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ENERGY PROPERTIES OF SYMMETRIC DEFORMABLE SYSTEMS

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Abstract: Energy methods for calculating structures, which have become popular for a century, are based on the Lagrange principle and have the meaning of equality of work of external forces and internal forces. Having proved their effectiveness in the overwhelming majority of problems of structural mechanics, they became the dominant approach in formulating the problems of studying solid deformable systems and gave rise to the main methodology for solving problems. As a result, a situation has arisen that the internal potential energy of a deformed body remains insufficiently studied.

The paper develops an approach to the study of the symmetric structure at critical levels of strain energy. The criterion of critical levels of strain energy, based on the concepts of "self-stress" ("self-balance") of a deformable body. Limiting values of the structure strain energy may get by varying the reactions and deflections in the nodal points. The extreme values of forces and displacements of the rods are calculated in matrix form from the values of nodal reactions (displacements).

Methodology for studying the energy properties of a system is shown on the examples of the study of symmetric rod systems without involving the concept of external forces. The technique is based on matrix methods of structural mechanics and the mathematical apparatus of eigenvalue problems. The comparison of structural design and structural analysis solution of structural mechanics tasks by traditional methods and with the proposed methodology is carried out.

Keywords: structural analysis, structural design, matrix methods, strain energy, critical strain energy levels, self-stress, reaction of structures, limit state design

ЭНЕРГЕТИЧЕСКИЕ СВОЙСТВА СИММЕТРИЧНЫХ ДЕФОРМИРУЕМЫХ СИСТЕМ

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Аннотация. Энергетические методы расчета конструкций, получившие распространение в течение столетия, основаны на принципе Лагранжа и имеют смысл равенства работ внешних и внутренних сил. Доказав свою эффективность в подавляющем большинстве задач строительной механики, они стали доминирующим подходом при постановке задач исследования твердых деформируемых систем и положили начало основной методологии решения задач. В результате возникла ситуация, когда внутренняя потенциальная энергия деформируемого тела остается недостаточно изученной.

В статье развивается подход к исследованию симметричной структуры при критических уровнях энергии деформации. Критерий критических уровней энергии деформации, основанный на понятиях "самонапряжения" ("саморавновесия") деформируемого тела. Предельные значения энергии деформации конструкции можно получить, изменяя реакции и прогибы в узловых точках. Экстремальные значения усилий и перемещений стержней вычисляются в матричной форме из значений узловых реакций (перемещений).

Методика изучения энергетических свойств системы показана на примерах исследования симметричных стержневых систем без привлечения понятия внешних сил. Методика основана на матричных методах строительной механики и математическом аппарате задач на собственные значения. Проведено сравнение решений проектных и поверочных задач строительной механики традиционными методами и по предлагаемой методике.

Ключевые слова: проектные и поверочные задачи строительной механики, матричные методы механики деформируемого тела, критические уровни энергии, самонапряжение, реакция упругой системы, предельное состояние

1. INTRODUCTION

The energetic methods of structural mechanics have proven their effectiveness in the minimum total strain energy form (problem in Lagrange form).Closely connected with the mathematical apparatus of the calculus of variations, they became the basis for obtaining approximate solutions in complex problems of stability and dynamics of structures, as well as solving nonlinear problems of deformation of spatial systems [1-5]. It is difficult to find such an area of structural mechanics where it would be possible to do without variational approaches and variational principles [6-10]. A special place is occupied by numerical methods for studying complex systems operating in the nonlinear stage of deformation [11-16], based on variational formulations of structural mechanics problems [17-24], which served as the basis for the creation of software systems using FEM, MGE and others methods of structural mechanics.

It should be noted that the overwhelming majority of the formulations of the problems mentioned above are based on the Laпrange approach, when the minimum of the total energy of the system leads to the equations of state of the system, which depend on the acting external loads. In this case, one might get the impression that the behavior of the system is completely determined by the magnitude, type and law of changes in the external load.

In a number of works [25,26] it is noted that such an approach does not always make it possible to calculate systems subject to the guiding loads, temperature and similar impacts. The necessity of formulating problems in the form of homogeneous systems of equations is noted, on the basis of which one can then restore the form of a possible external load.

