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RATIONAL DESIGN OF NONLINEAR-DEFORMABLE STRUCTURALLY HETEROGENEOUS ELEMENTS OF STRUCTURES

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Abstract: The article deals with the problem of rational design of a nonlinearly deformable heterogeneous composite Timoshenko rod under force impact. The rod has a structure that is symmetrical relative to the force plane, formed by the connection of quasi-homogeneous parts (phases, layers) with different physical properties, taking any geometric shape in space. The structural materials that form the rod have nonlinear elastic properties. To describe the main component of the stress tensor – the normal stress in the longitudinal direction – in each phase, the same type of approximation by entire rational polynomials is taken depending on the deformation. On their basis, compact nonlinear equations were obtained that connect integral forces with generalized deformations of the axial line of the rod. In this system, the rigidity characteristics of higher exponent are figured as coefficients.
Nonlinear equilibrium conditions written for the case of large displacements and rotation angles in combination with linear kinematic relations are resolved in the form of the initial parameter method.
Based on the strength condition written in the form of a quasi-uniaxial criterion, the designing criterion is formulated for the heterogeneous rod. This criterion is continuous along the longitudinal coordinates and it is discrete along the transverse coordinates. A two-stage algorithm is developed to solve the design problem of the rational design of a nonlinearly deformable layered rod. It makes it possible to identify the geometric functions of the longitudinal profiling of the rod layers presented in a discrete form. Resolution relations were obtained to find the functions of the width and height of the profiled layers.
Numerical results are presented to solve the design problem of calculating a compressed-bent I-section rod, in which the flanges and I-beam webs were made of various materials. The presence of geometric restrictions on variable values from below ensured non-degeneracy of the flanges at the pre-support areas. The three characteristic areas were detected in the rod with the implementation of the calculated continuous criterion in the form of two-, one- and zero-point conditions along the transverse coordinate. It is shown that the consideration of the shear stresses in the rod of this flexibility is not relevant.

Keywords: layered rod, physical nonlinearity, geometric nonlinearity, continuous criterion of strength, rational design

РАЦИОНАЛЬНОЕ ПРОЕКТИРОВАНИЕ НЕЛИНЕЙНО-ДЕФОРМИРУЕМЫХ СТРУКТУРНО-НЕОДНОРОДНЫХ ЭЛЕМЕНТОВ КОНСТРУКЦИЙ

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Аннотация: Рассматривается задача рационального проектирования нелинейно деформируемого неоднородного составного стержня Тимошенко при силовом воздействии. Стержень имеет структуру, симметричную относительно силовой плоскости, образованную соединением различных по физическим свойствам квазиоднородных частей (фаз, слоев), принимающих в пространстве произвольную геометрическую форму. Конструкционные материалы, образующие стержень, обладают нелинейно-упругими свойствами. Для описания основной компоненты тензора напряжений – нормального напряжения про-

дольного направления, в каждой фазе приняты однотипные аппроксимации целыми рациональным полиномами в зависимости от деформации. На их основе получены компактные нелинейные уравнения, связывающие интегральные усилия с обобщенными деформациями осевой линии стержня. В качестве коэффициентов в данной системе фигурируют жесткостные характеристики высших порядков.
Нелинейные условия равновесия, записанные для случая больших перемещений и углов поворота в сочетании с линейными кинематическими соотношениями разрешены в форме метода начальных параметров.
На основе условия прочности, записанного в форме квазиодноосного критерия, для неоднородного стержня сформулирован проектный критерий – непрерывный по продольной и дискретный по поперечной координатам. Разработан двухэтапный алгоритм решения проектной задачи рационального проектирования нелинейно деформируемого слоистого стержня. Он позволяет выявлять представленные в дискретной форме геометрические функции продольного профилирования слоев стержня. Получены разрешающие соотношения для нахождения функций ширины и высоты профилируемых слоев.
Приведены численные результаты решения проектной задачи расчета сжато-изогнутого стержня двутаврового сечения, в котором полки и стенка двутавра были выполнены из различных материалов. Наличие геометрических ограничений на варьируемые величины снизу обеспечивало невырождение полок в приопорных областях. В стержне было выявлено три характерных участка с выполнением расчетного непрерывного критерия в форме двух-, одно- и нульточечного условия по поперечной координате. Показано, что в стержне данной гибкости учет касательных напряжений не является актуальным.

Ключевые слова: слоистый стержень, физическая нелинейность, геометрическая нелинейность, непрерывный критерий прочности, рациональное проектирование

INTRODUCTION

Bearing structures used in construction, as a rule, experience complex thermal power and kinematic effects. The effective consideration of the operational requirements for structures currently dictates the use of new approaches and principles of mathematical modeling. Among them, we highlight the more significant ones, which allow radically increasing the performance of the structures. They are: 1). *The principle of structural heterogeneity* of the designed system, according to which the system is formed of parts (elements, blocks, layers, phases, etc.) made of various materials combined into a single structure. 2). *The principle of rational geometric profiling* of the system elements, as a result of which it acquires a calculated configuration (geometric, physical, structural). The combined use of these two principles allows us to create structural systems with a stress-strain state adapted to external impacts, and, as a result, to ensure regulatory requirements in the best possible way.

A large number of works known in the literature are devoted to the calculation of various heterogeneous structural systems, considering their physical structure (discrete- and dispersive-heterogeneous, reinforced, layered), structural purpose, physical features of the materials deformation, the nature of external impacts [1, 2], the differences in calculation approaches, hypotheses, and research methods. In most works, when solving the direct problems of stress-strain state calculation, the universal finite element method is used [3, 4, 5, 6, 7, 8, 9]. At the same time, analysis methods without the use of discretization procedures are also used to perform research tasks.

For example, [10] discusses the construction of an improved analysis of layered composites using a variation approach, considering piecewise linear displacements and shear effects. In the work [8], a layer-by-layer trigonometric approximation of shear deformations was performed with the satisfaction of the defining physical relations. The proposals to refine the Tymoshenko shear model are proposed in [6, 11]. The methods to homogenize the heterogeneous structures are also widely used [3, 4]. The model of the parametric method of cells, based on the theory of zigzag distributions of displacements by the thickness coordinate, was used in [12] in combination with spline approximations.

Modern materials from which composite systems are made often have nonlinear deformation properties. In order to take into account such effects, the work [13] proposes a universal method for their description and the formation of physical correlations in heterogeneous rods.

The methods of inverse problems - optimal and rational design - are of a particular interest in the structure design. They are rather complex in combination with the above-mentioned principles of mathematical modeling for heterogeneous designs. They are covered extremely insufficiently in the literature. In this direction, there is a number of works devoted to the search for rational geometry of elements in wooden [14] and concrete [15, 16] structures based on the strength criterion. In the works [17, 18, 19] an energy approach was used to assess the efficiency of the configuration and the internal structure of the construction. The possibilities to significantly increase the strength and rigidity of heterogeneous structures are presented in sources [7, 20, 21], and in work [22] an effective method based on a gradient reinforcement scheme is proposed for a hybrid heterogeneous rod.

It should be noted that the studies devoted to the comprehensive consideration of the above mentioned principles of modeling profiled heterogeneous rod elements of the structures are not presented in the literature.

MATERIALS AND METHODS OF THE RESEARCH

1. Mathematical model of a heterogeneous rod

The heterogeneous rod has a structure formed by the connection of quasi-uniform parts (phases) of finite sizes, different in physical properties, which takes a variety of geometric shapes in space (Fig. 1). With the use of certain technological methods, the phases are combined into a single heterogeneous structure. The rod can be a part of the rod system or be an independent structural element.

In the local coordinate system *xyz*, the longitudinal axis of the *s*-phase rod is aligned with the *x* axis, and the structure has symmetry properties relative to the *xy* plane, in which the rod experiences a straight longitudinal-transverse bend. A straight line having a arbitrary reference to the physical structure specified in a specific calculation is taken as the geometric axis of the rod [13].

