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ANALYTICAL ANALYSIS OF COMBINED FOUNDATION PLATES, SUBJECTED TO AN ACTION OF ANTISYMMETRIC LOADS

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Abstract: The combined foundation plates, having circular form and consisting of several parts which have different laws of thickness variation, are under study. The constructions under examination are subjected to an action of antisymmetric load distributed along circumferences according to the laws $cos\theta$ or $sin\theta$. Similar problems occur in the cases of seismic and wind loads. The interaction with elastic foundation is taking into account. The conditions of the construction's different parts conjugation are fulfilled. In this work for solution of such problems the analytical approach is used for the first time. The method of compensating loads (MCL) is applied. The solutions are obtained in closed form in terms of Bessel functions. As an example the foundation plate, consisting of two parts, is considered in detail. The inner part of the slab under study has the variable thickness, the outer one has the constant thickness.

Keywords: combined plates, antisymmetric loads, Bessel functions

АНАЛИТИЧЕСКИЙ РАСЧЕТ КОМБИНИРОВАННЫХ ФУНДАМЕНТНЫХ ПЛИТ ПРИ ДЕЙСТВИИ НА НИХ АНТИСИММЕТРИЧНЫХ НАГРУЗОК

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Аннотация: Рассматриваются комбинированные фундаментные плиты, представляющие собой конструкции, имеющие в плане круговую форму и состоящие из нескольких участков с различными законами изменения толщины. Изучается действие на подобные конструкции антисимметричных нагрузок, распределенных по окружностям по законам $cos\theta$ или $sin\theta$. Подобные задачи возникают при расчетах на действие ветровых и сейсмических нагрузок. Учитывается взаимодействие плиты с упругим основанием. Выполняются условия сопряжения отдельных участков. Впервые для решения подобных задач используется аналитическая методика. Используется метод компенсирующих нагрузок. Получены решения в замкнутом виде, выраженные в функциях Бесселя. В качестве примера изучается комбинированная плита, состоящая из двух участков. Внутренняя часть этой конструкции имеет переменную толщину, внешняя часть – постоянную толщину.

Ключевые слова: комбинированные плиты, антисимметричные нагрузки, функции Бесселя

1. INTRODUCTION

The constructions with piecewise thickness occur among modern structures and buildings. Among them foundation slabs of circular form and bottoms of cylindrical reservoirs are to be mentioned. The combined foundation plates subjected to an action of symmetric loads were studied in the works [1] and [2]. The inner parts of these plates had the variable thickness, the outer ones – the constant thickness. The influence of upper parts of structures and interaction with the elastic subgrade were examined. For the first time for analysis of similar constructions in [1] and [2] analytical method was applied. The solutions were obtained in closed forms in terms of Bessel functions.

Analytical methods for plates and shells analysis, in particular, connected with the use of special functions, occur in literature. The monographies [3] and [4] can be named. The book [5] is devoted to calculation problems of orthotropic and isotropic plates of variable thickness of different forms; the plates under study are subjected to an action of complicated loads.

In the present time the modern software allows to investigate various structures and building in detail. The numerical methods, in particular, the finite elements method, are widely utilized.

The buckling problems of orthotropic plates with various boundary conditions were considered in [6]. In the work [7] 3D free vibration problems of cross-ply laminated plates were under study. Vibrations of rectangular orthotropic and isotropic plates of linear thickness were investigated in [8].

Numerical analysis on experimental research on buckling of closed shallow conical shells under external pressure was given in [9]. The work [10] concerns free vibration analysis of a blade of variable thickness with arbitrary boundary conditions. The optimization problems of rectangular plates subjected to thermo-mechanical loads we examined in [11]. Comparative assessment of finite element modelling techniques for wind turbine rotors blades was given in [12]. Nonlinear primary resonance analysis of nanoshells was stated in [13]. Vibration problems of layered plates were considered in [14].

Statics, vibration and stability problems of rectangular plates with various boundary conditions were studied in the work [15]; the approximate analytical equation decomposition method was used.

The publications [16], [17], [18], [19] concerns the computation problems of combined plates with piecewise variable thickness, the exact analytical solutions in terms of special functions were obtained.

