DOI:10.22337/2587-9618-2024-20-1-14-20

MIXED SCHEME OF THE FINITE ELEMENT METHOD FOR CALCULATING CABLE SYSTEMS

Aleksandr V. Baenkhaev
Far Eastern Federal University, Vladivostok, RUSSIA

Abstract: A methodology for constructing variational formulations and solving geometrically nonlinear mixed problems by the finite element method is considered using the example of equilibrium of a flexible tensile thread. The results of solving the model problem are presented.

Keywords: geometrically nonlinear problem, equilibrium of a flexible tensile thread, variational formulation, finite element method

СМЕШАННАЯ СХЕМА МЕТОДА КОНЕЧНЫХ ЭЛЕМЕНТОВ ДЛЯ РАСЧЁТА ТРОСОВЫХ СИСТЕМ

А.В. Баенхаев
Дальневосточный федеральный университет, г. Владивосток, РОССИЯ

Аннотация: Рассматривается методика построения вариационных постановок и решения геометрически нелинейных смешанных задач методом конечных элементов на примере равновесия гибкой растяжимой нити. Приведены результаты решения модельной задачи.

Ключевые слова: геометрически нелинейная задача, равновесие гибкой растяжимой нити, вариационная постановка, метод конечных элементов

INTRODUCTION

The study of the behavior of cable systems is of great importance for solving many practical problems: underwater cable systems (hydrobiotechnical structures [1], fastening systems for floating oil and gas production platforms [2]), in the design and operation of overhead power lines, etc. Moreover, with the development of technology, the range of applications of the results and methods of dynamics and statics of cable systems is steadily expanding.

Differential formulations of problems in the mechanics of flexible threads and methods for solving them are given, for example, in [3]. The stress-strain state of the thread is described by the coordinates of the thread axis, tensile strain and force in the rope. Currently, to solve various problems of calculating cable systems, numerical methods based on differential formulations of problems are used, and modern numerical methods based on variational formulations of problems are not used.

In this work, based on a formal mathematical procedure [4], we show the formation of a variational formulation of the equilibrium problem of a flexible tensile thread and a method for solving a mixed problem by eliminating part of the unknowns not at the level of the entire region (as, for example, with block elimination in a resolving system of equations), and on the subdomain (at the level of the final element). This approach to solving a mixed problem for degenerate problems of structural mechanics was considered in [5]. The same, in essence, are hybrid finite element methods, for example [6].

This division of the problem into two levels (for the subdomain and the entire domain) makes it possible to construct an iterative procedure for solving a nonlinear problem, in which a variational equation at the subdomain (finite element) level is used to clarify the values of the efforts.

DIFFERENTIAL FORMULATION OF THE PROBLEM

Let us consider the equilibrium problem of a flexible thread, the shape of which is determined by the load and connections. The position of the flexible thread axis will be described by a vector of Cartesian coordinates

$$\mathbf{x} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

The stressed state of the thread is characterized by the force in the thread T $(T \geq 0)$. We denote the coordinate along the thread axis by s. The deformed state of the thread is characterized by relative linear deformation ε.

The thread arc differential in the stretched state ds is related to the thread arc differential in the initial state ds_0 by the relation:

$$ds = (1+\varepsilon)ds_0. \tag{1}$$

To form equations that take into account various types of boundary conditions, we will assume that one end of the thread is fixed, and we will apply a load to the other.

Then the differential formulation of the problem of a flexible tensile thread can be written as:

– static equation

$$\frac{d}{ds}\left(T\frac{d\mathbf{x}}{ds}\right) + \frac{\mathbf{p}_m}{1+\varepsilon} + \mathbf{p}_s = 0 \quad \text{or} \quad \frac{d}{ds_0}\left(\frac{T}{1+\varepsilon}\frac{d\mathbf{x}}{ds_0}\right) + \mathbf{p}_m + \mathbf{p}_s(1+\varepsilon) = 0 \tag{2}$$

– geometric equation

$$\left(\frac{d\mathbf{x}}{ds}\right)^T \cdot \frac{d\mathbf{x}}{ds} = 1 \quad \text{or} \quad \left(\frac{d\mathbf{x}}{ds_0}\right)^T \cdot \frac{d\mathbf{x}}{ds_0} = (1+\varepsilon)^2 \tag{3}$$