Thus, there is an insufficient study of the properties of the internal potential energy of deformable systems. The hypothesis that any deformable body has levels of critical energy values at which the system can change the form of the law of equilibrium states or a model of the structure, made it possible to obtain a criterion for critical levels of potential energy, which was tested on a number of well-known simple problems of stability and bending of rod systems [27].

This work is devoted to the study of the strain energy of symmetric rod systems.

2. EQUATIONS OF CRITICAL ENERGY OF SYMMETRIC STATICALLY INDETERMINATE SYSTEMS

In [28], the equations of state of a system with lumped parameters at critical energy levels were obtained, written in the matrix form

$$\left[\mathrm{L}\right]\left\{\delta\Phi_{\mathrm{k}}^{\mathrm{in}}\right\}=\left[\lambda_{0}^{\mathrm{L}}\right]\left\{\delta\Phi_{\mathrm{k}}^{\mathrm{in}}\right\}, \quad (1)$$

Here we have introduced the notation for the flexibility matrix $[\mathrm{L}]$ systems and eigenvalue matrices $\left[\lambda_0^{\mathrm{L}}\right]$ variations of effort. The eigenvalues $\lambda_{0i}^{\mathrm{L}}$ are the main values of the flexibility of the system, and the eigenvectors $\left\{\delta\Phi_{\mathrm{k}}^{\mathrm{in}}\right\}$ are the amplitude values of the distribution of self-stress forces. For variations of nodal displacements (system reactions), the relation for the eigenvalues of the flexibility and stiffness matrices is valid

$$[\lambda_{0i}^{\mathrm{K}}]=[(\lambda_{0i}^{\mathrm{L}})]^{-1}. \quad (2)$$

Here $\lambda_{0i}^{\mathrm{K}}$ are the eigenvalues of the stiffness matrix of the system.

The physical meaning of the derived equations (1), (2) is the state of self-stress in a statically indeterminate system at critical energy levels. The solution of the eigenvalue problem gives the main nodal displacements system and the corresponding vectors of the amplitude values of the nodal reaction forces. To understand the processes of changing the internal potential energy of an elastic rod structure, consider the three-rod system shown in Figure 1.

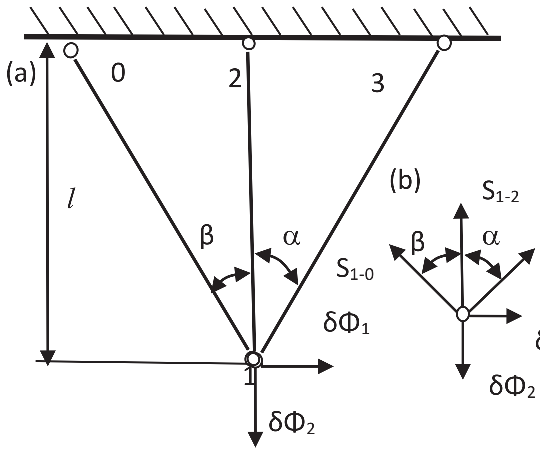

Figure 1. Three-rod system: (a) design model, (b) node 1

The internal potential energy of deformation of the system in matrix form can be represented as

$$2\{U\} = \{\xi\}^T [K]\{\xi\} = \{\Phi\}_{in}^T [L]\{\Phi\}_{in}. \quad (3)$$

It is easy to see that the levels of the internal potential energy of deformation are determined by the values of the stiffness (flexibility) matrices of the system. Strain energy at any level we may get multiplying the initial self-stress energy state at constant value. Let structure has elastic initial self-stress energy.

A symmetric redundant system has the same angles $\alpha = \beta = \pi/4$ of inclined rods and stiffness values $\eta_i = EA/E_iA_i$ $\eta_1 = \eta_2 = \eta_3 = 1$.