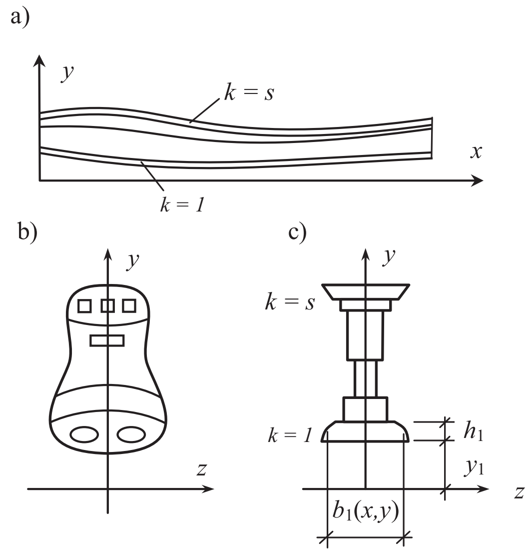

Figure 1. Heterogeneous rod scheme.
a – front projection of the rod in the plane xy,
b, c – variants of the rod cross section

We assume that the following restrictions are met with respect to the physical and geometric parameters of the rod:

1). Rod is thin $(h/l < 1/10)$ and submits to the hypotheses of S.P. Timoshenko. The validity of it is confirmed by numerical analysis [23]. 2). Structural materials that form the phases of the rod have nonlinear elastic properties described under single-axis stress state in the framework of the deformation theory of plasticity by finite dependences of the form $\sigma_k = \sigma_k(\varepsilon)$, $(k \in [1,s])$, which bind the deformation $\varepsilon = \varepsilon_x$ and conditional stress $\sigma_k = \sigma_{x,k}$ in the material of the k-phase under single-axis stress state. The physical parameters of these dependencies for different materials have values of the same order. 3). When forming the calculated dependencies, the deformations are considered small, and the displacements are considered large values, keeping to the relations

$$\varepsilon_x \ll 1,\ v \ll l,\ \sin\theta \approx \theta,\ \cos\theta \approx 1.$$

The dependences for values of longitudinal ε_x and transverse ε_y deformations, shear strain γ_{yx}, longitudinal u and transverse v displacements of the reference axis points, the angles of rotation of cross sections θ we present in the form

$$\begin{gathered} u(x,y)=u_0-\theta\, y, \quad v(x)=v_0, \\ \varepsilon_x(x,y)=\varepsilon_0-\kappa\, y, \quad \varepsilon_y(x,y)=0, \\ \gamma_{xy}(x,y)=\frac{dv_0}{dx}-\theta=\gamma_0(x), \\ \varepsilon_0(x)=\frac{du_0}{dx}, \quad \kappa(x)=\frac{d\theta}{dx}, \end{gathered} \tag{1}$$

where $u_0(x)$, $v_0(x)$ – displacement of the points of the geometric axis in the direction of the axes x, y; $\varepsilon_0(x)$, $\kappa(x)$ – deformation and curvature of the axis of the rod; $\gamma_0(x)$ – shear strain averaged along the y coordinate.

2. Physical dependencies

The expressions for integral force factors and differential equations of equilibrium of the rod are written in the form

$$[N,M,Q](x)=\sum_{k=1}^{s}\iint_{A_k}\left[\sigma_x^{(k)},-\sigma_x^{(k)}y,\tau_{yx}^{(k)}\right]dA, \tag{2}$$

$$\frac{dN}{dx}=-q_x, \quad \frac{dQ}{dx}=q_y, \quad \frac{dM}{dx}=Q+m_z.$$

Here q_x, q_y, m_z – are the distributed forces acting in the direction of the x, y and the moment in the xy plane; A_k – the square of the k-phase in the normal section.

The main component of the stress tensor for the k- phase under force action is represented in the form of decomposition

$$\sigma_x^{(k)}(\varepsilon)=p_{k,0}^{\pm}+p_{k,1}^{\pm}\varepsilon+p_{k,2}^{\pm}\varepsilon^2+\ldots=\sum_{i=0}^{r}p_{k,i}^{\pm}\varepsilon^i, \tag{3}$$

with the specified parameters $p_{k,i}^{+}$, $p_{k,i}^{-}$ for the tensile and compression areas.

The advantage of taking power functions in (3) is that the first three of them with numbers $i=0,1,2,...$ are traditionally used in various theories and have a definite physical meaning. The following allow you to refine the approximations of phase material diagrams. As special cases, the calculation model (3) describes: rigid plastic, linearly elastic, including - multimodule deformation.

When holding in (3) linear and quadratic terms, the power dependence of F.I. Gerstner, $\sigma=E\varepsilon+p\varepsilon^2$, used in [14] for the calculation of wooden frames with shifted centers of cross sections, as well as in the calculations of concrete elements and structures, is obtained, generalized to the case of difference resistance of materials [15, 16].

If the material deformation law can be approximated by an odd function, in (3) it is advisable to hold only terms with odd indices, thereby eliminating the need for special detection of the position of the neutral axis. Acceptance in (3) members with numbers gives the dependencies of A.R. Rzhanitsyn, P.A. Lukash.

Using stress approximations $\sigma_x^{(k)}(\varepsilon)$ *the Ilyushin function* ω_k used in nonlinear calculations can be found, which is entered based on the expression

$$\sigma_x^{(k)}(\varepsilon)=E_k\varepsilon(1-\omega_k). \tag{4}$$

Given that $p_{k,1}^{\pm}=E_k^{\pm}$ we receive

$$\omega_k(\varepsilon)=1-\sum_{i=1}^{r}\frac{p_{k,i}^{\pm}}{p_{k,1}^{\pm}}\varepsilon^{i-1}.$$

Substituting stresses (3) into the first two equations (2) when considering kinematic hypotheses (1), gives a system of nonlinear equations that connects the longitudinal force and the bending moment with generalized deformations ε_0, κ

$$\begin{cases} \sum_{i=0}^{r}\sum_{j=0}^{i}(-1)^j c_{ij} D_{ij}\varepsilon_0^{i-j}\kappa^j = N, \\ -\sum_{i=0}^{r}\sum_{j=0}^{i}(-1)^j c_{ij} D_{ij+1}\varepsilon_0^{i-j}\kappa^j = M. \end{cases} \quad (5)$$

Coefficients in (5) are integral *stiffness characteristics of a cross section of nonlinear elastic materials of* i *-physical and j-geometric order*.

$$D_{ij}(x) = \sum_{k=1}^{s} p_{k,i}^{\pm} \iint_{A_k} y^j dA = \sum_{k=1}^{s} D_{ij}^{(k)} . \quad (6)$$

Numerical coefficients c_{ij} ($j = 0,...,i$) – are *the Newton binomiall* coefficients $(a+b)^j$.

Calculating rigidity of the k -phase $D_{ij}^{(k)}$ in (6) it is necessary to perform an additional height breakdown $h_k = h_k^+ + h_k^-$ at the intersection of the layer with the neutral axis on the areas with the same deformation sign

$$D_{ij}^{(k)} = p_{k,i}^{+} \int_{h_k^+} b_k(x,y) y^j dy + p_{k,i}^{-} \int_{h_k^-} b_k(x,y) y^j dy .$$

The system of physical equations (5) is a generalization of known classical cases. Thus, when $r = 0$, the equations contain rigidity

$$D_{00} = \sum_{k=1}^{s} (\pm\sigma_{k,s}^{\pm} A_k) , \quad D_{01} = \sum_{k=1}^{s} (\pm\sigma_{k,s}^{\pm} S_k)$$

and describe the limit states of the layered section made of rigid plastic materials.

When holding in (3) linear members, the system (5) becomes quasi-linear with rigidity characteristics of the heterogeneous section from linearly elastic different-module materials

$$D_{10} = \sum_{k=1}^{s} E_k^{\pm} A_k , \quad D_{11} = \sum_{k=1}^{s} E_k^{\pm} S_k ,$$

$$D_{12} = \sum_{k=1}^{s} E_k^{\pm} I_k .$$

If you additionally perform reduction to the central axes (which gives $D_S = 0$, but it is not advisable when you solve inverse problems of the design type), we get the solution of S.P. Timoshenko.

