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LOCALIZATION OF SOLUTION OF THE PROBLEM OF THREE-DIMENSIONAL THEORY OF ELASTICITY WITH THE USE OF B-SPLINE DISCRETE-CONTINUAL FINITE ELEMENT METHOD

Marina L. Mozgaleva, Pavel A. Akimov

National Research Moscow State University of Civil Engineering, Moscow, RUSSIA

Abstract: Localization of solution of the problem of three-dimensional theory of elasticity with the use of B-spline discrete-continual finite element method (specific version of wavelet-based discrete-continual finite element method) is under consideration in the distinctive paper. The original operational continual and discrete-continual formulations of the problem are given, some actual aspects of construction of normalized basis functions of a B-spline are considered, the corresponding local constructions for an arbitrary discrete-continual finite element are described, some information about the numerical implementation and an example of analysis are presented.

Keywords: localization, wavelet-based discrete-continual finite element method, B-spline discrete-continual finite element method, discrete-continual finite element method, finite element method, B-spline, numerical solution, three-dimensional theory of elasticity, structural analysis

ЛОКАЛИЗАЦИЯ РЕШЕНИЯ ТРЕХМЕРНОЙ ЗАДАЧИ ТЕОРИИ УПРУГОСТИ НА ОСНОВЕ ВЕЙВЛЕТ-РЕАЛИЗАЦИИ ДИСКРЕТНО-КОНТИНУАЛЬНОГО МЕТОДА КОНЕЧНЫХ ЭЛЕМЕНТОВ С ИСПОЛЬЗОВАНИЕМ B-СПЛАЙНОВ

М.Л. Мозгалева, П.А. Акимов

Национальный исследовательский Московский государственный строительный университет, г. Москва, РОССИЯ

Аннотация: В настоящей статье рассматривается локализация решения трехмерной задачи теории упругости на основе вейвлет-реализации дискретно-континуального метода конечных элементов с использованием B-сплайнов. Приведены исходные операторные континуальная и дискретно-континуальная постановки задачи, рассмотрены некоторые актуальные вопросы построения нормализованных базисных функций B-сплайна, описаны соответствующие локальные построения для произвольного дискретно-континуального конечного элемента, представлены некоторые сведения о численной реализации и пример расчета.

Ключевые слова: локализация, вейвлет-реализация метода конечных элементов, дискретно-континуальный метод конечных элементов, метод конечных элементов, B-сплайны, численное решение, трехмерная задача теории упругости, расчеты конструкций

INTRODUCTION

Numerical solution of the problem of three-dimensional theory of elasticity with the use of B-spline version of so-called discrete-continual finite element method (DCFEM) [1-4] has been already considered in [1]. Systematic literature review has been presented in [1] as well. In this regard, let us immediately move on to the main material of the distinctive paper.

1. FORMULATIONS OF THE PROBLEM

Let the constancy of the parameters of the problem be in the direction corresponding to x_3 (main direction). The operational formulation of the problem with the use of so-called method of extended domain [5], taking into account the selection of the main direction, is determined by the equation [6]:

$$L\,\bar{u} = \bar{\tilde{F}}, \quad 0 \le x_1 \le \ell_1,\ 0 \le x_2 \le \ell_2,\ 0 \le x_3 \le \ell_3, \quad (1.1)$$

where we have

$$L = -L_{vv}\,\partial_3^2 + L_{uv}\,\partial_3 + L_{uu}; \quad (1.2)$$

$$L_{uu} = \partial_1^* \begin{bmatrix} 2\mu+\lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu \end{bmatrix} \partial_1 + \partial_1^* \begin{bmatrix} 0 & \lambda & 0 \\ \mu & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \partial_2 + \partial_2^* \begin{bmatrix} 0 & \mu & 0 \\ \lambda & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \partial_1 + \partial_2^* \begin{bmatrix} \mu & 0 & 0 \\ 0 & 2\mu+\lambda & 0 \\ 0 & 0 & \mu \end{bmatrix} \partial_2; \quad (1.3)$$