The present work considers the combined circular foundation plate, consisting of two parts. The plate's inner part has the variable thickness, the outer one has the constant thickness. The plate is subjected to an action of wind and seismic loads, which are considered as antisymmetric ones which vary according to the laws $cos\,\theta$ or $sin\,\theta$. The action of the construction's upper part on the foundation slab is analyzed. The properties of the elastic subgrade are described by Winkler's model.

2. THE ANALISYS OF THE INNER PART OF THE COMBINED PLATE

The inner part of the combined plate subjected to an action of antisymmetric loads is under study. The flexural rigidity of this part when $x_0 \le x \le x_1$ is varying according to the law

$$D = D_0 x^4 . \qquad (1)$$

The corresponding law of thickness variation is:

$$h = h_0 x^{4/3} . \qquad (2)$$

The antisymmetric distortion of the construction under study is caused by contour or lateral loads, varying according to the laws:

$$\begin{gathered} Q(\theta) = Q_1\, cos\,\theta, \\ M(\theta) = M_1\, cos\,\theta, \\ q(r,\theta) = q(r)\, cos\,\theta, \end{gathered} \qquad (3)$$

where $Q(\theta)$ and $M(\theta)$ are accordingly intensities of a contour transverse force and a contour bending moment.

It is assumed that the properties of the elastic subgrade are described by Winkler's model.

The differential equation, describing the antisymmetric bending of the circular plate of variable thickness, resting on an elastic Winkler's basis, is:

$$D\nabla^2\nabla^2 w+\frac{dD}{dx}\left\{2\frac{d^3w}{dx^3}+\frac{2+\sigma}{x}\frac{d^2w}{dx^2}-\right.$$
$$-\frac{3}{x^2}\frac{dw}{dx}+\left.\frac{3}{x^3}w\right\}+\frac{d^2D}{dx^2}\left\{\frac{d^2w}{dx^2}+\right. \quad (4)$$
$$\left.+\sigma\left(\frac{1}{x}\frac{dw}{dx}-\frac{1}{x^2}w\right)\right\}=(q(x)-cw)r_0^4,$$

where D is the cylindrical rigidity, c - modulus of subgrade, $x=\frac{r}{r_0}$, r_0 is the constant.

In our case the cylindrical rigidity varies according to the law (1). Then the equation (4) comes to the following form:

$$\frac{d^4w}{dx^4}+\frac{10}{x}\frac{d^3w}{dx^3}+\frac{(17+4\sigma)}{x^2}\frac{d^2w}{dx^2}-$$
$$-\frac{3(3-4\sigma)}{x^3}\frac{dw}{dx}+\frac{3(3-4\sigma)}{x^4}w+\frac{\beta^4}{x^4}w=\frac{qr_0^4}{D_0}, \quad (5)$$

where $\beta^4=\frac{cr_0^4}{D_0}$.

Homogeneous differential equation, corresponding to (5), is the equation of the Euler class [20].

By means of substitutions $z=e^z$ and $w=ue^{-z}$ the equation (5) can be reduced to the equation with constant coefficients.

As a result we get the following expression for the deflections of the plate:

$$w=x^{-1}\left[A_1x^{\alpha_1}+A_2x^{\alpha_2}+A_3x^{\alpha_3}+A_4x^{\alpha_4}\right]cos\,\theta, \quad (6)$$

where $\alpha_{1,2,3,4}=\pm\sqrt{2(2-\sigma)\pm\sqrt{2\sigma^2-\beta^4}}$.

It should be marked that $\alpha_1=-\alpha_2$, $\alpha_3=-\alpha_4$.

In the case when the value of β^4 is sufficiently large, the roots of the characteristic equation are mutually complex conjugate.

First the interaction of the inner part of the foundation slab and the upper part of the construction under study will be considered. For this aim the fundamental functions $w_i(x_1)$ for the solutions (6) are to be determined. Their properties are described in [5].

For determination of the fundamental functions the Cauchy functions $Z_i(x_0;x)(i=1,2,3,4)$ are at first to be obtained.

