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CONTACT PROBLEM FOR A SPHERICAL SHELL SUPPORTED ON TWO FLAT SURFACES

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Abstract: An effective numerical and analytical method is proposed for studying the force state inside the contact spots of elastic composite spherical shells resting on two flat surfaces inclined at an angle to each other. The method is based on constructing the influence function for the shell in an axisymmetric formulation, followed by its expansion into a Fourier series using the Legendre and Gegenbauer polynomials. A resolving system of equations for the corresponding coefficients is obtained. The developed methodology will allow for a more correct study of the dynamics of various robotic systems, the main controlled elements of which are elastic spherical shells that carry out complex kinematic movements on various surfaces. These systems, in particular, include a "butterfly robot", the main controlled element of which is a thin spherical shell, rolling with simultaneous spinning and sliding along two parallel plates, as if on rails.

Keywords: spherical composite shell; contact problem; theory of the multicomponent anisotropic dry friction

КОНТАКТНАЯ ЗАДАЧА ДЛЯ СФЕРИЧЕСКОЙ ОБОЛОЧКИ, ОПИРАЮЩЕЙСЯ НА ДВЕ ПЛОСКИЕ ПОВЕРХНОСТИ

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Резюме: Предложен эффективный численно-аналитический метод исследования силового состояния внутри пятен контакта упругих композитных сферических оболочек, опирающихся на две наклоненные под углом к друг другу плоские поверхности. Метод основан на построении функции влияния для оболочки в осесимметричной постановке, с последующим ее разложением в ряд Фурье по полиномам Лежандра и Гегенбауэра. Получена разрешающая система уравнений для соответствующих коэффициентов. Разработанная методика позволит провести более корректное исследования динамики различных робототехнических систем, основными управляемыми элементами которых являются упругие сферические оболочки, осуществляющие сложные кинематические движения по различным поверхностям. К числу этих систем, в частности, относится «робот-бабочка», основной управляемый элемент которого, тонкая сферическая оболочка, катящаяся с одновременным верчением и проскальзыванием по двум параллельным пластинам, как по рельсам.

Ключевые слова: сферическая композитная оболочка; контактная задача; теория многокомпонентного анизотропного сухого трения

INTRODUCTION

The main goal of the proposed research is to develop the effective numerical and analytical methods for studying the force state inside the contact spots of elastic composite spherical shells supported on two flat surfaces inclined at an angle to each other.

The contact problem for a spherical shell resting on two flat surfaces is generated by the need for a detailed study of the dynamics of various robotic systems in order to construct more effective control laws. The main controlled elements of these systems are elastic spherical shells that carry out complex kinematical movements on various surfaces [1-29].

In the case of non-point contact of rubbing bodies, the dynamics of the shells are significantly influenced by the effects of combined dry friction [1-29]. These effects arise due to the nonlinear relationship of the parameters describing the force state inside the contact spot (components of force and moment of friction) with kinematics parameters. To determine this relationship and calculate the coefficients of the so-called multicomponent friction model, knowledge of the laws of distribution of normal and tangential stresses inside the contact patch is required, taking into account the nature of the movement of the objects under study.

The task becomes much more difficult if the shell moves, contacting two surfaces at the same time, as is the case with the so-called “butterfly robot”. The main controlled element of the “butterfly robot” is a thin spherical shell, rolling with simultaneous spinning and sliding along two parallel plates, as if on rails.

The first developed control algorithms neglected the effects of combined dry friction under conditions of complex kinematics, which leads to their ineffectiveness for certain combinations of kinematical parameters of shell motion. To improve them, approximate models of dry friction inside contact patches were developed, under the assumption that an elastic ball is used as a controlled element, which allows the use of classical Hertz laws of distribution of normal contact stresses. As a result, effective approaches to constructing friction models for such systems were developed, based on simple asymptotic representations for distribution functions inside contact patches.