$$L_{uv} = \partial_1^* \begin{bmatrix} 0 & 0 & \lambda \\ 0 & 0 & 0 \\ \mu & 0 & 0 \end{bmatrix} + \partial_2^* \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \lambda \\ 0 & \mu & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & \mu \\ 0 & 0 & 0 \\ \lambda & 0 & 0 \end{bmatrix} \partial_1 - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \mu \\ 0 & \lambda & 0 \end{bmatrix} \partial_2; \quad (1.4)$$

$$L_{vv} = \begin{bmatrix} \mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & 2\mu+\lambda \end{bmatrix}; \quad (1.5)$$

$$\bar{\tilde{F}} = \theta \bar{F} + \delta_\Gamma \bar{f}; \quad (1.6)$$

$$\theta(x_1, x_2, x_3) = \begin{cases} 1, & (x_1, x_2, x_3) \in \Omega \\ 0, & (x_1, x_2, x_3) \notin \Omega; \end{cases} \quad (1.7)$$

$$\delta_\Gamma(x_1, x_2, x_3) = \partial\theta / \partial\bar{n}; \quad (1.8)$$

ℓ_1, ℓ_2, ℓ_3 are corresponding dimensions of extended domain (linear dimensions of considering structure); $x = (x_1,\ x_2,\ x_3)$; $x_1,\ x_2,\ x_3$ are Cartesian coordinates; μ and λ are Lame parameters; Ω is the domain, occupied by structure;

$$\Omega = \{\, (x_1,\ x_2):\ \ 0 < x_1 < \ell_1; \ \ 0 < x_2 < \ell_2; \ \ 0 < x_3 < \ell_3 \,\}; \quad ; (1.9)$$

$\theta(x_1, x_2, x_3)$ is characteristic function of domain Ω; $\delta_\Gamma = \delta_\Gamma(x_1, x_2, x_3)$ is the delta function of boundary $\Gamma = \partial\Omega$; $\bar{n} = [\, n_1 \ \ n_2 \ \ n_3 \,]^T$ is boundary normal vector; $\bar{F}$ is the load vector in domain Ω; $\bar{f}$ is the corresponding boundary load vector; $\bar{u}$ is the vector of displacements (unknown vector function),

$$\bar{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}; \quad (1.10)$$

$\partial_s = \partial / \partial x_s,\ s = 1, 2$; $\partial_s^* = -\partial / \partial x_s,\ s = 1, 2$.

Let us introduce the following notations

$$\bar{v} = \partial_3 \bar{u} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}; \quad (1.11)$$

$$\bar{u}' = \partial_3 \bar{u}; \quad \bar{v}' = \partial_3 \bar{v}. \quad (1.12)$$

Thus we can rewrite (1.1):

$$L_{uu}\,\bar{u} + L_{uv}\bar{v} - L_{vv}\partial_3 \bar{v} = \bar{\tilde{F}}. \quad (1.13)$$

Finally we obtain system of differential equations with operational coefficients:

$$\begin{cases} \overline{u}' = \overline{v} \\ \overline{v}' = L_{vv}^{-1} L_{uu} \overline{u} + L_{vv}^{-1} L_{uv} \overline{v} - L_{vv}^{-1} \overline{\overline{F}} \end{cases} \qquad (1.14)$$

or

$$\overline{z}' = \widetilde{L}\overline{z} + \overline{\overline{F}}\ , \qquad (1.15)$$

where

$$\widetilde{L} = \begin{bmatrix} 0 & E \\ L_{vv}^{-1} L_{uu} & L_{vv}^{-1} L_{uv} \end{bmatrix}; \quad \overline{\overline{F}} = \begin{bmatrix} 0 \\ -L_{vv}^{-1} \overline{\overline{F}} \end{bmatrix}; \qquad (1.16)$$

$$\overline{z} = \begin{bmatrix} \overline{u} \\ \overline{v} \end{bmatrix}; \quad E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \qquad (1.17)$$

The system of equations (1.15) is supplemented by boundary conditions, which are set in sections with coordinates $x_3^1 = 0$ and $x_3^2 = \ell_3$.