For calculation the properties of Wandermond's determinant are used. As a result we get:

$$Z_1(x_0;x)=\frac{x_0}{2(\alpha_3^2-\alpha_1^2)}\left\{\frac{\alpha_3^2}{\alpha_1}\left[(\alpha_1+1)x_0^{-\alpha_1}x^{\alpha_1-1}+\right.\right.$$
$$\left.+(\alpha_1-1)x_0^{\alpha_1}x^{-\alpha_1-1}\right]-\frac{\alpha_1^2}{\alpha_3}\left[(\alpha_3+1)x_0^{-\alpha_3}x^{\alpha_3-1}+\right.$$
$$\left.\left.+(\alpha_3-1)x_0^{\alpha_3}x^{-\alpha_3-1}\right]\right\};$$
$$Z_2(x_0;x)=\frac{x_0^2}{2(\alpha_3^2-\alpha_1^2)}\left\{\frac{(\alpha_3^2-3\alpha_1)}{\alpha_1}x_0^{-\alpha_1}x^{\alpha_1-1}-\right.$$
$$-\frac{(\alpha_3^2-3\alpha_1)}{\alpha_1}x_0^{-\alpha_1}x^{-\alpha_1-1}-\frac{(\alpha_1^2-3\alpha_3)}{\alpha_3}\times$$
$$\left.\times x_0^{-\alpha_3}x^{\alpha_1-1}+\frac{(\alpha_1^2-3\alpha_3)}{\alpha_3}x_0^{-\alpha_3}x^{-\alpha_3-1}\right\};$$
$$Z_3(x_0;x)=\frac{x_0^3}{2(\alpha_3^2-\alpha_1^2)}\left\{-\frac{x_0^{-\alpha_1}(6+\alpha_1)}{\alpha_1}x^{-\alpha_1-1}+\right. \quad (7)$$
$$+\frac{x_0^{-\alpha_1}(6-\alpha_1)}{\alpha_1}x^{-\alpha_1-1}+\frac{x_0^{-\alpha_3}(6+\alpha_3)}{\alpha_3}x^{\alpha_3-1}-$$
$$\left.-\frac{x_0^{-\alpha_3}(6-\alpha_3)}{\alpha_3}x^{-\alpha_3-1}\right\};$$
$$Z_4(x_0;x)=\frac{x_0^4}{2(\alpha_3^2-\alpha_1^2)}\left\{\frac{x_0^{-\alpha_1}x^{\alpha_1-1}}{\alpha_1}+\right.$$
$$\left.+\frac{x_0^{\alpha_1}x^{-\alpha_1-1}}{\alpha_1}+\frac{x_0^{-\alpha_1}x^{\alpha_3-1}}{\alpha_3}-\frac{x_0^{\alpha_3}x^{-\alpha_3-1}}{\alpha_3}\right\}.$$

As it was mentioned above for the case of sufficiently large β^4 the roots $\alpha_{1,2,3,4}$ are mutually complex conjugate. Then introducing the notations

$$\alpha_{1,2}=\varepsilon\pm\delta i,\ \alpha_{3,4}=-\varepsilon\pm\delta i$$

we write the general solution of the homogeneous differential equation, corresponding to (5), in the following form:

$$w(x) = x^{\varepsilon}\left[B_1 \,sin(\delta\, ln\, x) + B_2 \cos(\delta\, ln\, x)\right] + \\ + x^{-\varepsilon}\left[B_3 \,sin(\delta\, ln\, x) + B_4 \cos(\delta\, ln\, x)\right]. \tag{8}$$

Introducing the notation

$$sin(\delta\, ln\, x) = \alpha(x),\ \cos(\delta\, ln\, x) = \gamma(x).$$

We cite below one of Cauchy functions as an example:

$$Z_4(x_0;x) = \frac{x_0^4}{4\varepsilon\delta\left(\varepsilon^2+\delta^2\right)}\{x_0^{-\varepsilon}x^{\varepsilon}\times \\ \times\{-[\gamma(x_0)\delta+\varepsilon\alpha(x_0)]\gamma(x)+[\gamma(x_0)\varepsilon-\delta\alpha(x_0)]\times \\ \times\alpha(x)\}-x_0^{\varepsilon}x^{-\varepsilon}\{[-\gamma(x_0)\delta+\varepsilon\alpha(x_0)]\gamma(x)- \\ -[\varepsilon\gamma(x_0)+\delta\alpha(x_0)]\alpha(x)\}\}. \tag{9}$$