The used methodology for constructing friction models and determining their coefficients was previously developed to study the complex dynamics of pneumatic tires of various structures. In accordance with this technique, the distribution of normal pressure inside the contact patch is approximated by a linear approximation of the Taylor series expansion in polar or Cartesian coordinates inside the contact patch. In this case, the zero term of the expansion, which makes the main contribution to the calculation of the coefficients of friction models, is a stationary distribution (state of rest). The effectiveness of this approach made it possible to apply it in the case where a composite, spherical shell acts as the main controlled element. At the same time, the calculation of the stationary normal pressure distribution was made under the assumption that the shell is on a horizontal plane.

The presence of contact along two symmetrically located, non-horizontal areas significantly distinguishes the problem under consideration from those studied in previous works [30]. The normal reaction vector is no longer oppositely directed to the gravity vector (the weight of the shell), which, in turn, can lead to a significant change in the shape of the distribution of normal contact stresses inside the contact spots.

STATEMENT OF THE PROBLEM AND THE MAIN RESOLVING INTEGRAL EQUATION

To It is considering the contact problem for a spherical shell of the Timoshenko type [34, 35], supported on two flat solid surfaces intersecting at an opening angle α (Fig. 1). Here R and h are radius and shell thickness. It is assumed that $h << R$.

Let’s assume that the contact occurs under free slip conditions.

During deformation, two contact surfaces Ω_1 и Ω_2 are formed. The dimensions and positions of their boundaries are unknown in advance and must be determined in the process of solving the problem. The purpose of the contact problem is to determine contact pressures p_1 and p_2, distributed respectively over the surfaces Ω_1 and Ω_2.

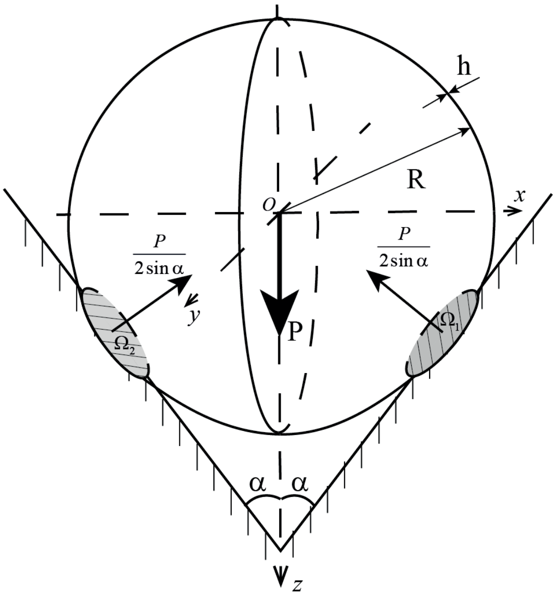

Figure 1. Contact problem

This problem is spatial but with two planes of symmetry: *Oxz* и *Oyz* (рис. 1). Consequently, $p_1 = p_2 = p$, as well as the size and geometry of the areas Ω_1 and Ω_2 are coincide. This fact makes it possible to reduce the dimension of the problem and present it as a superposition of two axisymmetric states with symmetry axes L_1 и L_2 (Fig. 2).

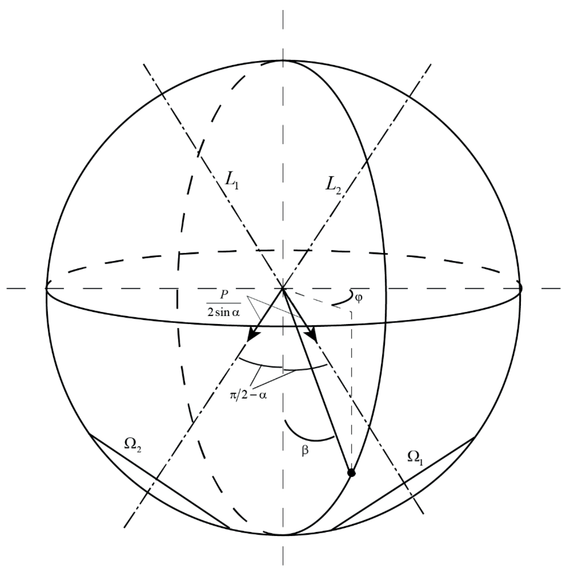

Figure 2. Superposition of axisymmetric states

Let's associate with the shell a spherical coordinate system with angles $\varphi \in [0, 2\pi)$ and $\beta \in [0, \pi]$ (Fig. 2). Further, when writing all equations and relationships, the following system of dimensionless quantities is used (if the quantities are written in the same way, they are indicated by the upper symbol «∗», which is omitted in the subsequent presentation):