2. SOME GENERAL ASPECTS OF FINITE ELEMENT APPROXIMATION

The discrete component of the numerical solution is represented by directions along the axes x_1 and x_2. The fulfilment within the element (rectangular element) is the same for all components of the vector function $\overline{U}$ (see (1.4)) Let's divide the interval $(0,\ \ell_1)$ into $N_{e,1}$ parts and interval $(0,\ \ell_2)$ into $N_{e,2}$ parts (sides of elements); $h_{e,1} = \ell_1 / N_{e,1}$ is the length of the side of the element in the direction x_1; $h_{e,2} = \ell_2 / N_{e,2}$ is the length of the side of the element in the direction x_2. Let us number the elements with a multi-index $i_e = (i_{e,1},\ i_{e,2})$, where $i_{e,1} = 1,\ ...,\ N_{e,1}$ is the number in the first direction (x_1); $i_{e,2} = 1,\ ...,\ N_{e,2}$ is the number in the second direction (x_2).

We will also divide each side of the element $i_e = (i_{e,1},\ i_{e,2})$ into and parts $N_{h,e,1}(i_{e,1})$ and $N_{h,e,2}(i_{e,2})$, respectively. It should be noted that $N_{h,e,1}(i_{e,1})$ and $N_{h,e,2}(i_{e,2})$ are of greater importance on the localization (of solution) elements than on the other elements. Let us denote the number of nodes on the element in the first and the second directions as

$$N_{p,1}(i_{e,1}) = N_{h,e,1}(i_{e,1}) + 1;$$
$$N_{p,2}(i_{e,2}) = N_{h,e,2}(i_{e,2}) + 1.$$

Thus, the total number of nodes on the element with this discretization is equal to

$$N_p(i_e) = N_{p,1}(i_{e,1}) N_{p,2}(i_{e,2}).$$

3. LOCAL CONSTRUCTIONS FOR ARBITRARY FINITE ELEMENT

Let us represent fulfilment on the element $i_e = (i_{e,1},\ i_{e,2})$ for the p-th component of the vector function ($p = 1,\ 2,\ 3$)

$$u_p(x_1,\ x_2) = w(t_1,\ t_2) = \\ = \sum_{j_2=1}^{N_{p,2}(i_{e,2})} \sum_{j_1=1}^{N_{p,1}(i_{e,1})} \varphi_{j_2}^{m_2(i_{e,2})}(t_2) \varphi_{j_1}^{m_1(i_{e,1})}(t_1) \cdot \alpha_p^{(j_2,j_1)}, \qquad (2.1)$$

where

$$x_1(i_{e,1}) \le x_1 \le x_1(i_{e,1}) + h_{e,1};$$
$$x_2(i_{e,2}) \le x_2 \le x_2(i_{e,2}) + h_{e,2};$$
$$t_1 = (x_1 - x_1(i_{e,1})) / h_{e,1}, \quad 0 \le t_1 \le 1;$$
$$t_2 = (x_2 - x_2(i_{e,2})) / h_{e,2}, \quad 0 \le t_2 \le 1;$$

$\varphi_j^m(t)$ is the j-th basis function of the B-spline of degree m (see, for example, [7, 8]).

In this case, we have the following relations:

$$\partial_1 = \frac{\partial}{\partial x_1} = \frac{1}{h_{e,1}} \cdot \frac{\partial}{\partial t_1}; \quad \partial_2 = \frac{\partial}{\partial x_1} = \frac{1}{h_{e,2}} \cdot \frac{\partial}{\partial t_2}; \qquad (2.2)$$

$$dx_1 = h_{e,1} \cdot dt_1; \quad dx_2 = h_{e,2} \cdot dt_2; \qquad (2.3)$$

Let us define the parameters $\alpha_p^{(j_2,j_1)}$ in terms of the nodal unknowns on the element $i_e=(i_{e,1},\ i_{e,2})$:

$$u_p^{(i_2,i_1)}(x_{i,1},\ x_{i,2})=w(t_{i,1},\ t_{i,2})=$$
$$=\sum_{j_2=1}^{N_{p,2}(i_{e,2})}\sum_{j_1=1}^{N_{p,1}(i_{e,1})}\varphi_{j_2}^{m_2(i_{e,2})}(t_{i,2})\varphi_{j_1}^{m_1(i_{e,1})}(t_{i,1})\cdot\alpha_p^{(j_2,j_1)};$$
$$x_{i_1}=x_1(i_{e,1})+\frac{h_{e,1}}{N_{p,1}(i_{e,1})-1}(i_1-1);$$
$$x_{i_2}=x_2(i_{e,2})+\frac{h_{e,2}}{N_{p,2}(i_{e,2})-1}(i_2-1);$$
$$t_1=(x_1-x_1(i_{e,1}))/h_{e,1}=$$
$$=\left[x_1(i_{e,1})+\frac{h_{e,1}}{N_{p,1}(i_{e,1})-1}(i_1-1)-x_1(i_{e,1})\right]/h_{e,1}=$$
$$=\frac{i_1-1}{N_{p,1}(i_{e,1})-1};$$
$$t_2=\frac{i_2-1}{N_{p,2}(i_{e,2})-1}.$$

We obtain a system of linear algebraic equations

$$T\overline{\alpha}_p^{i_e}=\overline{u}_p^{i_e}, \tag{2.4}$$

where we have

$T=\{t_{i,j}\}_{i,j=1,\dots,N_p(i_e)}$:

$$t_{i,j}=\varphi_{j_2}^{m_2(i_{e,2})}(t_{i_2})\cdot\varphi_{j_1}^{m_1(i_{e,1})}(t_{i_1})$$

does not depend on p;

$$\overline{\alpha}_p=\{\alpha_p^j\}=\{\alpha_p^{j_2,j_1}\};\quad \overline{u}_p^{i_e}=\{u_p^i\}=\{u_p^{i_2 i_1}\};$$
$$i=i_1+N_{p,1}(i_{e,1})\cdot(i_2-1),$$
$$i_2=1,\ \dots,N_{p,2}(i_{e,2}),\quad i_1=1,\ \dots,N_{p,1}(i_{e,1});$$
$$j=j_1+N_{p,1}(i_{e,1})\cdot(j_2-1),$$
$$j_2=1,\ \dots,N_{p,2}(i_{e,2}),\quad j_1=1,\ \dots,N_{p,1}(i_{e,1}).$$

Then we have

$$\overline{\alpha}_p^{i_e}=T^{-1}\overline{u}_p^{i_e}. \tag{2.5}$$

Let us introduce the following notations:

$$\overline{u}^{i_e}=\begin{bmatrix}\overline{u}^1\\ \vdots\\ \overline{u}^{N_p(i_e)}\end{bmatrix}$$

are the nodal values of the vector function on the element i_e;

$$\overline{u}^i=\begin{bmatrix}u_1^i\\ u_2^i\\ u_3^i\end{bmatrix}$$

is the value of the vector function at the i-th node on the element i_e;

$$\overline{u}_p^{i_e}=\begin{bmatrix}u_p^1\\ \vdots\\ u_p^{N_p(i_e)}\end{bmatrix}$$

are the nodal values of the p-th component of the vector function on the element i_e ($p=1,\ 2,\ 3$).

Let P be a permutation matrix such that

$$P\overline{u}^{i_e}=\begin{bmatrix}\overline{u}_1^{i_e}\\ \overline{u}_2^{i_e}\\ \overline{u}_3^{i_e}\end{bmatrix}. \tag{2.6}$$

Since for a permutation matrix $P^{-1}=P^T$, the inverse transformation has the form

$$\overline{u}^{i_e}=P^T\begin{bmatrix}\overline{u}_1^{i_e}\\ \overline{u}_2^{i_e}\\ \overline{u}_3^{i_e}\end{bmatrix}. \tag{2.7}$$

To construct local stiffness matrices corresponding to the operators L_{uu}, L_{uv}, L_{vv}, we consider bilinear forms, taking into account representation (2.1) and relations (2.2)-(2.3) on the form functions:

$$\begin{aligned}
&\overline{u}(x_1, x_2) = \overline{w}(t_1, t_2) = \\
&= \sum_{k_2=1}^{N_{p,2}(i_{e,2})} \sum_{k_1=1}^{N_{p,1}(i_{e,1})} \varphi_{k_2}^{m_2(i_{e,2})}(t_2)\varphi_{k_1}^{m_1(i_{e,1})}(t_1)\cdot \overline{\alpha}^k; \\
&\overline{y}(x_1, x_2) = \overline{q}(t_1, t_2) = \\
&= \sum_{k_2=1}^{N_{p,2}(i_{e,2})} \sum_{k_1=1}^{N_{p,1}(i_{e,1})} \varphi_{k_2}^{m_2(i_{e,2})}(t_2)\varphi_{k_1}^{m_1(i_{e,1})}(t_1)\cdot \overline{\beta}^k;
\end{aligned} \tag{2.8}$$