Further, using (7) and taking into account the properties of fundamental functions, we get the following expressions:

$$w_1(x_0;x) = Z_1(x_0;x) + \frac{\sigma}{x_0^2}Z_3(x_0;x) - \\ -\frac{(\sigma+2)}{x_0^3}Z_4(x_0;x); \\ w_2(x_0;x) = Z_2(x_0;x) - \frac{\sigma}{x_0}Z_3(x_0;x) + \\ +\frac{(\sigma+2)}{x_0^2}Z_4(x_0;x); \\ w_3(x_0;x) = -\frac{1}{D(x_0)}\left\{Z_2(x_0;x) - \left[\frac{1}{x_0} + \right.\right. \\ \left.\left. +\frac{1}{D(x_0)}\frac{dD(x_0)}{dx}\right]Z_4(x_0;x)\right\}; \\ w_4(x_0;x) = -\frac{1}{D(x_0)}Z_4(x_0;x). \tag{10}$$

It is assumed that the influence of the construction's upper part is transmitted on the foundation slab inner part as moments $M_A \,cos\,\theta$ and forces $Q_A \,cos\,\theta$, distributed along circumference with the radius x_A, where $x_0 < x_A < x_1$. As a result we can write the following expression for the deflection:

$$w_{inner} = \left[w_0 w_1(x_0;x) + \vartheta_0 r_0 w_2(x_0;x) - \\ - M_0 r_0^2 w_3(x_0;x) - Q_0 r_0^3 w_4(x_0;x) + \\ + M_A r_0^2 w_3(x_A;x) + Q_A r_0^3 w_4(x_A;x)\right] cos\,\theta = \\ = \left[w_I(x_0;x) + M_A r_0^2 w_3(x_A;x) + \\ + Q_A r_0^3 w_4(x_A;x)\right] cos\,\theta = \left[w_I(x_0;x) + \\ + w_{II}(x_A;x)\right] cos\,\theta, \tag{11}$$

where w_0, ϑ_0, M_0, Q_0 are accordingly the deflection, the angle, the moment, the force when $x = x_0$.

When for computation we assume that the influence of the upper part is transmitted on the foundation inner part as the load $q(x)\,cos\,\theta$, distributed over surface of the ring $x_A \le x \le x_B$. The result can be obtained by integrating in (11) the term containing Q_A.

3. THE COMPUTATION OF THE OUTER PART OF THE PLATE. THE BASIC SOLUTION

We go to the consideration of the outer part of the ring plate, subjected to an action of antisymmetric load. The thickness of the foundation slab part under study is constant. The general solution of the differential equation describing the antisymmetric bending of a circular plate resting on an elastic Winkler's subgrade is:

$$w = \left(A_1 u_1(x) + A_2 v_1(x) + A_3 f_1(x) + \\ + A_4 g_1(x)\right) cos\,\theta, \tag{12}$$

where u_1, v_1, f_1, g_1 are Bessel functions [4], [21]; A_1, A_2, A_3, A_4 - coefficients; $x = \frac{r}{l}$, $l = \sqrt[4]{\frac{D}{c}}$.

Method of compensating loads (MCL) [1], [2], [4] is used for the receiving of the solution. The basic and the compensating solutions are to be determined.

First the basic solution will be obtained.

When the circular plate is loaded by a concentrated force applied at the centre, the solution has the following form:

$$w = \frac{Pl^2}{4D} f_0(x). \tag{13}$$

The solution (13) mentioned above is the fundamental influence function when $P = 1$. Integrating this solution, we can obtain the basic solutions for consideration of several problems.

Let us assume that this part of the plate is subjected to the action of the load $q\cos\theta$, distributed along the circumference with the reduced radius α.

Further, the principle of addition of the effects will be used.

For the determination of the deflection of the point with the coordinates x and φ the following expression will be used:

$$w = \frac{q\alpha l^2}{4D} \times \times \int_0^{2\pi} f_0\left(\sqrt{\alpha^2 + x^2 - 2\alpha x \cos(\theta - \varphi)}\right) \cos\theta d\theta. \tag{14}$$

For this integral calculation the formula of the cylindrical functions addition is used:

$$Z_0\left(\sqrt{\alpha^2 + x^2 - 2\alpha x \cos(\theta - \varphi)}\right) = = 2\sum_{n=0}^{1}{}' J_n(\alpha) Z_m(x) \cos n(\theta - \varphi). \tag{15}$$

Here the symbol $'$ denotes that when $n = 0$ the coefficient $\frac{1}{2}$ should be introduced. The expression (15) is valid when $\alpha < x$. When $\alpha > x$ in the right part of the expression (15) the terms α and x must be replaced.