$$u^* = \frac{u}{R},\ w^* = \frac{w}{R}, p^* = \frac{p}{\lambda + 2\mu}\frac{R}{h},$$
$$\eta^2 = \frac{\mu}{\lambda + 2\mu},\ \gamma^2 = \frac{h^2}{12R^2},\ k^2 = \frac{5}{6} \tag{1.1}$$

where u, w are the tangential and normal movements; χ is the angle of rotation of the normal to the median surface before the deformation of the fiber due to shear deformations; λ, μ are the Lame's elastic parameters of the shell material, p is the normal pressure. Under these assumptions, the dimensionless radius of the shell is 1.

Additionally, two local spherical coordinate systems are introduced. Their center coincide with the center of the shell, and the axes relative to which the meridional angles are counted coincide with the axes of symmetry L_1 and L_2. Point $M(\beta, \varphi)$ on the surface of a sphere with coordinates β, φ, specified in the main coordinate system is characterized by meridional angles β_1 and β_2 in additional coordinate systems. The same point belongs to the secant planes Π_1 and Π_2 with unit normal vectors $\mathbf{n}_1 = (\cos\alpha, 0, \sin\alpha)^T$ and $\mathbf{n}_2 = (-\cos\alpha, 0, \sin\alpha)^T$ given in rectangular Decaurt coordinates systems *Oxyz* (Fig. 3). Then the connection of the angles $\beta_1 = \beta_1(\beta, \varphi)$ and $\beta_2 = \beta_2(\beta, \varphi)$ with coordinates β, φ are determined from the condition of intersection of the median surface of the shell with the planes Π_1 and Π_2:

$$\begin{aligned}&\left[\cos\left(\alpha-\beta_1\right)-\sin\beta\cos\varphi\right]\cos\alpha+\\&\left[\sin\left(\alpha-\beta_1\right)-\cos\beta\right]\sin\alpha=0,\ \beta_1>0;\\&\left[\cos\left(\alpha+\beta_2\right)-\sin\beta\cos\varphi\right]\cos\alpha+\\&\left[\sin\left(\alpha+\beta_2\right)-\cos\beta\right]\sin\alpha=0,\ \beta_2>0\end{aligned}\tag{1.2}$$

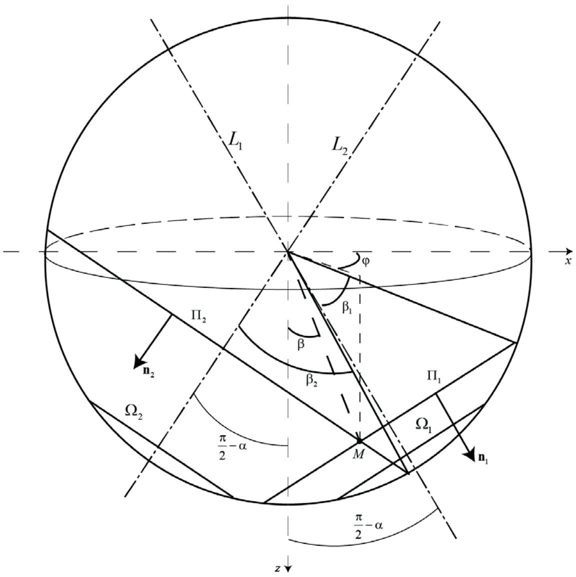

Figure 3. Relation of coordinates of a point on the shell surface

Thus, it is sufficient to construct a solution to the axisymmetric contact problem for a shell resting on a flat surface and being under the influence of a pressing force $Q=\dfrac{P}{2\sin\alpha}$. This solution depends only on the meridional angle β (in the local coordinate systems $\beta=\beta_1$ or $\beta=\beta_2$). Solving the original problem at an arbitrary point of the shell $M\left(\beta,\varphi\right)$ will represent a superposition of two solutions obtained from the solution of an axisymmetric problem in which the meridional angles β should be replaced by angles β_1 and β_2, connected with β and φ by formulas (1.2). In particular:

$$w(\beta,\varphi)=W\left[\beta_1\left(\beta,\varphi\right)\right]+W\left[\beta_2\left(\beta,\varphi\right)\right]\tag{1.3}$$

where $W\left(\beta\right)$ are the normal shell displacements obtained from the solution of the axisymmetric problem.

To solve the axisymmetric problem, we use a system of equations for the shell that takes into account the shear and rotational inertia of the cross sections according to the hypotheses of S.P. Tymoshenko [34, 35]:

$$\begin{gathered}\mathbf{LW}+\mathbf{p}=0,\ \mathbf{L}=\left(L_{ij}\right)_{3\times3},\\\mathbf{W}=\left(U,W,\mathrm{X}\right)^{\mathrm{T}},\ \mathbf{p}=\left(0,p,0\right)^{\mathrm{T}},\end{gathered}\tag{1.4}$$

Coefficients in formulas (1.4) are determined by the following formulas:

$$\begin{aligned}&L_{11}=\frac{\partial^2}{\partial\beta^2}+\mathrm{ctg}\beta\frac{\partial}{\partial\beta}+\eta^2\left(2-k^2\right)-\frac{1}{\sin^2\beta},\\&L_{12}=\left[2\left(1-\eta^2\right)+\eta^2k^2\right]\frac{\partial}{\partial\beta},\\&L_{13}=-\gamma^2\left(\frac{\partial^2}{\partial\beta^2}+\mathrm{ctg}\beta\frac{\partial}{\partial\beta}-\frac{1}{\sin^2\beta}\right)+\eta^2k^2,\\&L_{21}=-\left[2\left(1-\eta^2\right)+\eta^2k^2\right]\left(\frac{\partial}{\partial\beta}+\mathrm{ctg}\beta\right),\\&L_{22}=\eta^2k^2\left(\frac{\partial^2}{\partial\beta^2}+\mathrm{ctg}\beta\frac{\partial}{\partial\beta}\right)-4\left(1-\eta^2\right),\\&L_{23}=\eta^2k^2\left(\frac{\partial}{\partial\beta}+\mathrm{ctg}\beta\right),\ L_{31}=\gamma^{-2}L_{13},\\&L_{32}=-\eta^2k^2\gamma^{-2}\frac{\partial}{\partial\beta},\ L_{33}=-\gamma^{-2}L_{13}.\end{aligned}$$

Based on the principle of superposition [5, 31-36], the following integral representation is valid for normal $W\left(\beta\right)$ displacements of the shell:

$$W\left(\beta\right)=2\pi\int_0^{\pi}G_w\left(\beta,\xi\right)p\left(\xi\right)d\xi\tag{1.5}$$

where $G_w\left(\beta,\xi\right)$ is the influence function for the shell, which is the solution to the problem of the effect of a single concentrated force on the shell [31-36]:

$$\mathbf{LG} + \mathbf{p} = 0,$$
$$\mathbf{G} = \left(G_u, G_w, G_\chi\right)^{\mathrm{T}},\ \mathbf{p} = \left[0, \delta(\beta - \xi), 0\right]^{\mathrm{T}} \quad (1.6)$$

Taking into account the symmetry properties of the problem, it is sufficient to determine the distribution of contact pressure over the Ω_1 domain, since in domain Ω_2 the distribution will be similar.