– physical equation

$$\varepsilon = \frac{T}{EA}, \tag{4}$$

– kinematic boundary condition at $s = 0$

$$\mathbf{x}\Big|_{s=0} = \mathbf{x}_0, \tag{5}$$

– static boundary condition at $s = l$

$$T\frac{d\mathbf{x}}{ds}\bigg|_{s=l} = \mathbf{F}_l \quad \text{or} \quad \frac{T}{1+\varepsilon}\frac{d\mathbf{x}}{ds_0}\bigg|_{s=l} = \mathbf{F}_l \tag{6}$$

Here

$\mathbf{p}_m = \{p_{m1} p_{m2} p_{m3}\}^T$ – vector of distributed rope mass loads;

$\mathbf{p}_s = \{p_{s1} p_{s2} p_{s3}\}^T$ – vector of distributed surface loads on the thread;

$\mathbf{x}_0 = \{x_{01} x_{02} x_{03}\}^T$ – a given coordinate vector at the starting point of the rope,

$\mathbf{F}_l = \{F_{l1} F_{l2} F_{l3}\}^T$ – load at the free end of the thread;

EA – thread stiffness.

The division of the load into mass and surface is made to take into account the reduction in mass load per unit length of the thread during tension.

BASIC INTEGRAL FORMULA

To construct variational formulations of problems that reflect the energy properties of the system under study, it is necessary that the structure of the equations admits a certain integral formula, similar to the formula for integration by parts and based on a differential operator included in the static and geometric equation of the problem.

Let's introduce arbitrary vectors

$$\mathbf{a} = \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} \quad \text{и} \quad \mathbf{b} = \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix}$$

with components – functions of the coordinate s_0. Let these functions have the required degree of smoothness to ensure the ability to perform the required operations. Then the formula for integration by parts is valid for them:

$$\int_0^l \left(\frac{d\mathbf{a}}{ds_0}\right)^T \mathbf{b}\, ds = \mathbf{a}^T \mathbf{b}\Big|_0^l - \int_0^l \mathbf{a}^T \frac{d\mathbf{b}}{ds_0}\, ds_0. \qquad (7)$$

Integral formula (7) transfers the differentiation operation from vector **a** to vector **b**. Let's write vectors **a** and **b** through an arbitrary vector $\overline{\mathbf{x}}$ and function $\overline{T}$ in the form:

$$\mathbf{a} = \overline{\mathbf{x}} \quad \text{и} \quad \mathbf{b} = \frac{\overline{T}}{1+\dfrac{\overline{T}}{EA}} \cdot \frac{d\overline{\mathbf{x}}}{ds_0}$$

and substituting into (7), we obtain the *basic integral formula* for the equilibrium problem of a tensile flexible thread:

$$\begin{aligned} &\int_0^l \frac{\overline{T}}{1+\dfrac{\overline{T}}{EA}} \left(\frac{d\overline{\mathbf{x}}}{ds_0}\right)^T \frac{d\overline{\mathbf{x}}}{ds_0}\, ds_0 = \\ &= \overline{\mathbf{x}}^T \frac{\overline{T}}{1+\dfrac{\overline{T}}{EA}} \frac{d\overline{\mathbf{x}}}{ds_0}\Bigg|_0^l - \\ &-\int_0^l \overline{\mathbf{x}}^T \frac{d}{ds_0}\left(\frac{\overline{T}}{1+\dfrac{\overline{T}}{EA}} \frac{d\overline{\mathbf{x}}}{ds_0}\right) ds_0. \end{aligned} \qquad (8)$$

It is obvious that formula (8) is also valid in the case when the true values of the coordinate vector **x** and force T are substituted into it:

$$\begin{aligned} &\int_0^l \frac{T}{1+\dfrac{T}{EA}} \left(\frac{d\mathbf{x}}{ds_0}\right)^T \frac{d\mathbf{x}}{ds_0}\, ds_0 = \mathbf{x}^T \frac{T}{1+\dfrac{T}{EA}} \frac{d\mathbf{x}}{ds_0}\Bigg|_0^l - \\ &-\int_0^l \mathbf{x}^T \frac{d}{ds_0}\left(\frac{T}{1+\dfrac{T}{EA}} \frac{d\mathbf{x}}{ds_0}\right) ds_0. \end{aligned}$$