Assuming

$$Z_0 = H_0^{(1)}\left(\sqrt{i}\sqrt{\alpha^2 + x^2 - 2\alpha x \cos(\theta - \varphi)}\right)$$

we get the following expressions fulfilling the integration and separating the real and the imaginary parts:

when $x \le \alpha$

$$w = w_{\rm I}^{(0)} = \frac{\pi\alpha q l^3}{2D}[u_1(x) f_1(\alpha) - v_1(x) g_1(\alpha)] \times \times \cos\theta, \tag{16}$$

when $x \ge \alpha$

$$w = w_{\rm II}^{(0)} = \frac{\pi\alpha q l^3}{2D}[u_1(\alpha) f_1(x) - v_1(\alpha) g_1(x)] \times \times \cos\varphi. \tag{17}$$

Using the Wronskian of the Bessel equation it can be shown that the received solutions satisfy to the conjugation conditions when $x = \alpha$.

The problem of infinite plate subjected to an action of the load $\psi(x)\cos\theta$, where $\psi(x)$ is the given function, is considered for the obtaining of the basic solution.

When the construction under study is subjected to an action of the load $q(\varphi)$ distributed along the circumference with the reduced radius α, we will present the load in the following form:

$$q = a_1 \cos\theta + b_1 \sin\theta$$

and using the formulae (16) and (17) we get the solutions:

when $\alpha \le x$

$$w=\frac{\pi\alpha l^3}{2D}\times$$
$$\times\{a_1[u_1(\alpha)f_1(x)-v_1(\alpha)g_1(x)]sin\varphi+ \quad (18)$$
$$+b_1[u_1(\alpha)f_1(x)-v_1(\alpha)g_1(x)]cos\varphi\},$$

when $\alpha\geq x$

$$w=\frac{\pi\alpha l^3}{2D}\times$$
$$\times\{a_1[u_1(x)f_1(\alpha)-v_1(x)g_1(\alpha)]sin\varphi+ \quad (19)$$
$$+b_1[u_1(x)f_1(\alpha)-v_1(x)g_1(\alpha)]cos\varphi\}.$$

For the basic solution receiving for the load $q=F(x,\theta)$ the formulae (18) and (19) should be used. The load can be represented in the form:

$$q=x(a_1\,cos\,\theta+b_1\,sin\,\theta) \quad (20)$$

or

$$q=x^{-1}(a_1\,cos\,\theta+b_1\,sin\,\theta). \quad (21)$$

The integrating for these cases is highly simple. The formulae for Bessel functions derivation are:

$$\left.\begin{aligned}&\frac{d}{dz}\left[z^nJ_n(z)\right]=z^nJ_{n-1}(z),\\&\frac{d}{dz}\left[z^nH_n^{(1)}(z)\right]=z^nH_{n-1}^{(1)}(z).\end{aligned}\right\} \quad (22)$$

Let us assume $z=x\sqrt{i}$. Fulfilling integration and separating real and imaginary parts, we get the following expression:

$$\left.\begin{aligned}&\int xu_0(x)dx=\frac{x}{\sqrt{2}}[u_1(x)+v_1(x)],\\&\int xv_0(x)dx=-\frac{x}{\sqrt{2}}[u_1(x)-v_1(x)],\end{aligned}\right\} \quad (23)$$

$$\left.\begin{aligned}&\int xf_0(x)dx=\frac{x}{\sqrt{2}}[f_1(x)+g_1(x)],\\&\int xg_0(x)dx=-\frac{x}{\sqrt{2}}[f_1(x)-g_1(x)].\end{aligned}\right\} \quad (24)$$

The solution for the case of the load varying according to the law (20) and distributed over a surface of the ring $\alpha_1\leq x\leq\alpha_2$ is to be received. For this aim the expressions (16) and (17) are to be integrated. In the mentioned expressions the load q must be replaced by the elementary load $q_0\alpha dz=q_0\alpha d\alpha$. It should be marked that the multiplier $\frac{l^4}{D}$, which is equal to $\frac{1}{c}$, appears.