Depending on the properties of phase materials, different (not necessarily sequential) sets of homogeneous blocks can be used in (5).

When organizing numerical procedures in iterative algorithms, physical dependencies (3) are presented in quasi-linear form

$$\sigma_x^{(k)}(\varepsilon) = \bar{E}_k(\varepsilon) \cdot \varepsilon \quad (7)$$

with variable secant elastic module depending on the deformation

$$\bar{E}_k(\varepsilon) = \sum_{i=0}^{r} p_{k,i}^{\pm} \varepsilon^{i-1} . \quad (8)$$

Applying expression (7) for forces (2) gives the system of quasi-linear equations

$$\begin{cases} \bar{D}_A \varepsilon_0 - \bar{D}_S \kappa = N, \\ -\bar{D}_S \varepsilon_0 - \bar{D}_I \kappa = M, \end{cases} \quad (9)$$

with functional *secant stiffness characteristics* $\bar{D}$

$$\left[\bar{D}_A, \bar{D}_S, \bar{D}_I\right](x,\varepsilon) = = \sum_{i=1}^{r}\sum_{j=0}^{i-1}(-1)^j c_{i-1,j}\left[D_{ij}, D_{ij+1}, D_{ij+2}\right]\varepsilon_0^{i-j-1}\kappa^j , \quad (10)$$

depending on generalized deformations ε_0 , κ .

In expressions (10) higher-exponent characteristics are used (6).

For odd cubic approximation $\sigma_x^{(k)} = p_{k,1}^{\pm}\varepsilon + p_{k,3}^{\pm}\varepsilon^3$ stiffness expressions (10) take the form

$$\begin{aligned} \bar{D}_A &= D_{10} + \varepsilon_0^2 D_{30} - 2\varepsilon_0\kappa D_{31} + \kappa^2 D_{32}, \\ \bar{D}_S &= D_{11} + \varepsilon_0^2 D_{31} - 2\varepsilon_0\kappa D_{32} + \kappa^2 D_{33}, \\ \bar{D}_I &= D_{12} + \varepsilon_0^2 D_{32} - 2\varepsilon_0\kappa D_{33} + \kappa^2 D_{34}. \end{aligned} \quad (11)$$

3. Shear stress

Obtaining shear stresses under equilibrium conditions, with non-linear dependencies (3), is a laborious procedure that is not advisable for secondary stress components of energy significance. More efficient and consistent with the initial assumptions is an alternative approach based on the equilibrium condition of the upper displaced part of $y \in [y, y_{s+1}]$ the rod element with a length of *dx*

$$\tau_{yx}^{(k)}(x,y) = -\frac{1}{b_k(x,y)}\frac{\partial}{\partial x}\left(\int_{y}^{y_{s+1}(x)} \sigma_x(x,y)b(x,y)dy\right). \quad (12)$$

We approximate the linear shear force (derivative of the integral in (12)) in the form

$$\frac{\partial N^{\sec}(x,y)}{\partial x} = b_0 \tau_0(x) f_\tau(y) \quad (13)$$

using a dimensionless function of the transverse distribution of shear forces $f_\tau(y)$, that satisfies the boundary conditions $f_\tau(y_1) = f_\tau(y_{s+1}) = 0$. The parameter $\tau_0(x)$ characterizes the longitudinal distribution of shear forces, and b_0 is a geometric parameter. Combining (12), (13), excluding the parameter τ_0 using condition (2) we obtain the formula of the shear stress

$$\tau_{yx}^{(k)}(x,y) = \frac{Q(x)}{b_k(x,y)}\frac{f_\tau(y)}{F_\tau}, \quad F_\tau = \int_{y_1}^{y_{s+1}} f_\tau dy. \quad (14)$$

In heterogeneous structures which have fragments of Z-layering (Fig. 2), for which lines $y(z) = \text{const}$ cross different phases, the formula (14) should be improved by introducing the shear stiffness coefficient at the *y* level

$$k_\tau(y,z) = \frac{G_k(y,z)}{\sum_j G_k(y,z) b_j(y)}.$$

Here, the summation is performed on the phases crossed by the line $y(z) = \text{const}$. Assuming, in accordance with the accepted hypotheses, the constancy of shears (1), we obtain shear stresses in Z-layered structures

$$\tau_{yx}^{(k)}(x,y,z) = \frac{Q(x) f_\tau(y)}{F_\tau}\frac{G_k(y,z)}{\sum_j G_j(y,z) b_j(x,y)}.$$

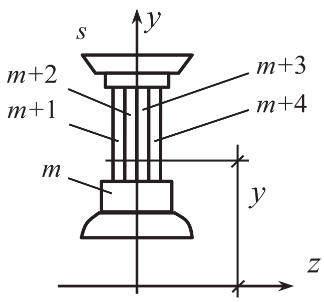

Figure 2. Cross-layered structure with Z-layered fragment

When the shear stiffness of the section is found, we assume the equivalence of the Ilyushin functions in the dependencies (4) and in the law for shear stresses, by

$$\tau_{yx}^{(k)} = G_k(1-\omega_k)\gamma_{yx}.$$

The increment of the elastic deformation energy in the heterogeneous section during the shear is presented in two forms: through the integral force *Q* and stress $\tau_{yx}^{(k)}$

$$Q\Delta\gamma_0 = \sum_{k=1}^{s}\iint_{A_k} \tau_{yx}^{(k)} \Delta\gamma_{yx} dA. \quad (15)$$

Here $\Delta\gamma_0 = \Delta Q / \bar{D}_Q(\gamma)$ is the deformation increment of the average shear in the Timoshenko rod, $\bar{D}_Q$ is section shear stiffness, $\Delta\gamma_{yx}(x,y) = \Delta\tau_{yx}^{(k)} / G_k / (1-\omega_k)$ – shear strain increment at a heterogeneous section point. Substituting stress (14) into (15), we obtain variable *secant shear stiffness*

$$\bar{D}_Q = \frac{F_\tau^2}{\sum_{k=1}^{s} \int_{y_k}^{y_{k+1}} \frac{f_\tau^2}{b_k G_k (1-\omega_k)} dy}, \qquad (16)$$

with which the third physical relation is recorded

$$\bar{D}_Q \gamma_0 = Q, \qquad (17)$$

complements the system of equations (5), and binds shear strain γ_0 with transverse force Q.

4. Relations for internal forces and displacements

The differential equations of equilibrium of the rod in the deformed state are

$$\begin{cases} N' + (\theta Q)' = -q_x, \\ Q' - (\theta N)' = q_y, \qquad (\ldots)' = \frac{\partial}{\partial x}. \\ M' = Q + m_z, \end{cases}$$

Their integration considering small values of the first order when the following conditions are satisfied

$$N(0) = N_0 - Q_0\theta_0, \; Q(0) = Q_0 + N_0\theta_0, \; M(0) = M_0$$

gives the expressions of internal force factors in the form of the initial parameters method. Applying the method of iterative refinement, we write them for the t-step in the form

$$N^{[t]}(x) = N_0^{[t]} - Q_0^{[t]}\theta^{[t-1]} - F_x - \theta^{[t-1]}F_y - \int_{x_{qx}}^{x} q_x dx - \theta^{[t-1]} \int_{x_{qy}}^{x} q_y dx,$$

$$Q^{[t]}(x) = N_0^{[t]}\theta^{[t-1]} + Q_0^{[t]} - \theta^{[t-1]}F_x + F_y - \theta^{[t-1]} \int_{x_{qx}}^{x} q_x dx + \int_{x_{qy}}^{x} q_y dx, \qquad (18)$$

$$M^{[t]}(x) = M_0^{[t]} + N_0^{[t]} v^{[t-1]} + Q_0^{[t]} x + M_z + \int_{x_m}^{x} m_z dx + F_y(x - x_{F_y}) - F_x[v^{[t-1]}(x) - v^{[t-1]}(x_{F_x})] - \int_{x_{qx}}^{x} q_x(\xi)[v^{[t-1]}(x) - v^{[t-1]}(\xi)]d\xi + \int_{x_{qy}}^{x} q_y(\xi)(x-\xi)d\xi.$$

In case there are several similar loads on the rod, in (18) several terms of the corresponding type should be used. The differentiation of the influence areas of loads is completed with the use of the Heaviside step function (to reduce the entry not specified in (18)).

