadratic equation with the positive root

$$z^+(x,t) = \frac{1+\sqrt{1+4e^{\lambda t}(e^{\lambda x}-1)}}{2e^{\lambda t}}.$$

Characteristic variables are expressed using the formulae

$$t_0 = \frac{1}{\lambda}\ln\frac{2e^{\lambda t}}{1+\sqrt{1+4e^{\lambda t}(e^{\lambda x}-1)}},$$

$$\tau = t - \frac{1}{\lambda}\ln\frac{2e^{\lambda t}}{1+\sqrt{1+4e^{\lambda t}(e^{\lambda x}-1)}}.$$

Solution (13) and (14) takes the form

$$S = S_m\left(e^{-\lambda x} - \frac{\sqrt{1+4e^{\lambda t}(e^{\lambda x}-1)}+1}{2e^{\lambda t}e^{\lambda x}}\right), \quad (17)$$

$$C = \frac{1}{\sqrt{1+4e^{\lambda t}(e^{\lambda x}-1)}}. \quad (18)$$

On the concentration front $t = x$ the solution is

$$C = \frac{1}{2e^{\lambda x}-1}, \quad S = 0.$$

When $x = const,\ t \to \infty$ the solutions tend to the limits $C \to 0,\ S \to e^{-\lambda x}$. The asymptotics of the solution at $t \to \infty$ has the form

$$C = \frac{e^{-\frac{\lambda t}{2}}}{2\sqrt{e^{\lambda x}-1}} + O\left(e^{-\frac{3\lambda t}{2}}\right),$$

$$S = e^{-\lambda x} - e^{-\frac{\lambda t}{2}}\sqrt{e^{-\lambda x}-e^{-2\lambda x}} + O\left(e^{-\frac{3\lambda t}{2}}\right).$$

3. NUMERICAL RESULTS

Two-dimensional graphs of the solution to the cake filtration problem in a porous medium make it possible to determine the dynamics (at a constant coordinate *x*) and the profiles (at a constant time *t*) [25, 26]. The model is calculated at $\lambda = 1,\ S_m = 1$. Fig. 1 shows the dependence of concentrations on time at various points of the porous medium.

According to Fig. 1. when $t < x$ at the considered point *x* of the porous medium there are no particles and the solution is zero. At the moment $t = x$ the concentration front arrives at point *x* and the concentration of suspended particles *C* changes abruptly. At $t > x$ at point *x* the suspended and retained particles are present and the solution is positive. When moving away from the inlet of the porous medium, i.e. as *x* increases, the concentrations of suspended and deposited particles decrease exponentially.

At a fixed time *t,* non-zero concentrations of suspended and retained particles are located behind the concentration front at $x < t$ (Fig. 2).

Graphs of the limit retention profiles at $t \to \infty$ for various λ are shown in Fig. 3.

In the cake filtration model, retention profiles decrease exponentially and decrease with increasing λ.