Further we integrate the received solutions within the limits $\alpha=\alpha_1$ and $\alpha=\alpha_2$.

When $x\leq\alpha_1<\alpha_2$ the solution has the following form:

$$w=\frac{\pi q}{2c\sqrt{2}}\Big(\big\{\alpha_2^2[f_2(\alpha_2)+g_2(\alpha_2)]-$$
$$-\alpha_1^2[f_2(\alpha_1)+g_2(\alpha_1)]\big\}u_1(x)+ \quad (25)$$
$$+\big\{\alpha_2^2[f_2(\alpha_2)-g_2(\alpha_2)]-\alpha_1^2\times$$
$$\times[f_2(\alpha_1)-g_2(\alpha_1)]\big\}v_2(x)\Big)cos\varphi;$$

when $x\geq\alpha_2>\alpha_1$

$$w=\frac{\pi q}{2c\sqrt{2}}\Big(\big\{\alpha_2^2[u_2(\alpha_2)+v_2(\alpha_2)]-$$
$$-\alpha_1^2[u_2(\alpha_1)+v_2(\alpha_1)]\big\}f_1(x)+ \quad (26)$$
$$+\big\{\alpha_2^2[u_2(\alpha_2)-v_2(\alpha_2)]-\alpha_1^2\times$$
$$\times[u_2(\alpha_1)-v_2(\alpha_1)]\big\}g_1(x)\Big)cos\varphi;$$

when $\alpha_1\leq x\leq\alpha_2$

$$w = \frac{\pi q}{2\sqrt{2}c}\{\alpha_2^2[f_2(\alpha_2)+g_2(\alpha_2)]\times \times u_1(x)+\alpha_2^2[f_2(\alpha_2)-g_2(\alpha_2)]\times \times v_1(x)-\alpha_1^2[u_2(\alpha_1)+v_2(\alpha_1)]f_1(x)- -\alpha_1^2[u_2(\alpha_1)-v_2(\alpha_1)]g_1(x)-2x\frac{\sqrt{2}}{\pi}\}cos\,\varphi. \quad (27)$$

These formulae give the expressions for the deflections of the foundation plate.

4. THE OUTER PART OF THE PLATE. THE COMPENSATING SOLUTION

The compensating solution should be determined for the plate's outer part investigation. This solution satisfies to boundary conditions and with the basic solutions satisfies to the resolving differential equation of the problem. For the foundation slab under study, which is subjected to an action of antisymmetric loads, the case of free edges is more actual. Below, however, another cases of boundary conditions will be considered.

The sought solution can be represented as the result of an action of two compensating loads q_1 and q_2, which are applied along the concentric circumferences with the reduced radius α_1 and α_2. Let us note the reduced radius of the outer contour as β.

a) First it is assumed that the outer boundary of the plate, subjected to an action of antisymmetric load, is clamped. For example it is possible for the plate when $x=\beta$ connected with the upper part of the construction which is represented by the shell of the rotation and subjected to an action of seismic or wind loads.

Let us present the deflection w_0 and the angle $\frac{\partial w_0}{\partial x}$ from the basic solution in the following form:

$$w_0 = A_1\,sin\,\varphi + B_1\,cos\,\varphi, \quad (28)$$

$$\frac{\partial w_0}{\partial x} = C_1\,sin\varphi + D_1\,cos\varphi. \quad (29)$$

The compensating loads q_1 and q_2, which with the sum of the basic solution are satisfying to the case of the clamped boundary, are to be determined.