Taking into account relations (2)-(6), we obtain:

$$\begin{aligned} &\int_0^l T(1+\varepsilon)\, ds_0 = \\ &= \mathbf{x}^T \mathbf{F}_l\Big|_{s_0=l} - \mathbf{x}_0^T \mathbf{R}_0 + \int_0^l \mathbf{x}^T \left[\mathbf{p}_m + \mathbf{p}_s(1+\varepsilon)\right] ds_0, \end{aligned} \qquad (9)$$

where

$\mathbf{R}_0 = T\dfrac{d\mathbf{x}}{ds}\Big|_{s=0}$ – connection reaction vector.

In this case, the formula takes on a mechanical meaning. Let's imagine two states of the system. First, the thread is folded at the origin of the coordinate system. The second state is when the thread takes on a deformed position that corresponds to the given fastenings and load. The work done by the internal force T to unwind the thread from the first state to the second (the left side of the equality) is equal to the work that external forces do during these movements (the right side of the equality).

VARIATIONAL FORMULATION OF THE PROBLEM

Taking into account that when solving the problem, kinematic boundary conditions are not difficult to take into account, we set a geometrically possible variation to the coordinate vector of the thread **x**:

$$\overline{\mathbf{x}} = \mathbf{x} + \delta\mathbf{x}, \qquad \delta\mathbf{x}\big|_{s_0=0} = 0 \qquad (10)$$

and force T – arbitrary variation:

$$\overline{T} = T + \delta T \qquad (11)$$

Let us substitute them into the main integral identity (8) and after transformations we obtain a *variational equation for the equilibrium problem of an extensible flexible thread* in the form:

$$\frac{1}{2}\int_0^l \frac{\delta T}{1+\dfrac{T}{EA}}\cdot\left[\left(\frac{d\mathbf{x}}{ds_0}\right)^T \frac{d\mathbf{x}}{ds_0}-\left(1+\frac{T}{EA}\right)^2\right]ds_0 + \\ +\int_0^l \left(\frac{\delta d\mathbf{x}}{ds_0}\right)^T \frac{T}{1+\dfrac{T}{EA}}\frac{d\mathbf{x}}{ds_0}ds_0 - \\ -\int_0^l \delta\mathbf{x}^T\left[\mathbf{p}_m+\mathbf{p}_s\cdot\left(1+\frac{T}{EA}\right)\right]ds_0 - \\ -\delta\mathbf{x}^T\mathbf{F}_l\Big|_{s_0=l}=0 \tag{12}$$

or the stationary condition of the functional:

$$B(\mathbf{x},T)= \\ \frac{1}{2}\int_0^l \frac{T}{1+\dfrac{T}{EA}}\left[\left(\frac{d\mathbf{x}}{ds_0}\right)^T \frac{d\mathbf{x}}{ds_0}-\left(1+\frac{T}{EA}\right)^2\right]ds_0 - \\ \int_0^l \mathbf{x}^T\left[\mathbf{p}_m+\mathbf{p}_s\cdot\left(1+\frac{T}{EA}\right)\right]ds_0-\mathbf{x}^T\mathbf{F}_l\Big|_{s=l}=\text{stat.}, \tag{13}$$

defined on the class of vectors $\mathbf{x}$ satisfying kinematic boundary conditions (5).

METHOD FOR SOLVING THE MIXED VARIATIONAL PROBLEM

Functional (13) is mixed. The unknowns are the forces T and the coordinates of the thread axis $\mathbf{x}$. To solve this mixed problem, we use the technique proposed in [5], in which the computational domain is divided into subdomains, and it is assumed that part of the equations of the mixed method is satisfied in each such subdomain separately.