$$x_1(i_{e,1}) \le x_1 \le x_1(i_{e,1}) + h_{e,1}; \quad x_2(i_{e,2}) \le x_2 \le x_2(i_{e,2}) + h_{e,2};$$

$$0 \le t_1,\ t_2 \le 1;$$

$$\overline{\alpha}^k = \begin{bmatrix} \alpha_1^k \\ \alpha_2^k \\ \alpha_3^k \end{bmatrix}; \quad \overline{\beta}^k = \begin{bmatrix} \beta_1^k \\ \beta_2^k \\ \beta_3^k \end{bmatrix};$$

$$k = k_1 + N_{p,1}(i_{e,1})\cdot(k_2 - 1);$$

$$k_2 = 1,\dots,N_{p,2}(i_{e,2}); \quad k_1 = 1,\dots,N_{p,1}(i_{e,1});$$

$$\begin{aligned}
&B_{vv}(\overline{u},\overline{y}) = (L_{vv}\overline{u},\overline{y}) = \\
&= \int_{x_2(i_{e,2})}^{x_2(i_{e,2})+h_{e,2}} \int_{x_1(i_{e,1})}^{x_1(i_{e,1})+h_{e,1}} (\begin{bmatrix} \mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & 2\mu+\lambda \end{bmatrix} \overline{u},\overline{y}) dx_1 dx_2 = \\
&= h_{e,2}h_{e,1}\int_0^1\int_0^1 (\begin{bmatrix} \mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & 2\mu+\lambda \end{bmatrix} \overline{u},\overline{y}) dt_1 dt_2 = \\
&= \sum_{i_2=1}^{N_{p,2}(i_{e,2})} \sum_{i_1=1}^{N_{p,1}(i_{e,1})} \sum_{j_2=1}^{N_{p,2}(i_{e,2})} \sum_{j_1=1}^{N_{p,1}(i_{e,1})} \gamma_{i,j} \times \\
&\times(\begin{bmatrix} \mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & 2\mu+\lambda \end{bmatrix} \overline{\alpha}^j, \overline{\beta}^i),
\end{aligned}$$

where we have

$$\begin{aligned}
\gamma_{i,j} &= h_{e,1}\int_0^1 \varphi_{j_1}^{m_1(i_{e,1})}(t_1)\cdot\varphi_{i_1}^{m_1(i_{e,1})}(t_1)dt_1 \times \\
&\times h_{e,2}\int_0^1 \varphi_{j_2}^{m_2(i_{e,2})}(t_2)\cdot\varphi_{i_2}^{m_2(i_{e,2})}(t_2)dt_2.
\end{aligned} \tag{2.9}$$

Let us denote

$$\Gamma = \{\gamma_{i,j}\}_{i,j=1,\dots,N_p(i_e)}; \quad G = (T^{-1})^T \Gamma T^{-1}. \tag{2.10}$$

Then, taking into account dependences (2.5) and (2.6), the following chain of equalities is valid