The determination of displacements is performed using quasi-linear equations (9). Expressing from (9), (17) generalized deformations through *secant section pliability*

$$\bar{\delta}_A = \frac{1}{\bar{D}_A - \bar{D}_S^2 / \bar{D}_I}, \; \bar{\delta}_S = \frac{\bar{D}_S}{\bar{D}_I \bar{D}_A - \bar{D}_S^2}, \; \bar{\delta}_I = \frac{1}{\bar{D}_I - \bar{D}_S^2 / \bar{D}_A}, \; \delta_Q = \frac{1}{\bar{D}_Q} \qquad (19)$$

and substituting them into kinematic dependencies (1), we obtain the angle of rotation and displacements at t-step

$$\theta^{[t]}(x) = \theta_0^{[t]} + \int_0^x \delta_S^{[t-1]} N^{[t-1]} dx + \int_0^x \delta_I^{[t-1]} M^{[t-1]} dx,$$

$$u^{[t]}(x) = u_0^{[t]} + \int_0^x \delta_A^{[t-1]} N^{[t]} dx + \int_0^x \delta_S^{[t-1]} M^{[t]} dx, \qquad (20)$$

$$v^{[t]}(x) = v_0^{[t]} + \theta_0^{[t]} x + \int_0^x \int_0^x \delta_S^{[t-1]} N^{[t-1]} dx dx + \int_0^x \int_0^x \delta_I^{[t-1]} M^{[t-1]} dx dx + \int_0^x \delta_Q^{[t-1]} Q^{[t-1]} dx.$$

Substituting here the internal forces (18) gives the final expressions of displacements (they are not given here), which depend on the initial parameters $N_0^{[t]}$, $Q_0^{[t]}$, $M_0^{[t]}$, $u_0^{[t]}$, $\theta_0^{[t]}$, $v_0^{[t]}$, loads,

secant pliability (19) and displacements in the previous step.
Initial parameters are from six boundary conditions

$$N^{[t]}(x_*) + \theta^{[t-1]}(x_*)Q^{[t]}(x_*) = \mp F_{x*},$$
$$\theta^{[t-1]}(x_*)N^{[t]}(x_*) - Q^{[t]}(x_*) = \mp F_{y*},$$
$$M^{[t]}(x_*) = \pm m_{z*}, \qquad (21)$$

when recording which the upper signs are used on the left end, and the lower signs on the right.

5. Rational design of layered rod based on continuous strength criterion

In a cross-layered rod at a given force effect, it is required to determine a particular set of geometric functions from the full set

$$r_k(x), \quad r \in [b,h], \ (k = 1,...,s), \qquad (22)$$

internal force factors (18) and displacements (20), satisfying boundary conditions (21). In addition, geometric functions must satisfy the constraints

$$r_k \geq r_{\min}, \quad r \in [b,h]. \qquad (23)$$

The strength condition of the material of the k-layer with the calculated resistance R_k is presented in the form

$$\frac{1+\phi_k}{2}\sqrt{\sigma_x^2 + \beta_k \tau_{yx}^2} + \frac{1-\phi_k}{2}|\sigma_x| \leq R_k^{\pm}.$$

Presenting stresses σ_x and R_k with the use of (7) through secant moduli of elasticity

$$\bar{E}_k(\varepsilon) = \frac{\sigma_x^{(k)}}{\varepsilon}, \quad \bar{E}_k(\varepsilon_{\text{adm}}^{(k)}) = \frac{R_k}{\varepsilon_{\text{adm}}^{(k)}}, \qquad (24)$$

we obtain the strength condition recorded through the deformation

$$\text{sgn}(\varepsilon_x)\,\varepsilon_x(x,y) \leq \mu_\tau^{(k)}\mu_\varepsilon^{(k)}\varepsilon_{\text{adm}}^{(k)\pm}, \ (k = 1,...,s). \qquad (25)$$

Here $\varepsilon_{\text{adm}}^{(k)\pm}$ is allowable deformation levels of the material of the k-layer under tensile and compression;

$$\mu_\tau^{(k)}(x,y) =$$
$$= \frac{1}{2\phi_k}\left[(1+\phi_k)\sqrt{1-\phi_k\beta_k\left(\frac{\tau_{yx}(x,y)}{R_k^{\pm}}\right)^2} + \phi_k - 1\right] \qquad (26)$$
$$\mu_\tau^{(k)}(x,y) \in (0,1] -$$

the coefficient considering the decrease of the allowable material deformation $\varepsilon_{\text{adm}}^{(k)\pm}$ at shear stresses; and

$$\mu_\varepsilon^{(k)}(x,y) = \frac{\bar{E}_k(\varepsilon_{\text{adm}}^{(k)\pm})}{\bar{E}_k(\varepsilon)} \in (0,1] - \qquad (27)$$

the coefficient reflecting non-linearity of stress function $\sigma_x(\varepsilon)$, recorded through secant moduli of elasticity at finite (limit) and current points (24).
Based on the strength condition for the biaxial stress state (25), determine the *continuous multipoint design* criterion (CDC) of the heterogeneous non-linearly deformed the s-phase rod.
Definition. The generalized limit state in the area of heterogeneous nonlinear elastic rod that occupies the space X, Y, is realized if condition (25) is performed in the form of the strict equality

$$\text{sgn}(\varepsilon_x)[\varepsilon_0(x) - \kappa(x)y] = \mu_\tau^{(k)}\mu_\varepsilon^{(k)}\varepsilon_{\text{adm}}^{(k)\pm},$$
$$x \in X, \quad y = \xi_1,...,\xi_{n_y} = Y_* \in Y$$

continuously along the longitudinal coordinate $x \in X$, in discrete number of levels n_y along transverse coordinate $y = \xi_1,...,\xi_{n_y} = Y_* \in Y$. For all other points $x \in X$, $y \notin Y_*$ condition (25) is real in the form of inequality.
The multiplicity Y_* in the cross-layered structures is typically formed from the layer interfaces $y_1,..., y_{s+1}$. The multiplicity of the criterion

n_y depending on the conditions of the problem, takes values from 1 to $s+1$, while the criterion is referred to as n_y - point CDC. In general, to satisfy these equalities, except for the varying geometric functions, requires a change in the physical structure of the rod. For fixed-structure rods, the maximum multiplicity n_y is two.

Application *s*+1- and *s* - point criterion. As $s+1$ variable parameters (22) we take the width of external layers b_1, b_s and the height h_k of all, except one, layers ($k=1,..,k_0-1,k_0+1,..s$). The height h_{k_0} and width of the inner layers are set. According to CDC we compose a system $n_y = s+1$ of equations (23), for certainty, considering the case of different-valued deformations in the section with a change of sign within the layer k_0

$$\begin{cases} \varepsilon_0(x)-\kappa(x)y_1 = \pm\,\mu_\tau^{(1)}\mu_\varepsilon^{(1)}\varepsilon_{\text{adm}}^{(1)\pm}, \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ \varepsilon_0(x)-\kappa(x)y_{k_0} = \pm\,\mu_\tau^{(k_0)}\mu_\varepsilon^{(k_0)}\varepsilon_{\text{adm}}^{(k_0)\pm}, \\ \varepsilon_0(x)-\kappa(x)y_{k_0+1} = \mp\,\mu_\tau^{(k_0)}\mu_\varepsilon^{(k_0)}\varepsilon_{\text{adm}}^{(k_0)\mp}, \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ \varepsilon_0(x)-\kappa(x)y_{s+1} = \mp\,\mu_\tau^{(s)}\mu_\varepsilon^{(s)}\varepsilon_{\text{adm}}^{(s)\mp}. \end{cases} \tag{28}$$

Here, with positive (negative) curvature, the upper (lower) signs are taken. The relationships for other types of strain distribution are also written analogically. With unambiguous distributions k_0 and k_0+1 both equations are replaced by one, and the criterion becomes the *s*-point. At the same time, the height of one of the layers is excluded from the variable parameter.