Let's divide the thread into n subareas - elements. We will denote l^r the length of the element with number r (r = 1,2,…,n). Let us assume that the variational equation is satisfied on each element:

$$\int_{l^r}\delta T\frac{1}{1+\dfrac{T}{EA}}\left[\left(\frac{d\mathbf{x}}{ds_0}\right)^T\frac{d\mathbf{x}}{ds_0}-\left(1+\frac{T}{EA}\right)^2\right]ds_0 = \\ =0,\quad r=1,2,...,n \tag{14}$$

Then, to solve the entire variational problem (12), it remains to consider the integral identity:

$$\int_0^l \left(\frac{\delta d\mathbf{x}}{ds_0}\right)^T \frac{T}{1+\dfrac{T}{EA}}\frac{d\mathbf{x}}{ds_0}ds_0 - \\ -\int_0^l \delta\mathbf{x}^T\left[\mathbf{p}_m+\mathbf{p}_s\left(1+\frac{T}{EA}\right)\right]ds_0 - \\ -\delta\mathbf{x}^T\mathbf{F}_l\Big|_{s_0=l}=0, \tag{15}$$

defined on the class of vectors $\mathbf{x}$ satisfying kinematic boundary conditions (5).

FINITE ELEMENT DISCRETIZATION

Let us construct a finite element method scheme based on the variational formulation of problem (14)-(15). Let us define an approximation of the vector $\mathbf{x}$ and force T on the finite element in the form:

$$\mathbf{x}=\mathbf{N}^r\mathbf{X}^r,\qquad T=[1]\cdot t^r \qquad \in l^r, \tag{16}$$

where $\mathbf{N}^r$ is the matrix of shape functions; $\mathbf{X}^r$ – vector of coordinates of nodes of element r; t^r is the value of the force in element r (force T is assumed to be a piecewise constant function).

Let us substitute (16) into the first variational equation (14)

$$\delta t^r\left(\left(\mathbf{X}^r\right)^T\frac{1}{1+\dfrac{t^r}{EA}}\int_{l^r}\left(\frac{d\mathbf{N}^r}{ds_0}\right)^T\frac{\mathbf{N}^r}{ds_0}ds_0\mathbf{X}^r - \\ -\int_{l^r}\left(1+\frac{t^r}{EA}\right)ds_0\right)=0,\quad r=1,2,...,n$$

and, due to the arbitrariness of the variation of efforts, we obtain

$$\left(\mathbf{X}^r\right)^T \mathbf{H}^r\mathbf{X}^r - l^r\left(1+\frac{t^r}{EA}\right)=0,\ r=1,2,...,n, \quad (17)$$

where $\mathbf{H}^r = \dfrac{1}{1+\dfrac{t^r}{EA}}\int_{l^r}\left(\dfrac{d\mathbf{N}^r}{ds_0}\right)^T\left(\dfrac{d\mathbf{N}^r}{ds_0}\right)ds_0.$

The resulting systems of equations at the finite element level (17) contain the unknown forces t^r and the squares of the nodal coordinates $\mathbf{X}^r$.
Let us present the remaining variational equation (15) as a sum over finite elements:

$$\sum_{r=1}^{n}\left[\int_{l^r}\left(\frac{\delta d\mathbf{x}}{ds_0}\right)^T\frac{T}{1+\dfrac{T}{EA}}\frac{d\mathbf{x}}{ds_0}ds_0 - \right.$$
$$-\int_{l^r}\delta\mathbf{x}^T\left(\mathbf{p}_m+\mathbf{p}_s\left(1+\frac{T}{EA}\right)\right)ds_0 -$$
$$\left.-\delta\mathbf{x}^T\mathbf{F}_l\Big|_{s_0=l}\right]=0.$$