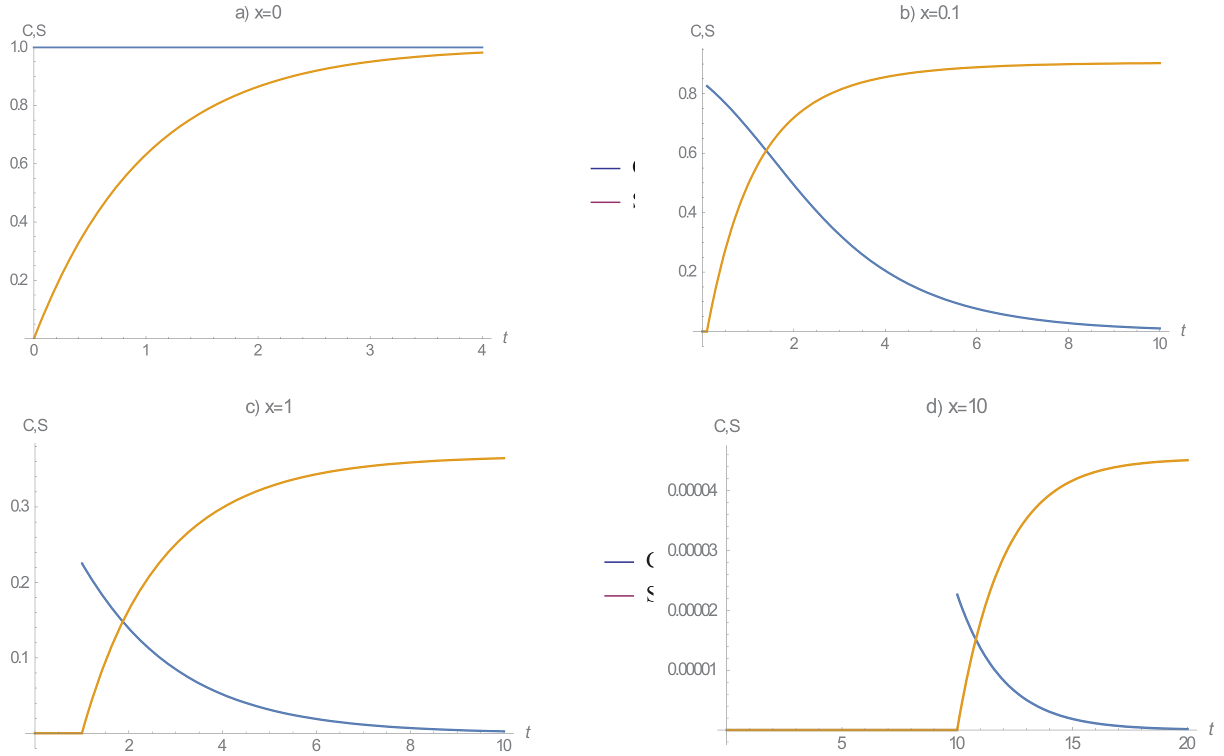

Figure 1. Dynamics of suspended and retained particles concentrations. a) $x=0$ b) $x=0.1$ c) $x=1$ d) $x=10$