The solution of the problem will be performed with phased accounting of the values of the coefficients $\mu_\tau^{(k)}(x,y)$, $\mu_\varepsilon^{(k)}(x,y)$ at the computational points. At the first stage, their values are taken equal to one.

▪ Using k_0 and k_0+1 the equations of the system (28), we determine the curvature and deformation

$$\kappa_*(x) = \frac{\pm\mu_\tau^{(k_0)}\mu_\varepsilon^{(k_0)}\varepsilon_{\text{adm}}^{(k_0)\pm} \pm \mu_\tau^{(k_0)}\mu_\varepsilon^{(k_0)}\varepsilon_{\text{adm}}^{(k_0)\mp}}{h_{k_0}},$$
$$\varepsilon_{0*} = \frac{\pm\mu_\tau^{(k_0)}\mu_\varepsilon^{(k_0)}\varepsilon_{\text{adm}}^{(k_0)\pm} y_{k_0+1} \pm \mu_\tau^{(k_0)}\mu_\varepsilon^{(k_0)}\varepsilon_{\text{adm}}^{(k_0)\mp} y_{k_0}}{h_{k_0}} \tag{29}$$

Then, using the remaining equations, we find $(s-1)$ of the desired heights

$$h_{k+1} = \frac{\pm\mu_\tau^{(k_0)}\mu_\varepsilon^{(k_0)}\varepsilon_{\text{adm}}^{(k+1)\mp} \mp \mu_\tau^{(k_0)}\mu_\varepsilon^{(k_0)}\varepsilon_{\text{adm}}^{(k)\mp}}{\kappa_*},$$
$$k = k_0,..,s-1, \tag{30}$$
$$h_{k-1} = \frac{\pm\mu_\tau^{(k_0)}\mu_\varepsilon^{(k_0)}\varepsilon_{\text{adm}}^{(k-1)\pm} \mp \mu_\tau^{(k_0)}\mu_\varepsilon^{(k_0)}\varepsilon_{\text{adm}}^{(k)\pm}}{\kappa_*},$$
$$k = k_0,..,2\,.$$

If received $h_k < 0$ we need to change the physical structure of the package or reduce the multiplicity of the criterion.

▪ The internal force factors at the *t*-step in the design area are written in the form (18), and the functions of displacements and rotation angles, assuming $\varepsilon_0 = \varepsilon_{0*}$, $\kappa = \kappa_*$, in the form

$$u^{[t]}(x) = u_0{}^{[t]} + \int_0^x \varepsilon_{0*}^{[t]}dx, \quad \theta^{[t]}(x) = \theta_0{}^{[t]} + \int_0^x \kappa_*^{[t]}dx,$$

$$v^{[t]}(x) = v_0{}^{[t]} + \theta_0{}^{[t]}x + \int_0^x dx\int_0^x \kappa_*^{[t]}dx - Q_0{}^{[t]}\int_0^x \delta_Q^{[t-1]}dx -$$

$$-F_y\int_{x_{Fy}}^x \delta_Q^{[t-1]}dx - \int_{x_{q_y}}^x \delta_Q^{[t-1]}\int_{x_{q_y}}^x q_y dxdx\,.$$

▪ Physical relations (5), when substituting limit generalized deformations (29), give a system of equations to determine the two remaining variable parameters

$$\begin{cases} \Psi_N(N,\varepsilon_{0*},\kappa_*,b_1,b_s,h_1,...,h_s,x)=0, \\ \Psi_M(M,\varepsilon_{0*},\kappa_*,b_1,b_s,h_1,...,h_s,x)=0. \end{cases} \tag{31}$$

- Calculation of stiffness characteristics $\bar{D}_A$, $\bar{D}_S$, $\bar{D}_I$ (10), $\bar{D}_Q$, shear (14) and normal stresses

$$\sigma_x^{(k)}(x,y)=\sum_{i=0}^{r} p_{k,i}^{\pm}[\varepsilon_0(x)-\kappa(x)y]^i .$$

- Verification of constraint (23) and criteria to achieve the required accuracy of calculations.

Application of two-point criterion. As variable parameters, any two *geometric functions* from set (22) are accepted. In this case, a two-point CDC and a system (23) are used in the form of two equations recorded for the two $\xi_1, \xi_2 = Y_*$ most dangerous levels

$$\varepsilon_{0*}-\kappa_* y_i = \pm\mu_\tau^{(k)}\mu_\varepsilon^{(k)}\varepsilon_{\text{adm}}^{(k)\pm}, \quad (32)$$
$$i=\xi_1,\ \xi_2,\quad x\in X_*,\quad y_i\in Y_* .$$

The difference between two-point setting is the ability to perform the solution for any given physical structure of the layered package.

After determination of limit deformation and curvature from (32), further steps bypassing (30), are performed similarly to the procedure for multipoint criterion.

The obtained solution is realized if the obtained geometric functions continuously along the x coordinate satisfy the problem constraints (23). Otherwise, we need to decrease the multiplicity and switch to the single-point criterion ($n_y = 1$). Then, for the corresponding function (for example, the first one), the equality is assumed

$$r_{i,k_1}(x)=r_{\min} \quad (33)$$

with simultaneous setting of deformation

$$\varepsilon_{i,\xi_2}(x)=\pm\mu_\tau^{(k)}\mu_\varepsilon^{(k)}\varepsilon_{\text{adm},\xi_2}^{(k)\pm} . \quad (34)$$

Substituting (33), (34) into (31) gives a mixed system of resolving equations for the geometric function $r_{k_2}(x)$ and deformation $\varepsilon_{i,\xi_1}(x)$:

$$\begin{cases}\Psi_N(N_i,\varepsilon_{i,\xi_1},r_{i,k_2},x)=0,\\ \Psi_M(M_i,\varepsilon_{i,\xi_1},r_{i,k_2},x)=0.\end{cases}$$

If the two constraints cannot be satisfied (23), a zero-point ($n_y = 0$) inequality criterion (25) is applied at the area. Taking in this case for the desired geometric equality functions

$$r_{i,k_1}(x)=r_{\min},\quad r_{i,k_2}(x)=r_{\min},$$

we have a direct formulation of the problem.

Scheme to solve the problem of rational design. Using preliminary information about the stress-strain state of the rod, we will select sections in it (with indefinite boundaries) with the implementation of CDC of various degrees of multiplicity. Then we add to them the areas with the same laws of effort. The total number of areas is denoted by n_u. For each of them, based on (18), (20), we compose expressions of efforts and displacements containing constants $N_{i,0}$, $Q_{i,0}$, $M_{i,0}$, $u_{i,0}$, $\theta_{i,0}$, $v_{i,0}$ ($i=1,...,n_u$). Secant pliability (19) and displacements in the right parts should be considered known according to the calculation results at the previous step.

To determine $6n_u$ the constant, $n_y n_u$ of variable functions and $n_u - 1$ boundaries of the areas, we will form groups of conditions. *The first group* of conditions, a total number $7n_u - 1$, is invariant to the type of the design criterion and consists of six boundary conditions for the rod as a whole (21) and $6(n_u - 1)$ conditions for the conjugation of power and kinematic functions at the boundaries of the areas. This group of conditions is supplemented, formed on the basis of constraints (23), $n_u - 1$ equation to define x_i – boundaries of the area,

$$r_i(x_i)=r_{\min},\quad i=1,...,n_u-1,\quad r\in[b,h].$$

The second group of $n_y n_u$ conditions depending on the accepted design criterion, is represented by n_y equations (23), (31) for each of n_u design areas.