Using the approximation of the unknowns in the form (16), we obtain

$$\sum_{r=1}^{n}\left\{\delta\left(\mathbf{X}^r\right)^T\left[\frac{t^r}{1+\dfrac{t^r}{EA}}\int_{l^r}\left(\frac{d\mathbf{N}^r}{ds_0}\right)^T\frac{d\mathbf{N}^r}{ds_0}ds_0\mathbf{X}^r - \right.\right.$$
$$-\int_{l^r}\left(\mathbf{N}^r\right)^T\left(\mathbf{p}_m+\mathbf{p}_s\left(1+\frac{t^r}{EA}\right)\right)ds_0 -$$
$$\left.\left.-\left(\mathbf{N}^r\right)^T\mathbf{F}_l\Big|_{s_0=l}\right]\right\}=0$$

or

$$\sum_{r=1}^{n}\left(\left(\delta\mathbf{X}^r\right)^T\left(\mathbf{H}_t^r\mathbf{X}^r-\mathbf{P}^r\right)\right)=0 \quad (18)$$

where

$\mathbf{H}_t^r = t^r\mathbf{H}^r,$

$$\mathbf{P}^r = \int_{l^r}\left(\mathbf{N}^r\right)^T\left(\mathbf{p}_m+\mathbf{p}_s\left(1+\frac{t^r}{EA}\right)\right)ds_0+\left(\mathbf{N}^r\right)^T\mathbf{F}_l\Big|_{s_0=l}$$

Performing the usual procedure of the finite element method to form the global matrix $\mathbf{H}_t$ and the load vector $\mathbf{P}$, taking into account the connections between the finite elements and kinematic boundary conditions, we come to the solution of a system of algebraic equations for the unknowns – the vector of nodal coordinates $\mathbf{X}$:

$$\mathbf{H}_t\mathbf{X}=\mathbf{P} \quad (19)$$

The system of equations contains unknown coordinates in the nodes of the system $\mathbf{X}$ and unknown forces in the elements t^r (implicitly in the matrix $\mathbf{H}_t$).
The final system of algebraic equations has the form

$$\begin{cases}\left(\mathbf{X}^r\right)^T\mathbf{H}^r\mathbf{X}^r - l^r\left(1+\dfrac{t^r}{EA}\right)=0,\ \ r=1,...,n \\ \mathbf{H}_t\mathbf{X}=\mathbf{P}\end{cases} \quad (20)$$

ITERATION PROCEDURE

Let us reduce equation (17) to the form:

$$\frac{\left(\mathbf{X}^r\right)^T\mathbf{H}^r\mathbf{X}^r}{l^r\left(1+\dfrac{t^r}{EA}\right)}=1,\quad r=1,2,...,n, \quad (21)$$

The iterative procedure for solving the problem is as follows. The initial values of the forces in the $t_{(0)}^r$ elements are set. The system of algebraic equations (19) is solved, linear with respect to the unknown coordinates $\mathbf{X}$. To clarify the values of t^r, relation (21) is used in the form:

$$t_{(k)}^r = t_{(k-1)}^r \cdot \sqrt{\frac{\left(\mathbf{X}^r\right)^T \mathbf{H}^r \mathbf{X}^r}{l^r \left(1+\frac{t_{(k-1)}^r}{EA}\right)}}, \quad r = 1, 2, ..., n, \qquad (22)$$

where k is the iteration number.

With updated values of t^r, the system of algebraic equations (19) is solved again. Iterations continue until the required accuracy of force calculation is achieved.

SOLUTION OF THE MODEL PROBLEM OF EQUILIBRIUM OF FLEXIBLE TENSILE THREAD

Examples of solving model problems for a flexible *inextensible* thread are given in [7]. Calculations have shown that the proposed finite element method for an inextensible thread allows already in the first iterations to achieve the correct position of the ropes in accordance with the specified loads and connections, and about 10-15 iterations are enough to calculate the forces in the thread with acceptable accuracy.

In this paper, the problem of the equilibrium of a flexible *tensile* thread for different values of the stiffness of the *EA* thread is solved. The results are shown in Figure 1.