Due to the symmetry of the structure, the flexibility matrix is diagonal, which indicates that the eigenvalues of the flexibility matrix are located on the main diagonal.

$$[L] = \frac{l}{EA}\begin{vmatrix} 1.414 & 0 \\ 0 & 0.5858 \end{vmatrix}. \quad (4)$$

It is easy to write out the eigenvectors of the external compliance matrix

$$\left[\vartheta^L\right] = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}, \quad (5)$$

that is, the initially selected axes are the principal axes of the system's ultimate energy. The stiffness matrix of the system is also diagonal and containing the eigenvalues

$$[K] = \frac{EA}{l}\begin{vmatrix} 0.7072 & 0 \\ 0 & 1.707 \end{vmatrix}, \quad (6)$$

with their corresponding eigenvectors

$$\left[\vartheta^K\right] = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}. \quad (7)$$

The ellipsoid of limiting displacements has a maximum radius equal to the first eigenvalue of the flexibility matrix $\delta_{11} = \lambda_{max}^L = 1.414l/EA$, (see Figure 2).

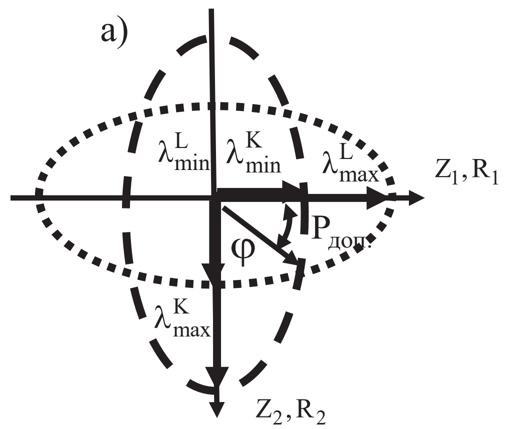

Figure 2. Ellipses of external flexibility (points), and external rigidity (dotted line) of the system for the first state of self-stress

The minimum radius of flexibility is located along the second vector of variations of external forces $\delta_{22} = \lambda_{min}^L = 0.5858l/EA$. Only two states of self-tension of the system correspond to these values.

For the stiffness matrix, we have an ellipse of the limiting stiffness of the structure with the maximum radius $k_{22} = \lambda_{max}^K = 1.707EA/l$ located along the second direction of the reactive forces of the system, and the minimum radius

$k_{11} = \lambda^{K}_{min} = 0.7072EA/l$ located along the first direction (see Figure 2). As we can see, the axes of ellipses are mutually perpendicular.

To demonstrate the physical meaning of the results obtained, we represent the system in the form of an absolutely rigid disk, and we refer the elastic properties of the system to the support rods (see Fig. 3).

The flexibility and stiffness vectors shown in node 1 represent the maximum and minimum values of flexibility (stiffness) of the structure.

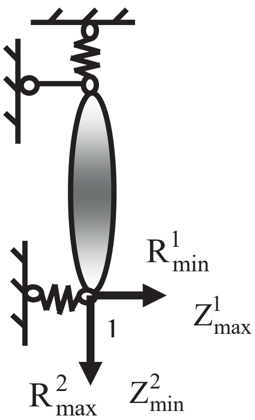

Figure 3. The physical meaning of the eigenvalues of the stiffness and flexibility matrices

In the matrix expressions of the force method and the displacement method, which are consequences of the expressions for the stationarity of the total energy of system (1), we have

$$[K]\{\xi\} = \{\Phi\}_{ex}; [L]\{\Phi\}_{in} = \{\xi\}. \qquad (8)$$

For the symmetric structures that we are considering, the principal vectors coincide with the originally selected axes and we can write that

$$\begin{aligned} &[\lambda^{K}_{\substack{min,\\max}}]\{\vartheta^{K}\} = \{R_{\substack{min,\\max}}\} = -\{\Phi_{ex}\}; \\ &[\lambda^{L}_{\substack{max,\\min}}]\{\vartheta^{L}\} = \{Z_{\substack{max,\\min}}\} = -\{\xi\}. \end{aligned} \qquad (9)$$