6. Forms of resolving equations

Varying the width of the layers. The desired functions of the width of two layers with numbers k_1, k_2 we set in the form

$$b_k(x,y)=b_k^x(x)\,\overline{b}_k^y(y),\quad (k=k_1,k_2)$$

with the desired layer width function $b_k^x(x)$ and the defined dimensionless transverse profiling function $\overline{b}_k^y(y)$.

The resolution system (31), derived from the physical equalities (9), takes the form

$$\begin{cases}\varphi_{11}b_{k_1}^x+\varphi_{12}b_{k_2}^x=\varphi_1(x),\\ \varphi_{21}b_{k_1}^x+\varphi_{22}b_{k_2}^x=\varphi_2(x),\end{cases}\qquad(35)$$

$$\varphi_{11}=\sum_{i=1}^{r}\sum_{j=0}^{i}(-1)^j c_{i,j}\varepsilon_{0*}^{i-j}\kappa_*^j\overline{D}_{i,j}^{(k_1)},$$

$$\varphi_{12}=\sum_{i=1}^{r}\sum_{j=0}^{i}(-1)^j c_{i,j}\varepsilon_{0*}^{i-j}\kappa_*^j\overline{D}_{i,j}^{(k_2)},$$

$$\varphi_{21}=-\sum_{i=1}^{r}\sum_{j=0}^{i}(-1)^j c_{i,j}\varepsilon_{0*}^{i-j}\kappa_*^j\overline{D}_{i,j+1}^{(k_1)},$$

$$\varphi_{22}=-\sum_{i=1}^{r}\sum_{j=0}^{i}(-1)^j c_{i,j}\varepsilon_{0*}^{i-j}\kappa_*^j\overline{D}_{i,j+1}^{(k_2)},$$

$$\varphi_1=N-\sum_{i=1}^{r}\sum_{j=0}^{i}(-1)^j c_{i,j}\varepsilon_{0*}^{i-j}\kappa_*^j\overline{D}_{i,j}^{(0)},$$

$$\varphi_2=M+\sum_{i=1}^{r}\sum_{j=0}^{i}(-1)^j c_{i,j}\varepsilon_{0*}^{i-j}\kappa_*^j\overline{D}_{i,j+1}^{(0)},$$

$$\overline{D}_{ij}^{(k)}=p_{k,i}^{\pm}\int_{h_k}\overline{b}_k^{(y)}y^j\,dy,$$

$$D_{ij}^{(0)}(x)=\sum_{k\notin k_1,k_2}p_{k,i}^{\pm}\int_{h_k}b_k^x(x,y)y^j\,dy.$$

Varying the height of the layers. When varying the heights $h_{k_1}(x)$, $h_{k_2}(x)$ of the numbered layers k_1, k_2 we obtain a non-linear system of resolution equalities (31) of degree n_g+1, where n_g – is the highest geometric order of the stiffness characteristic (6). Matrix of Jacobi, used in solving system (31) has components

$$\frac{\partial\Psi_N}{\partial h_l}=\sum_{i=0}^{\eta}\sum_{j=0}^{i}(-1)^j c_{ij}\frac{\partial D_{ij}}{\partial h_l}\varepsilon_{0*}^{i-j}\kappa_*^j,$$

$$\frac{\partial\Psi_M}{\partial h_l}=\sum_{i=0}^{\eta}\sum_{j=0}^{i}(-1)^{j+1} c_{ij}\frac{\partial D_{ij+1}}{\partial h_l}\varepsilon_{0*}^{i-j}\kappa_*^j,.\qquad(36)$$

$$\left(l=k_1,k_2\right)$$

$$\frac{\partial D_{ij}}{\partial h_l}=\sum_{k=1}^{s}p_{k,i}^{\pm}\left[\frac{\partial y_{k+1}}{\partial h_l}b_k(x,y_{k+1})y_{k+1}^j-\frac{\partial y_k}{\partial h_l}b_k(x,y_k)y_k^j\right].$$

Depending on the number of the variable layer and the position of the reference surface, we have derivatives $\partial y_k/\partial h_l=-1,0,1$. For example, for a three-layer section that has the dimensions of rectangular sections of layers: $b_k(x)$, $h_k(x)$ ($k=1,2,3$), of which all functions are defined, and $h_1(x)$ and $h_3(x)$ are to be determined, taking on the basis of (3) the quadratic law, binding the reference surface to the middle of the inner layer, we obtain

$$[y_1,y_2,y_3,y_4]=$$
$$=[-h_1-0{,}5h_2;-0{,}5h_2;0{,}5h_2;0{,}5h_2+h_3]$$

$$\frac{\partial}{\partial h_1}[y_1,y_2,y_3,y_4]=[-1;\,0;\,0;\,0],$$

$$\frac{\partial}{\partial h_1}[D_{10},D_{11},D_{12}](x)=p_{11}b_1[1,y_1,y_1^2],$$

$$\frac{\partial}{\partial h_3}[y_1,y_2,y_3,y_4]=[0;\,0;\,0;\,1],$$

$$\frac{\partial}{\partial h_3}[D_{10},D_{11},D_{12}](x)=p_{31}b_3[1,y_4,y_4^2],$$

$$\frac{\partial}{\partial h_1}[D_{20},D_{21},D_{22},D_{23}](x)=p_{12}b_1[1,y_1,y_1^2,y_1^3],$$

$$\frac{\partial}{\partial h_3}[D_{20},D_{21},D_{22}](x)=p_{32}b_3[1,y_4,y_4^2].$$

Thus we obtain the components of Matrix of Jacobi (36)

$$\frac{\partial \Psi_N}{\partial h_1} = p_{11} b_1 (\varepsilon_{0*} - \kappa_* y_1) + p_{12} b_1 (\varepsilon_{0*} - \kappa_* y_1)^2 ,$$

$$\frac{\partial \Psi_N}{\partial h_3} = p_{31} b_3 (\varepsilon_{0*} - \kappa_* y_4) + p_{32} b_3 (\varepsilon_{0*} - \kappa_* y_4)^2 ,$$

$$\frac{\partial \Psi_M}{\partial h_1} = -p_{11} b_1 y_1 (\varepsilon_{0*} - \kappa_* y_1) - p_{12} b_1 y_1 (\varepsilon_{0*} - \kappa_* y_1)^2 ,$$

$$\frac{\partial \Psi_M}{\partial h_3} = -p_{31} b_3 y_4 (\varepsilon_{0*} - \kappa_* y_4) - p_{32} b_3 y_4 (\varepsilon_{0*} - \kappa_* y_4)^2 .$$

RESULTS AND DISCUSSION

Fig. 3 shows the design scheme of the compressed-bent rod. We assume that initially the rod was homogeneous, it had a prismatic shape with a rectangular cross section. Then, in order to increase the bearing capacity, it was strengthened by adding two outer plates made of another (stronger) material. As a result, the rod took the form of a heterogeneous I-beam.