Using the influence function $G_w(\beta,\xi)$, integral representation (1.5) and formulas (1.2-1.3) it is possible to construct the main resolving integral equation of the contact problem:

$$2\pi \int_0^\pi G(\beta,\varphi,\xi)\, p(\xi)\, d\xi = w(\beta,\varphi),$$
$$G(\beta,\varphi,\xi) = G_w\left[\beta_1(\beta,\varphi),\xi\right] + \qquad (1.7)$$
$$+ G_w\left[\beta_2(\beta,\varphi),\xi\right]$$

Displacement $w(\beta,\varphi)$ at the area of contact Ω_1 are determined by the distance (taking into account the sign) from the point $M(x_0, y_0, z_0)$ at the sphere surface with coorinates: $x_0 = \sin\beta\cos\varphi$, $y_0 = \sin\beta\sin\varphi$, $z_0 = \cos\beta$ to plane Ω_1 (Fig.4): $x\cos\alpha + z\sin\alpha - 1 + w_0 = 0$

$$w(\beta,\varphi) = 1 - \sin\beta\cos\varphi\cos\alpha - \qquad (1.8)$$
$$- \cos\beta\sin\alpha - w_0$$

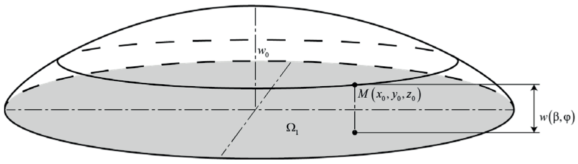

Figure 4. Determination of normal displacements in the contact area

Thus, the main resolving integral equation of the contact problem takes the form:

$$2\pi \int_0^\pi G(\beta,\varphi,\xi)\, p(\xi)\, d\xi = 1 - w_0 -$$
$$- \sin\beta\cos\varphi\cos\alpha - \cos\beta\sin\alpha, \qquad (1.9)$$
$$(\beta,\varphi):\ 1 - \sin\beta\cos\varphi\cos\alpha -$$
$$- \cos\beta\sin\alpha - w_0 \le 0$$

This equation is a one-dimensional integral equation with respect to the contact pressure distribution function $p(\xi)$. To solve it, we need to determine the kernel $G(\beta,\varphi,\xi)$ of this integral equation.

THE INFLUENCE FUNCTION FOR SHELL

The influence function $G_w(\beta,\xi)$ for the shell in the axisymmetric formulation is constructing using Fourier series expansions in Legendre $P_n(\cos\beta)$ and Gegenbauer $-\sin\beta C_{n-1}^{3/2}(\cos\beta)$ polynomials [34]:

$$\begin{Vmatrix} G_u(\beta,\xi) \\ G_\chi(\beta,\xi) \end{Vmatrix} = -\sum_{n=1}^{\infty} \begin{Vmatrix} G_{un}(\xi) \\ G_{\chi n}(\xi) \end{Vmatrix} C_{n-1}^{3/2}(\cos\beta) \times$$
$$\times - \sin\beta,$$
$$\begin{pmatrix} G_w(\beta,\xi) \\ \delta(\beta-\xi) \end{pmatrix} = \sum_{n=0}^{\infty} \begin{pmatrix} G_{wn}(\xi) \\ \delta_n(\xi) \end{pmatrix} P_n(\cos\beta), \qquad (2.1)$$
$$\frac{2\delta_n(\xi)}{2n+1} = \int_0^\pi \delta(\beta-\xi) P_n(\cos\beta)\sin\beta\, d\beta =$$
$$= \frac{2n+1}{2} P_n(\cos\xi)\sin\xi.$$