In this case, the vectors $\{R_{min}\}$ and $\{Z_{max}\}$ are mutually perpendicular, which makes it possible to assess the values of the limiting stiffness and flexibility of the system, as well as to compare with the acting load. If we assume that the loss of the bearing capacity by the system occurs from the loss of strength according to the first hypothesis, then in the direction of minimum stiffness (1 axis) the reaction of the system according to (9)

$$[\lambda^{K}_{min}]\{\vartheta^{K}\} = \{R_{min}\} = \sigma_1 A. \qquad (10)$$

Then the displacement of the system in the direction of axis 2 will have the maximum possible value

$$\{Z_{max}\} = \{\vartheta^{K}\} = \frac{\sigma_1 l}{0.7071E}. \qquad (11)$$

Substituting the resulting value in the expression for maximum flexibility

$$[\lambda^{L}_{max}]\{\vartheta^{L}\} = \{Z_{max}\} = \frac{\sigma_1 l}{0.7071E}, \qquad (12)$$

we get an expression for the minimum force in a rod located along the first axis

$$\{R_{min}\} = \{\vartheta^{L}\} = \frac{\sigma_1 A}{0.7071 \times 1.414} = \sigma_1 A. \qquad (13)$$

This confirms the correctness of our assumption about the reciprocity of the values of the reactions and displacements of the system in the main directions, as well as the results known from the theory of eigenvalue problems. Let us show what types of stress states of the system are realized for the obtained flexibility (stiffness) matrix of the system. By varying the values of the reaction of the system in the direction of the first axis indicated in Figure 3, we obtain the internal forces in the rods of the structure for the first state of self-stress. So, for the first direction of the unit vector of the internal reaction $\delta\Phi^{1}_{in} = 1$ according to

$$[S] = [B]^{-1}[A]^{T}[L]\{\delta\Phi_{in}\}, \qquad (14)$$

we get forces

$$S_{1-2} = 0,\ S_{1-0} = -S_{1-3} = 0.7071. \qquad (15)$$

That is, the first state of self-stress corresponds to the case of compression of the right inclined bar and the tension left one. There is no force in the vertical middle bar. The values obtained from single impacts have the dimension [unit strength]. For the second state of self-stress, in the case of variation of system nodal reaction in the direction of the second eigenvector $\delta\Phi_{in}^{2} = 1$, we have the following distribution of efforts

$$S_{1-2} = -0.5858,\ S_{1-0} = S_{1-3} = 0.2929, \qquad (16)$$

when two inclined rods are tensioned with the same force, and the vertical one is compressed. Note that the obtained results (15), (16) completely coincide with the values of the efforts given in [29] for solving a similar problem. Here it should be noted that the sign of the efforts is important only for choosing the type of limiting state: loss of strength during tension (compression) or stability of the rod.

The limiting value of internal efforts $S_{1-0}^{cr} = 0,7071P^{cr}$ obtained at the first stage (until the moment the link is turned off from work) is selected in accordance with the formulation of the limiting state. If this is the limiting state in the elastic stage of the work of a brittle material, then $P_{e}^{cr} = A\sigma_{I}$, where the limiting value of the load is the product of the cross-sectional area of the bar A and the equivalent stress according to the first hypothesis of strength σ_{I}. Then the condition for the onset of the limiting state will have the form

$$\sigma_{1-0}^{cr} = \frac{S_{1-0}^{cr}}{A} = 0.7071\sigma_{1}. \qquad (17)$$

If elastic - plastic work of the material is allowed, then the limiting stress σ_{t}, and the value of the limiting load $P_{t}^{cr} = A\sigma_{t}$. Similarly, we do for the cases of bar stability, crack theory and other types of stress state. If we now compare the value of a single external load applied in the main direction of the maximum flexibility of the system (minimum stiffness), then the condition for the equality of its minimum reactive stiffness of the structure gives

$$\{\overline{F}^{cr}\} = -[A]\{S_{max}\}. \qquad (18)$$

Then, taking into account (9), we obtain

$$\{\overline{F}^{cr}\} = -\{\delta\Phi_{in}^{max}\} = -[\lambda_{max}^{K}]\{\vartheta^{K}\}. \qquad (19)$$

Since the vector $\{\vartheta^{K}\}$ coincides in direction with the vector of the unit load $\{\overline{F}^{cr}\}$, the difference between the maximum response of the system and the external load that can be applied in the node in the direction of maximum flexibility of the structure, is equal to the first eigenvalue (6) of the flexibility matrix.