As a result we get the following system of equations:

$$\alpha_1\int_0^{2\pi} q_1(\theta) f_0\left(\sqrt{\alpha_1^2+\beta^2-2\alpha_1\beta cos(\theta-\varphi)}\right)d\theta + +\alpha_2\int_0^{2\pi} q_2(\theta) f_0\left(\sqrt{\alpha_2^2+\beta^2-2\alpha_2\beta cos(\theta-\varphi)}\right)d\theta + +\frac{4D}{l^2}w_0 = 0, \quad (30)$$

$$\alpha_1\int_0^{2\pi} q_1(\theta)\frac{\partial}{\partial x} f_0\left(\sqrt{\alpha_1^2+\beta^2-2\alpha_1\beta cos(\theta-\varphi)}\right)d\theta + +\alpha_2\int_0^{2\pi} q_2(\theta)\frac{\partial}{\partial x} f_0\left(\sqrt{\alpha_2^2+\beta^2-2\alpha_2\beta cos(\theta-\varphi)}\right)d\theta + +\frac{4D}{l^2}\frac{\partial w_0}{\partial x} = 0. \quad (31)$$

The given above integral equations (30) and (31) express the conditions of equality of the values of deflections and angles to zero at the contour.

Further using the formulae of Bessel functions addition the integration is fulfilled [21]. The functions q_1 and q_2 are to be expanded into trigonometrical series. As a result we get a system of algebraic equations with respect to the coefficients of these series. Here this system is not cited.

The compensating solution can be written in the following form when $x \le \alpha$:

$$w_k = (a_1\,sin\,\varphi + b_1\,cos\,\varphi)u_1(x) + +(c_1\,sin\,\varphi + d_1\,cos\,\varphi)v_1(x). \quad (32)$$

The coefficients of the expression (32) can be determined from the following equations, using (30) and (31):

$$\left.\begin{aligned} a_1 u_1(\beta) + c_1 v_1(\beta) + A_1 = 0, \\ a_1 u_1'(\beta) + c_1 v_1'(\beta) + C_1 l = 0; \end{aligned}\right\} \quad (33)$$

$$\left.\begin{aligned} b_1 u_1(\beta) + d_1 v_1(\beta) + B_1 = 0, \\ b_1 u_1'(\beta) + d_1 v_1'(\beta) + D_1 l = 0. \end{aligned}\right\} \quad (34)$$

Solving these systems, we obtain:

$$a_1 = -\frac{C_1 l v_1(\beta) - A_1 v_1'(\beta)}{v_1(\beta) u_1'(\beta) - u_1(\beta) v_1'(\beta)}, \quad (35)$$

$$b_1 = -\frac{D_1 l v_1(\beta) - B_1 v_1'(\beta)}{v_1(\beta) u_1'(\beta) - u_1(\beta) v_1'(\beta)}, \quad (36)$$

$$c_1 = \frac{C_1 l u_1(\beta) - A_1 u_1'(\beta)}{v_1(\beta) u_1'(\beta) - u_1(\beta) v_1'(\beta)}, \quad (37)$$

$$d_1 = \frac{D_1 l u_1(\beta) - B_1 u_1'(\beta)}{v_1(\beta) u_1'(\beta) - u_1(\beta) v_1'(\beta)}. \quad (38)$$

We receive introducing (35)-(38) into expression (32):

$$\begin{aligned} w_k = & \frac{1}{v_1(\beta) u_1'(\beta) - u_1(\beta) v_1'(\beta)} \times \\ & \times (\{[C_1 l v_1(\beta) - A_1 v_1'(\beta)] sin\,\varphi + \\ & + [D_1 l v_1(\beta) - B_1 v_1'(\beta)] cos\,\varphi\} u_1(x) - \\ & - \{[C_1 l u_1(\beta) - A_1 u_1'(\beta)] sin\,\varphi + \\ & + [D_1 l u_1(\beta) - B_1 u_1'(\beta)] cos\,\varphi\} v_1(x)). \end{aligned} \quad (39)$$

b) The case when the outer boundary is simply supported is under study. The deflections are represented in the form (28). The radial bending moments on the contour are represented in the form:

$$M_r = M_1\, sin\,\varphi + L_1\, cos\,\varphi. \quad (40)$$

The following notations are introduced:

$$u_1^{(M)}(\beta) = -u_1(\beta) - \frac{1-\sigma}{\beta}\left[v_1'(\beta) - \frac{1}{\beta} v_1(\beta)\right], \quad (41)$$

$$v_1^{(M)}(\beta) = v_1(\beta) - \frac{1-\sigma}{\beta}\left[u_1'(\beta) - \frac{1}{\beta} u_1(\beta)\right], \quad (42)$$

$$f_1^{(M)}(\beta) = -f_1(\beta) - \frac{1-\sigma}{\beta}\left[g_1'(\beta) - \frac{1}{\beta} g_1(\beta)\right], \quad (43)$$

$$g_1^{(M)}(\beta) = g_1(\beta) - \frac{1-\sigma}{\beta}\left[f_1'(\beta) - \frac{1}{\beta} f_1(\beta)\right]. \quad (44)$$