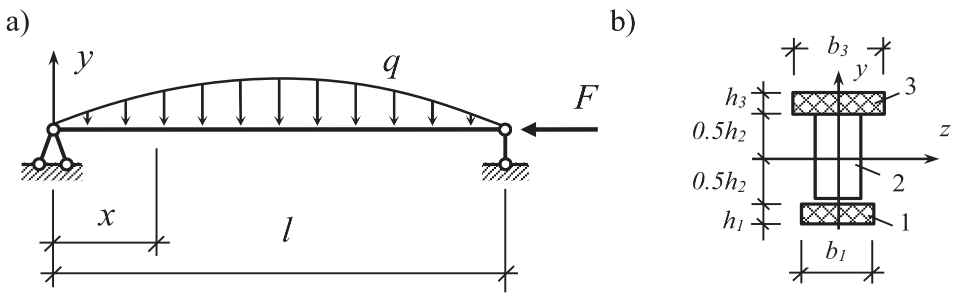

Figure 3. Design scheme of the compressed-bent rod (a); cross section (b)

Table. Data on the geometric and physical parameters of the layered rod

k	Type of the material	$p_{k,1}$ hPa	$p_{k,3}$ hPa	$\varepsilon^{(k)}_{\text{adm}}$	b_k mm	h_k mm
1	1	22	-162000	0,0053	$b_1(x)$	10
2	2	11	-105000	0,0045	50	300
3	1	22	-162000	0,0053	$b_1(x)$	10

It is required, at given loads, structure and materials, to perform rational profiling of the width $b_1(x)$, $b_3(x)$ of the outer layers using two-point strength criterion (32) and the satisfaction of limitations (23): $b_1(x) \geq b_{\min}$, $b_3(x) \geq b_{\min}$. Minimum values of the width are defined $b_{\min} = b_2$.

The rod with the length $l = 6$ m is loaded with axial force $F = 60$ kN and transverse load of variable intensity $q = 18\sin(\pi x / l)$ kN/m. The y axis is compatible with the force plane, and the x axis is compatible with the center axis of the inner ($k = 2$) layer. The diagrams of material deformation of I-beam elements are represented by identical symmetric cubic dependencies $\sigma_x^{(k)} = p_{k,1}\varepsilon + p_{k,3}\varepsilon^3$ (3) with two parameters. Table shows the physical and geometric parameters of the three layers, fig. 4 shows the approximations of the function $\sigma(\varepsilon)$ with the limit points marked.

The stress-strain state functions of the rod are represented in discrete form at m = 21 design points

$$\Phi \in [N, Q, M, \theta, v, \varepsilon_0, \kappa, b].$$

The solution of the design problem is possible to comply on the basis of a two-stage algorithm. *The first stage* is to solve *the direct problem* on the basis of physical and geometric nonlinearity with fixed design parameters – discrete functions b_1, b_3. At this stage, secant stiffness characteristics (11), which depend on generalized deformations and internal forces, are specified iteratively. The latter depends on movements and rotation angles.

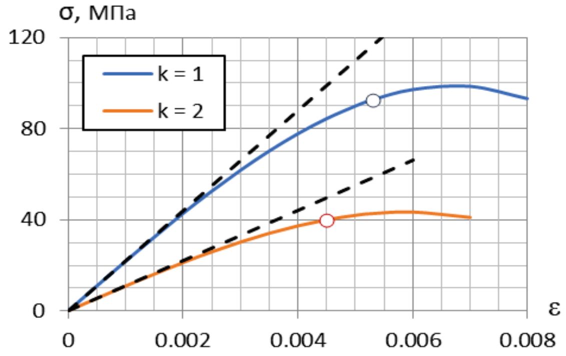

Figure 4. Graphs $\sigma_x(\varepsilon)$ for the material of the I-beam flange (k = 1) and the wall (k = 2)

Considering boundary conditions

$$N_0 = -F\,,\quad Q_0 = \frac{q_0 l}{\pi},\quad M_0 = 0\,,$$

using expressions (18) for t_1-step we obtain the calculated expressions of internal forces

$$N^{[t_1]}(x) = -F - \frac{q_0 l}{\pi}\theta^{[t_1-1]}\cos\frac{\pi x}{l}\,,$$
$$Q^{[t_1]}(x) = \frac{q_0 l}{\pi}\cos\frac{\pi x}{l} - F\theta^{[t_1-1]}\,, \qquad (37)$$
$$M^{[t_1]}(x) = -Fv^{[t_1-1]} + \frac{q_0 l^2}{\pi^2}\sin\frac{\pi x}{l}\,.$$

At *the second stage (design stage)* solution of the system of equations (35) the discrete design functions are obtained b_1, b_3 considering the calculated (32) on the limit values of generalized deformations ε_{0*}, κ_*. At this stage, the internal forces are taken from the previous stage. Further, the constraints (23) are checked and, if they are violated for one or (and) another parameter in the interval $x \in X_r$, the equality is accepted $b_{k,j} = b_{\min}$ ($k = 1, 3;\ j \in X_r$). The final check of the calculation is the control of accuracy achievement of iterative calculation of the required parameters values of the width of the layers $b_{1,j}$, $b_{3,j}$.

Initially, the calculation was performed without considering the influence of shear stresses of (32) $\mu_\tau^{(k)} = \mu_\varepsilon^{(k)} = 1$. According to the specified limit deformations $\varepsilon_{\text{adm}}^{(k)}$ (Table) the polygon of limit deformations is formed (fig. 5). It shows the limit line of deformations $\varepsilon_*(y) = \varepsilon_{0*} - \kappa_* y$, it has internal contacts with the polygon at two points $y_{\xi_1} = y_2$, $y_{\xi_2} = y_3$, located in the second layer. The coordinates of the layer boundaries are indicated by y_j ($j = 1,\ldots,4$). In this problem, the limit state regulates the inner layer. External layers ($k = 1$, $k = 3$) are in the pre-limit state.
Under the accuracy equal to 0.01, the computational procedure required $t_1 = 4 \div 7$ refinement iterations at the first stage and $t_2 = 5$ at the second stage.
The influence of geometric nonlinearity due to the presence of interchange in the force formulas (37), led to an increase in the bending moment by 11.3% in the middle of the span and a shear force by 12.0% in the support sections. Physical nonlinearity caused a decrease in secant stiffness characteristics $\bar{D}_A$, $\bar{D}_S$, $\bar{D}_I$ (11) in relation to the stiffness of linear theory D_A, D_S, D_I by values 9.1; 16.0; 14.1% respectively in section $x = l/2$.

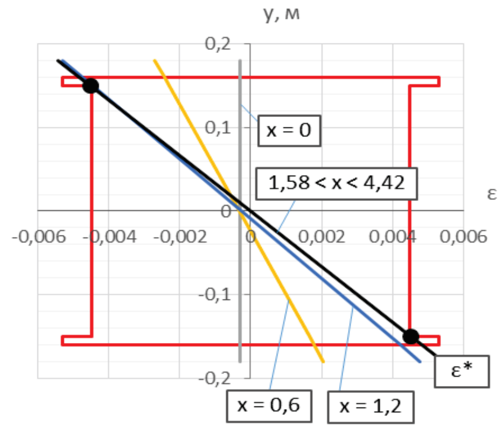

Figure 5. Polygon of limit deformations of the three-layer section with deformation curves ε(y)

The calculated width profiles of the outer layers $b_1(x)$, $b_3(x)$ are shown in Fig. 6. In the middle part of the rod ($1.58 \le x \le 4.42$) under the action of large bending moments, both limits (23):

$b_1 \geq 0.05$ m, $b_3 \geq 0.05$ m, were performed and the criterion (32) was implemented in the form of two-point ($n_y = 2$).

The decrease of the bending moments observed at the approach to the support led to the decrease of the calculated width of the layers. And, on two symmetric areas $0.93 \leq x \leq 1.58$, $4.42 \leq x \leq 5.07$ were obtained $b_1 < 0.05$ м, that demanded the acceptance of $b_1 = 0.05$ m. Further, this situation was in addition shown also for width of the upper layer on two pre-support areas $0 \leq x \leq 0.93$, $5.07 \leq x \leq 6$. At these areas, it was accepted $b_1 = b_3 = 0.05$ m. м. These areas with the indication of the multiplicity degree of the strength criterion is shown in Fig. 6.

Figure 5 shows deformation lines $\varepsilon(y)$ in sections $x = 0; 0.6; 1.2$ and on the area $1.58 \leq x \leq 4.42$. The entire middle area $1.58 \leq x \leq 4.42$ is characterized by *invariant* deformation distribution with fulfillment of the two-point strength criterion. The line for section $x = 1.2$ reflects one-point, and for sections $x = 0; 0.6$ reflects zero-point criteria.