Since the forces in the rods from a unit external load, and the forces corresponding to the vector of maximum flexibility of the system, are applied towards each other, then the internal forces in the rods of the system will differ by the value of their eigenvalues for the specified loading options. Obviously, the result obtained will be valid when symmetric systems are loaded with a load of any size. Therefore, the ellipses of the critical values of flexibility and stiffness can serve as limit curves that limit the permissible values of external loads in the nodes of the structure. The condition of the permissible external load follows from the geometric relationships of the ellipse (see Fig. 2)

$$P_{доп.} \le \frac{R_{max}R_{min}}{\sqrt{R_{max}^{2}(\sin\varphi)^{2} + R_{min}^{2}(\cos\varphi)^{2}}}. \qquad (20)$$

Similarly, you can determine the value of the limiting displacement of node 2. In the considered example, the value of the largest force in the structure bar from the external limiting nodal unit load is determined by (19). The maximum value of the reaction in direction 1 reaches $k_{11} = \lambda^{K}_{min} = 0.7072EA/l$. It is easy to check that for, $\{\delta\Phi in\} = 1$

$$[S] = [C]\ [A]^{T}[L][\lambda^{K}]\left\{\vartheta^{K}_{\substack{max,\\min}}\right\} \qquad (21)$$

we get the same values of efforts presented in (15), (16). That is, with the magnitude of the external load, which coincides with the magnitude of the reaction of the system, the structure does not have a ultimate factor of safety. Note that A.R. Rzhanitsyn gave an example of a system that did not have a safety factor, and it was symmetric [29].

3. EXAMPLES OF SOLUTION OF CONSTRUCTION MECHANICS PROBLEMS FOR SYMMETRIC SYSTEMS

Example 3.1. Structural analysis problem.
For the system shown in Figure 1, with geometric parameters $\alpha = \beta = \pi/4$, $l = 2m$, identical areas of the circular cross-section of rods ($A = 0.785*10^{-4}m^{2}$), diameter $d = 10mm$, modulus of elasticity $E = 2,1*10^{5}MPa$, and yield strength $\sigma_t = 240$ MPa, check the load-carrying capacity of the system, loaded by $P_1 = 10кN,\ P_2 = 20кN$.

Solution 3.1.1 Comparison of external load and system response to variations in external influences.
We find the resultant external forces and the angle of its inclination to the horizontal axis, as well as the normalized value of the load.

$$P = \sqrt{20^{2} + 10^{2}} = 22.36кN,$$
$$tg\varphi = 20/10 = 2, \varphi = 63.43^{0},\ \overline{P} = P/22.36 = 1.0.$$

We calculate the maximum and minimum radii of gyration of the ellipse of the system's reaction according to the method described above.
We get, $R_{max} = 1.707EA\vartheta^{K}/l$, $R_{min} = 0.7072EA\vartheta^{K}/l$. Recall that due to the symmetry of the problem, we have $\vartheta^{K} = \sigma_t l/0.7071E$. Then the values of the reactive components of the system $R_{max} = 2.414\sigma_t A, R_{min} = \sigma_t A$.
We find the dimensionless value of the permissible load according to (14), per unit of movement of the system node, and having the same direction as the given load

$$\overline{P}_{доп.} = 2.205.\ \overline{P}_{доп.} = \frac{P_{доп.}}{\sigma_t A}$$

Then the greatest value of the reactive nodal force created by the system in the direction of the acting external load will be $P_{доп.} = \sigma_t A_1 = 38.622кN$. The value of the resulting specified external load $P = 22.36кN$, which is less than the permissible value obtained by (20). The system will not lose its bearing capacity from a given load. According to the set task, this means that no yield stress will arise in any of the rods of the system. The structure has a margin of safety for the load (or residual resource of the bearing capacity), and can be subjected to the optimization procedure according to the criteria: equal strength of the rods, minimum weight at a given load, etc.