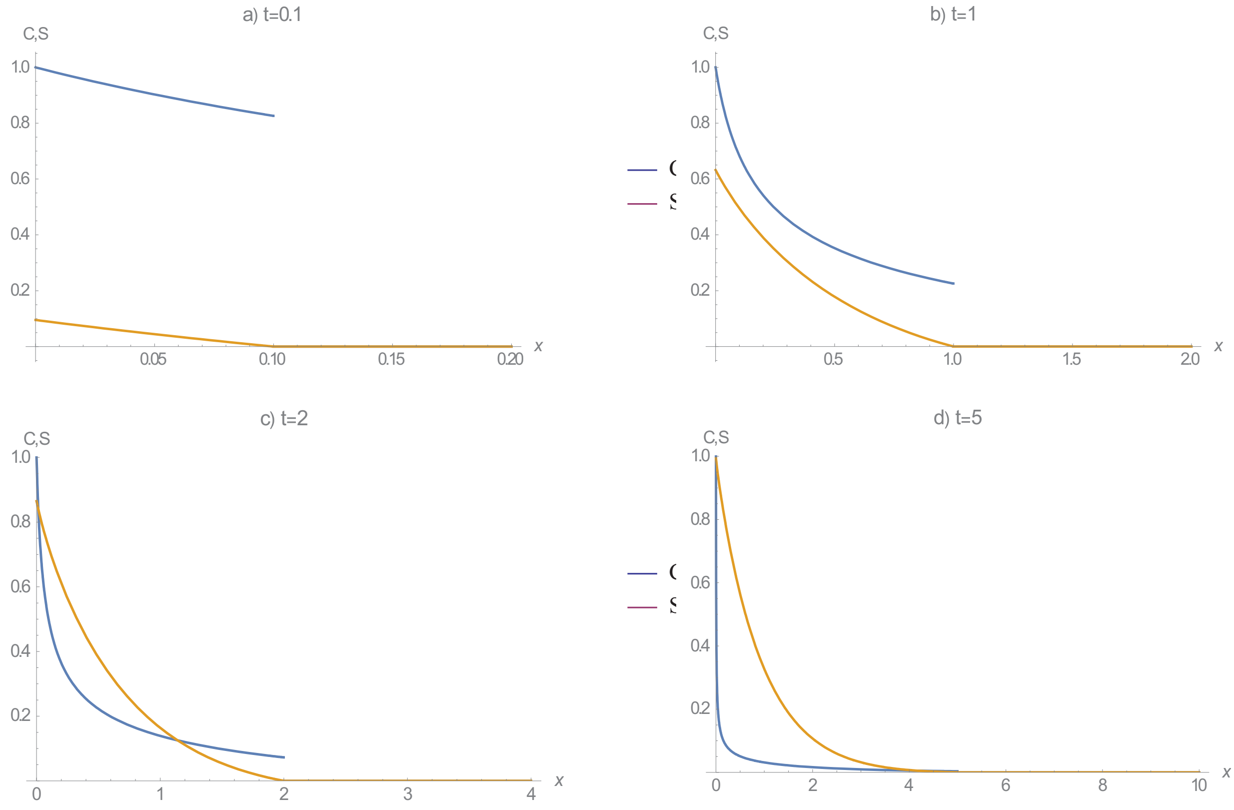

Figure 2. Profiles of suspended and retained particles. a) $t=0.1$ b) $t=1$ c) $t=2$ d) $t=5$.

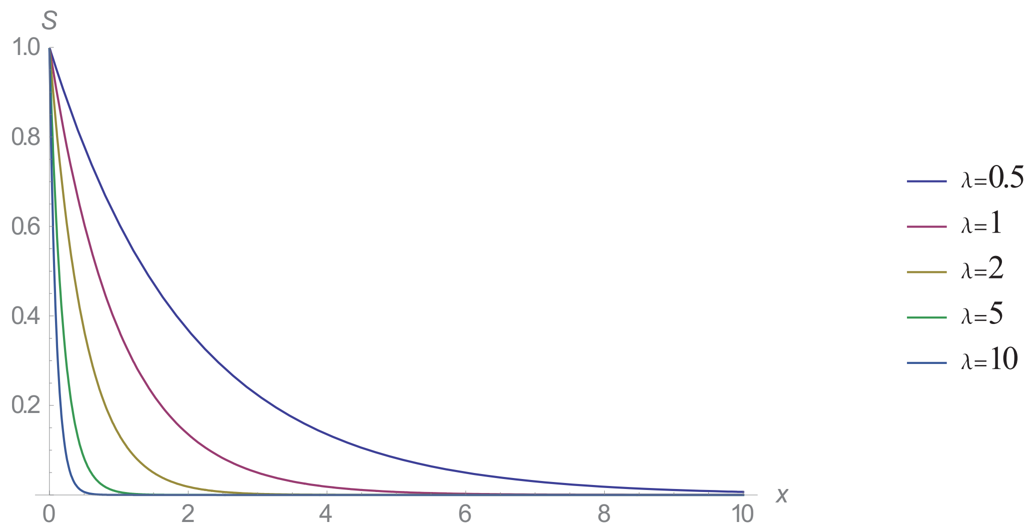

Figure 3. Limit retention profiles at different filtration coefficients.

DISCUSSION

The proposed cake filtration model is a modification of the deep bed filtration model. Thus, both different types of filtration processes are described in a unified way by population balance equations.

At high fluid speeds, trapped particles can be detached from the framework of the porous medium and become suspended again [14, 15, 27, 28]. The model with sediment erosion will be considered separately.

Two free parameters can adapt the model to the results of laboratory experiments [29, 30].

4. CONCLUSIONS

The study of the cake filtration model allows us to draw the following conclusions.

- Cake filtration model is constructed as a modified deep bed filtration model.
- Particles are deposited on the surface and near the inlet of the porous medium.
- The concentration of suspended particles decreases with time and tends to zero.
- An exact solution in explicit form is obtained.
- The asymptotics of the solution at $t \to \infty$ is constructed.

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