Substituting (2.1) into (1.6) taking into account the known relations for Legendre and Gegenbauer polynomials [35] leads to a system of equations with respect to the coefficients of the series (2.1):

$$
\begin{gathered}
\mathbf{L}_n\mathbf{G}_n+\mathbf{p}_n=0,\ \mathbf{L}_n=\left(L_{nij}\right)_{3\times3},\\
\mathbf{G}_n=\left(G_{un},G_{wn},G_{\varphi n}\right)^{\mathrm{T}},\\
\mathbf{p}_n=\left[0,\delta_n(\xi),0\right]^{\mathrm{T}},\\
L_{n11}=\eta^2\left(2-k^2\right)-n\left(n+1\right),\\
L_{n12}=2\left(1-\eta^2\right)+\eta^2k^2,\\
L_{n13}=\gamma^2n\left(n+1\right)+\eta^2k^2,\\
L_{n21}=\left[2\left(1-\eta^2\right)+\eta^2k^2\right]n\left(n+1\right),\\
L_{n22}=-\eta^2k^2n\left(n+1\right)-4\left(1-\eta^2\right),\\
L_{n23}=-\eta^2k^2n\left(n+1\right),\ L_{n31}=\gamma^{-2}L_{n13},\\
L_{n32}=-\eta^2k^2\gamma^{-2},\ L_{n33}=-\gamma^{-2}L_{n13}
\end{gathered}
\tag{2.2}
$$

The coefficients of the series for the influence function are finding from the solution of system (2.2):

$$
\begin{gathered}
G_{wn}\left(\xi\right)=\frac{Q_{2n}}{Q_{3n}}\delta_n\left(\xi\right),\\
Q_{2n}=Q_2\left[n\left(n+1\right)\right],\ Q_{3n}=Q_3\left[n\left(n+1\right)\right],\\
Q_2\left(l\right)=\left(\eta^2k^2+\gamma^2l\right)\left[2\eta^2+l\left(\gamma^2-1\right)\right],\\
Q_3\left(m\right)=k^2\eta^2l^3\gamma^2\left(\gamma^2-1\right)+2\gamma^2l^2\times\\
\times\left[\eta^4\left(k^2+2\right)-2\eta^2\left(\gamma^2+1\right)+2\gamma^2\right]+\\
+\eta^2l\left[k^2\eta^4-\left(\left(\gamma^2+1\right)k^2+2\gamma^2\right)4\eta^2+\right.\\
\left.\left(4\gamma^2k^2+8\gamma^2\right)\right]+8\eta^4k^2\left(1-\eta^2\right)
\end{gathered}
\tag{2.3}
$$

Thus, the desired influence function is a series (2.1) with coefficients determined by formula (2.3).

RESOLVING SYSTEM OF EQUATIONS

The expansion of contact pressure $p\left(\xi\right)$ in a series in terms of Legendre polynomials has the form:

$$
p\left(\xi\right)=\sum_{m=0}^{\infty}p_mP_m\left(\cos\xi\right)
\tag{3.1}
$$

Substituting representations (2.1), (2.2) and (2.3) into equation (1.9), taking into account the orthogonality property for Legendre polynomials [31, 34] leads to the equation:

$$
\begin{gathered}
2\pi\sum_{m=0}^{\infty}\frac{Q_{2m}}{Q_{3m}}p_m\left\{P_m\left[\cos\beta_1\left(\beta,\varphi\right)\right]+\right.\\
\left.+P_m\left[\cos\beta_2\left(\beta,\varphi\right)\right]\right\}=1-w_0-\\
-\sin\beta\cos\varphi\cos\alpha-\cos\beta\sin\alpha
\end{gathered}
\tag{3.2}
$$

The solution to equation (3.2) can be obtained by reducing it to a system of algebraic equations for the coefficients p_m.

CONCLUSIONS

An effective numerical and analytical method is proposed for studying the force state inside the contact spots of elastic composite spherical shells supported on two flat surfaces inclined at an angle to each other.

The method is based on constructing the influence function for the shell in an axisymmetric formulation, followed by its expansion into a Fourier series using the Legendre and Gegenbauer polynomials. A resolving system of equations for the corresponding coefficients is obtained.

The developed methodology will allow for a more correct study of the dynamics of various robotic systems, the main controlled elements of which are elastic spherical shells that carry out complex kinematical movements on various surfaces. These systems, in particular, include a “butterfly robot”, the main controlled element of which is a thin spherical shell, rolling with simultaneous spinning and sliding along two parallel plates, as if on rails

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