Solution 3.1.2. Comparison of the forces in the rods from a given normalized load and the forces derived from self-tension of the system.
We normalize the external load by the resulting value and obtain the vector of external influences in the form

$$\{\overline{P}\} = \{P\}/22.36 = \{0.4472, 0.8945\}^{T}.$$

We determine the internal forces in the rods of the system from the normalized vector of the external load by the formula

$$\left[S_P\right] = -\left[B\right]^{-1}\left[A\right]^T\left[L\right]\overline{\{P\}}. \qquad (22)$$

We obtain the distribution of efforts from the dimensionless components of the external load vector in the form

$$\left[\overline{S}^P\right] = \begin{vmatrix} 0.5782 \\ 0.524 \\ -0.05423 \end{vmatrix}. \qquad (23)$$

Find the vector of internal efforts from the nodal reaction vector of the system

$$\left[\overline{S}_{\delta\Phi_{in}}\right] = \begin{vmatrix} 1.0 \\ 0.5858 \\ -0.4141 \end{vmatrix}. \qquad (24)$$

From the results obtained, it follows that the force in the 0-1 rod from the external load is less than the force from self-stress in the same rod $\overline{S}_{0-1}^P \succ \overline{S}_{0-1}^{\delta\Phi_{in}}$. Therefore, the specified bar should not lose strength from a given load. Compared efforts can be presented in dimensional units. Internal force from external load in the rod 0-1

$$S_{0-1}^P = 0.5782 \times P = 12.93\text{кN}.$$

The magnitude of the force in the same rod from the nodal reaction of the system

$$S_{0-1}^{\delta\Phi_{in}} = \sigma_t A = 18.84\text{кN}.$$

The permissible maximum value of the force in the rod from the value of the reaction of the system is greater than the force from the given load by 5.91кN.

Solution 3.1.3. Comparison of stresses in rods from a given load and allowable stresses (design resistances) according to the traditional method.

According to the algorithm of matrix structural mechanics, according to the formula

$$\left[S\right] = -\left[C\right]\left[A\right]^T\left[L\right]\{P\}. \qquad (25)$$

Here is $\left[C\right]$ - the matrix of internal flexibility; $\left[A\right]$ - static matrix; $\left[L\right]$ - is the flexibility matrix of the system, $\{P\} = \{10;20\}(\text{kN})$ - is the load vector.

Determine internal forces in rods, structures (in kN)

$$\left[S\right] = \begin{vmatrix} 12.93 \\ 11.72 \\ -1.213 \end{vmatrix}. \qquad (26)$$

$$\sigma_{1-0}^{cr} = \frac{S_{1-0}^{cr}}{A} = 164.7\text{MPa}, \qquad (27)$$

which is less than the yield stress of the rod material. That is, the rod, and therefore the system, will not lose the non-existing ability.

Example 3.2. Structural design problem

For the system shown in Figure 1, with geometrical parameters, $\alpha = \beta = \pi/4$, $l = 2\text{m}$, modulus of elasticity E=2.1*10^5 MPa, and yield point $\sigma_t = 240$ MPa, select the area of a rod of circular cross-section from a given load $P_1 = 10\text{кN},\ P_2 = 20\text{кN}$.

Solution 3.2.1. Traditional method of strength of materials.

Based on the results obtained in Section Solution 3.3, we obtain from expression (23)

$$A = \frac{S_{1-0}^{cr}}{\sigma_t} = 0.5388 * 10^{-4}\text{m}^2. \qquad (38)$$

We get the diameter of the rod $d = 8.3\text{mm}$.

Solution 3.2.2. The use of forces in the rods from the normalized values of the load.

For the obtained value of the forces in the rods (23), we find the cross-sectional area

$$A = \frac{S_{1-0}^{cr}}{\sigma_t} = 0.5388 * 10^{-4}\text{m}^2 \qquad (39)$$

4. THE CONCLUSIONS

The study of the strain energy of deformable systems shows the possibilities of new formulations of problems in structural mechanics, based on the concept of "self-stress" or "self-balance" of a structure. Bringing the stiffness properties of the system to characteristic nodes and then varying the nodal reactions of the structure or nodal displacements makes it possible to construct ellipses of restrictions on possible displacements (nodal reactions) acting on the structure. The study of the limiting values of the strain energy of deformation leads to problems on the eigenvalues for the stiffness (flexibility) matrices of the structure, which makes it possible to assess the limiting possibilities of perceiving external influences before determining the stresses in the elements of the system. The solved examples of structural design and structural analysis tasks of structural mechanics show coincidence of the results with the tasks solved by the classical method.

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