The following compensating solution is obtained:

$$\begin{aligned} w_k = & -\frac{1}{u_1(\beta) u_1^{(M)}(\beta) - v_1(\beta) v_1^{(M)}(\beta)} \times \\ & \times \Bigg(\Big\{ [M_1 \frac{l^2}{D} v_1(\beta) + A_1 u_1^{(M)}(\beta)] sin\,\varphi + \\ & + [L_1 \frac{l^2}{D} v_1(\beta) + B_1 u_1^{(M)}(\beta)] cos\,\varphi \Big\} u_1(x) - \\ & - \Big\{ [M_1 \frac{l^2}{D} u_1(\beta) + A_1 v_1^{(M)}(\beta)] sin\,\varphi + \\ & + [L_1 \frac{l^2}{D} u_1(\beta) + B_1 v_1^{(M)}(\beta)] cos\,\varphi \Big\} v_1(x) \Bigg). \end{aligned} \quad (45)$$

c) The case of free contour of the fundament plate is under consideration. It is known that the bending moment and the reduced transverse force are equal to zero at the contour.
The boundary bending moment is defined by the expression (40). The boundary reduced transverse force will be written in the form:

$$Q - \frac{dH}{dS} = N_1\, sin\,\varphi + O_1\, cos\,\varphi. \quad (46)$$

The following notation, taking into account (40), and (46) are introduced:

$$u_1^{[Q]}(\beta) = -u_1'(\beta) - (1-\sigma)\frac{1}{\beta^2}\left[v_1'(\beta) - \frac{v_1(\beta)}{\beta}\right], \quad (47)$$

$$v_1^{[Q]}(\beta) = v_1'(\beta) - (1-\sigma)\frac{1}{\beta^2}\left[u_1'(\beta) - \frac{u_1(\beta)}{\beta}\right], \quad (48)$$

$$f_1^{[Q]}(\beta) = -f_1'(\beta) - (1-\sigma)\frac{1}{\beta^2}\left[g_1'(\beta) - \frac{g_1(\beta)}{\beta}\right], \quad (49)$$

$$g_1^{[Q]}(\beta) = g_1'(\beta) - (1-\sigma)\frac{1}{\beta^2}\left[f_1'(\beta) - \frac{f_1(\beta)}{\beta}\right]. \quad (50)$$

As a result we get the compensating load:

$$w_k = -\frac{l^2}{D}\frac{1}{u_1^{[Q]}(\beta)v_1^{[M]}(\beta) - u_1^{[M]}(\beta)v_1^{[Q]}(\beta)} \times \\ \times \Big\{\Big([lN_1 u_1^{[M]}(\beta) - M_1 u_1^{[Q]}(\beta)]\sin\varphi + \\ + [lO_1 u_1^{[M]}(\beta) - L_1 v_1^{[Q]}(\beta)]\cos\varphi\Big) u_1(x) - \\ - \Big([lN_1 u_1^{[M]}(\beta) - M_1 v_1^{[Q]}(\beta)]\sin\varphi + \\ + [lO_1 v_1^{[M]}(\beta) - L_1 v_1^{[Q]}(\beta)]\cos\varphi\Big) v_1(x)\Big\}. \quad (51)$$

The solution for the outer part, according to the MCL, is represented by the sum of the basic and the compensating solutions [22], [23]:

$$w = w_0 + w_k. \quad (52)$$

The solution of the problem under study can be presented as the sum of the expressions (11) and (52) for the study of bend of the whole combined plate.

5. THE CONCLUSIONS

The work receives the exact solution of the antisymmetric deformation of the combined plate which inner part has variable thickness and the outer one – the constant thickness. The solution is obtained in closed form in terms of Bessel functions. Method of compensating loads (MCL) is used. An influence of the construction's upper part is taken into account. The received results can be used for the analysis of the combined plates subjected to an action of seismic and wind loads.

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