Fig. 7 shows the curves of normal stresses $\sigma_x(x, y)$, calculated in the cross sections of the rod with coordinates $x = 0; 0.6; 1.2$ and in the design area $1.58 \leq x \leq 4.42$. The curves have a nonlinear character, as the stresses increase, the degree of nonlinearity increases. This is clearly visible in the area of the second layer $-0.15 \leq y \leq 0.15$. Nonlinearity is also manifested in the outer layers, but due to their low height, it is less noticeable. To illustrate the nonlinearity of stress laws in the outer layers, the sections of the curves are formally continued beyond these layers and are shown in dotted lines.

At the boundaries of the layers, when the physical parameters of the materials are changed $p_{k,1}$, $p_{k,3}$ (Table), according to the formula (3), breaks are observed at the curves $\sigma_x(y)$. At all points of the heterogeneous rod volume, the strength limit is performed $\operatorname{sgn}(\varepsilon_x)\,\sigma_x^{(k)}(x, y) \leq \mu_\tau^{(k)} \mu_\varepsilon^{(k)} R^{(k)\pm}$, in which the design stress are found according to the laws specified for stresses (Fig. 4) and the values of limit deformations (Table): $R^{(k=1)\pm} = \pm 92.5$ mPa, $R^{(k=2)\pm} = \pm 40.0$ mPa. In accordance with the two-point criterion of strength at the design area $1.58 \leq x \leq 4.42$ in two levels according to the y coordinate, the continuous equality of stresses is realized

$$\sigma_x^{(2)}(x; -0.15) = 40 \text{ mPa},$$
$$\sigma_x^{(2)}(x; +0.15) = \text{-}40 \text{ mPa}.$$

The analysis of the stress state showed that for the heterogeneous rod at a) a specified ratio of geometric parameters $l/(h_1 + h_2 + h_3) = 18.75$ and b) specified loads, the consideration of shear stresses and shear deformations is not relevant. In support of this, Fig. 7 shows the curve of shear stress (14), that acts in the support section of the rod with the greatest shear force, as well as the analysis of the coefficient $\mu_\tau(x, y_3)$ (26) is performed in the criterion (32). When calculating it, according to Mises' theory, it was admitted $\phi_k = 1$, $\beta_k = 3$. For the calculated level $y = y_3$ the function is obtained $\mu_\tau^{(2)}(x; 0.15)$ (Fig. 8). Its accounting makes amendments to the results of the calculation less than 0.03 %, which indicates the validity of the assumption $\mu_\tau^{(k)} = 1$.

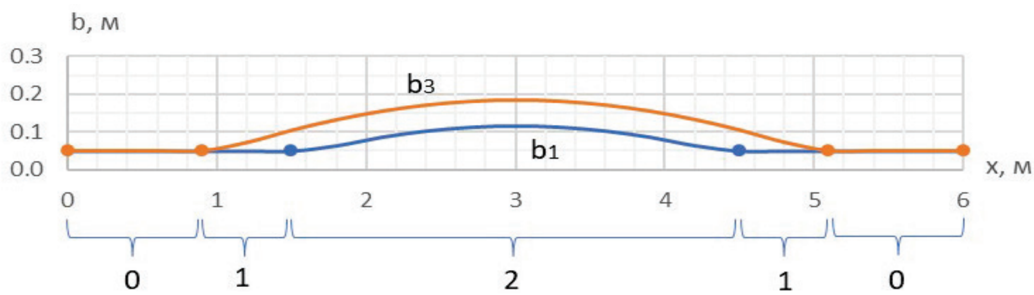

Figure 6. Design profiling of the external layers width b_1, b_3 by two-point strength criterion

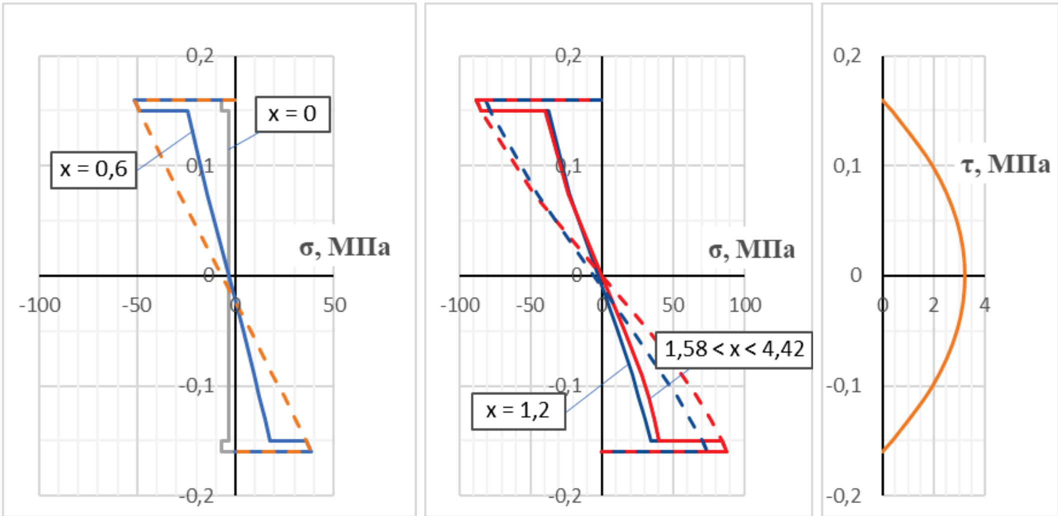

Figure 7. Curves of normal stresses $\sigma_x(x,y)$ *in the cross sections of the rod with coordinates* $x = 0; 0.6; 1.2$ *(m) and in the design area* $1.58 \le x \le 4.42$ *(m). Curve* $\tau_{yx}(0,y)$ *in the support section of the rod at* $Q = Q_{\max}$

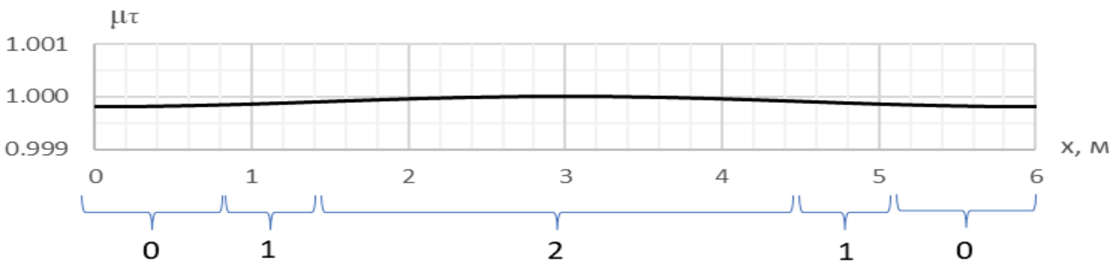

Figure 8. Graph of a correction factor change $\mu_\tau(x, y_3)$ *to consider the shear stresses in the strength criterion*

In other cases, the consideration of shear stresses and shear deformations can be completed according to the algorithm described above. Such cases are characterized by: a) the specifics of loading, if there are areas with a dangerous combination of several conditions, including shear forces, in the design areas; b) the consideration of rods of relatively shorter length, which will lead to an increase in the relative contribution of shear stresses in the strength criterion; c) the structure of the rod sections that has local thin-walled areas with significant shear stresses.

CONCLUSION

The developed method allows solving design nonlinear problems for heterogeneous rods with finding a rational profile (geometric configuration) of the rod layers. A computationally convenient formulation of physical dependencies in combination with an iterative method to clarify the functions of internal forces and displacements makes it possible to reduce the complexity in the problem solution of rational design of a nonlinearly deformable heterogeneous rod, i.e. a structural element of engineering structure.

The developed design can be used to solve problems of strengthening structures by adding external layers-plates to the elements of the initial system, which has the configuration and location determined by the solution of the opposite problem based on specified requirements for strength, rigidity and others.

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