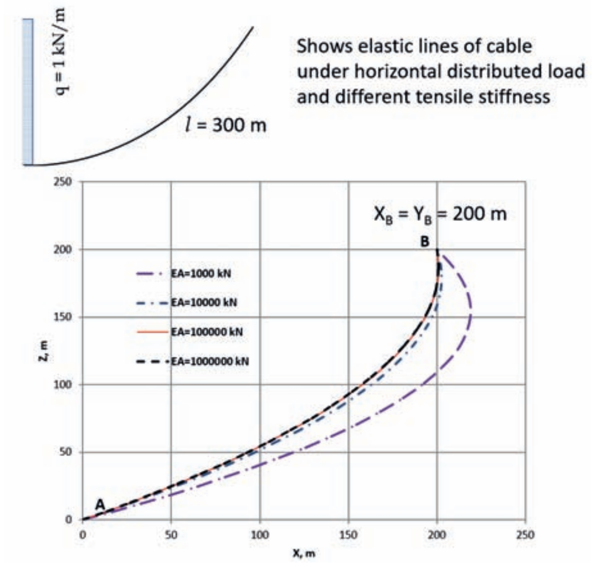

Figure 1. Elastic lines of cable

CONCLUSION

The proposed finite element method allows solving problems of statics of any cable systems and gives good results even with a small number of iterations.

The considered technique for constructing a finite element method for solving the equilibrium problem of a flexible tensile thread can also be applied to solve other nonlinear problems.

REFERENCES

1. **Stotsenko A.A**. Hydrobiotechnical constructions. Vladivostok: Far Eastern State University Press, 1984. 136 p.
2. **Sukhorukov A.L**. The Theory of Underwater Cable Systems and its Engineering Applications. Moscow: FIZMATLIT, 2017. 272 p.
3. **Merkin D.V.** Introduction into the mechanics of flexible string. Moscow: Nauka, 1980. 240 p.
4. **Rozin L.A.** Variation Formulations of the Problems for the Elastic Systems. Leningrad: Leningrad State University Press, 1978. 224 p.
5. **Rozin L.A., Baenkhaev A.V**. Mixed schemes of the finite element method and their application to the solution of the problems of the theory of elasticity // News of the All-Union Scientific Research Institute of Hydraulic Engineering named after B.E. Vedeneeva. 1986. Vol. 194. pp. 79-84.
6. **Pian T.H.H.** State-of-the-art development of hybrid/mixed finite element method // Finite Elements in Analysis and Design. 1995, Vol. 21, Issues 1-2, pp. 5–20. https://doi.org/ 10.1016/0168-874X(95)00024-2
7. **Baenkhaev A.V.** Finite element simulation of flexible ropes. // APCSCE 2018. IOP Conf. Series: Materials Science and Engineering, 2018. Vol. 456, 012101. https://doi.org/ 10.1088 /1757-899X/456/1/012101

СПИСОК ЛИТЕРАТУРЫ

1. **Стоценко А.А.** Гидробиотехнические сооружения. Владивосток: Изд-во Дальневосточного ун-та. 1984. 136 с.
2. **Сухоруков А.Л**. Теория подводных тросовых систем и её инженерные приложения. М.: ФИЗМАТЛИТ. 2017. 272 с.
3. **Меркин Д.В.** Введение в механику гибкой нити. М.: Наука. 1980. 240 с.
4. **Розин Л.А.** Вариационные постановки задач для упругих систем. Л.: Изд-во Ленингр. ун-та. 1978. 224 с.
5. **Розин Л.А., Баенхаев А.В.** Смешанные схемы МКЭ и их применение к решению задач теории упругости // Известия ВНИИГ. Л.: Энергоатомиздат. 1986. Т. 194. С. 79-84.
6. **Pian T.H.H.** State-of-the-art development of hybrid/mixed finite element method // Finite Elements in Analysis and Design. 1995. Vol. 21, Issues 1-2, pp. 5–20. https://doi.org/10.1016/0168-874X(95)00024-2
7. **Baenkhaev A.V**. Finite element simulation of flexible ropes. // APCSCE 2018. IOP Conf. Series: Materials Science and Engineering. 2018, Vol. 456, 012101. https://doi.org/10.1088/1757-899X/456/1/012101

Aleksandr V. Baenkhaev — Ph. D., Associate Professor, Department of Geoinformation Technologies, Far Eastern Federal University (FEFU), 10, Ajax, Russky Island, Vladivostok, 690922, Russia, baenkhaev.av@dvfu.ru, +74232652424

Баенхаев Александр Викторович — кандидат технических наук, доцент департамента геоинформационных технологий, Дальневосточный федеральный университет (ДВФУ), 10, п. Аякс, остров Русский, г. Владивосток, 690922, Россия, baenkhaev.av@dvfu.ru, +74232652424