$$\begin{aligned}
&B_{vv}(\overline{u},\overline{y}) =< L_{vv}\overline{u},\overline{y} >= \mu(\Gamma\overline{\alpha}_1^{i_e},\overline{\beta}_1^{i_e}) + \\
&+ \mu(\Gamma\overline{\alpha}_2^{i_e},\overline{\beta}_2^{i_e}) + (2\mu+\lambda)(\Gamma\overline{\alpha}_3^{i_e},\overline{\beta}_3^{i_e}) = \\
&= \mu((T^{-1})^T\Gamma T^{-1}\overline{u}_1^{i_e},\overline{y}_1^{i_e}) + \\
&+ \mu((T^{-1})^T\Gamma T^{-1}\overline{u}_2^{i_e},\overline{z}_2^{i_e}) + \\
&+ (2\mu+\lambda)((T^{-1})^T\Gamma T^{-1}\overline{u}_3^{i_e},\overline{z}_3^{i_e}) = \\
&= \left(\begin{bmatrix} \mu G & 0 & 0 \\ 0 & \mu G & 0 \\ 0 & 0 & (2\mu+\lambda)G \end{bmatrix}\begin{bmatrix} \overline{u}_1^{i_e} \\ \overline{u}_2^{i_e} \\ \overline{u}_3^{i_e} \end{bmatrix}, \begin{bmatrix} \overline{z}_1^{i_e} \\ \overline{z}_2^{i_e} \\ \overline{z}_3^{i_e} \end{bmatrix}\right) = \\
&= \left(P^T \begin{bmatrix} \mu G & 0 & 0 \\ 0 & \mu G & 0 \\ 0 & 0 & (2\mu+\lambda)G \end{bmatrix} P\overline{u}^{i_e}, \overline{z}^{i_e} \right) = \\
&= (K_{vv}^{i_e}\overline{u}^{i_e},\overline{z}^{i_e}).
\end{aligned}$$

Thus, the local stiffness matrix $K_{vv}^{i_e}$ corresponding to the operator L_{vv} has the form

$$K_{vv}^{i_e} = P^T \begin{bmatrix} \mu G & 0 & 0 \\ 0 & \mu G & 0 \\ 0 & 0 & (2\mu+\lambda)G \end{bmatrix} P. \tag{2.11}$$

Similarly, one can obtain local stiffness matrices corresponding to the operators L_{uu} and L_{uv}. We present here the final formulas.
The operator L_{uu} corresponds to

$$K_{uu}^{i_e} = K_{uu,1}^{i_e} + K_{uu,2}^{i_e} + K_{uu,3}^{i_e} + K_{uu,4}^{i_e}, \qquad (2.12)$$

where we have

$$K_{uu,1}^{i_e} = P^T \begin{bmatrix} (2\mu+\lambda)G_1 & 0 & 0 \\ 0 & \mu G_1 & 0 \\ 0 & 0 & \mu G_1 \end{bmatrix} P;$$

$$G_1 = (T^{-1})^T \Gamma_1 T^{-1}; \quad \Gamma_1 = \{\gamma_{1,i,j}\}_{i,j=1,\dots,N_p(i_e)};$$

$$\gamma_{1,i,j} = \frac{1}{h_{e,1}} \int_0^1 \frac{d\varphi_{j_1}^{m_1(i_{e,1})}}{dt_1}(t_1) \cdot \frac{d\varphi_{i_1}^{m_1(i_{e,1})}}{dt_1}(t_1) dt_1 \times \\ \times h_{e,2} \int_0^1 \varphi_{j_2}^{m_2(i_{e,2})}(t_2) \cdot \varphi_{i_2}^{m_2(i_{e,2})}(t_2) dt_2; \qquad (2.13)$$

$$K_{uu,2}^{i_e} = P^T \begin{bmatrix} 0 & \lambda G_2 & 0 \\ \mu G_2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} P;$$

$$G_2 = (T^{-1})^T \Gamma_2 T^{-1}; \quad \Gamma_2 = \{\gamma_{2,i,j}\}_{i,j=1,\dots,N_p(i_e)};$$

$$\gamma_{2,i,j} = \int_0^1 \varphi_{j_1}^{m_1(i_{e,1})}(t_1) \cdot \frac{d\varphi_{i_1}^{m_1(i_{e,1})}}{dt_1}(t_1) dt_1 \times \\ \times \int_0^1 \frac{d\varphi_{j_2}^{m_2(i_{e,2})}}{dt_2}(t_2) \cdot \varphi_{i_2}^{m_2(i_{e,2})}(t_2) dt_2; \qquad (2.14)$$

$$K_{uu,3}^{i_e} = P^T \begin{bmatrix} 0 & \mu G_3 & 0 \\ \lambda G_3 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} P;$$

$$G_3 = (T^{-1})^T \Gamma_3 T^{-1}; \quad \Gamma_3 = \{\gamma_{3,i,j}\}_{i,j=1,\dots,N_p(i_e)};$$

$$\gamma_{3,i,j} = \int_0^1 \frac{d\varphi_{j_1}^{m_1(i_{e,1})}}{dt_1}(t_1) \cdot \varphi_{i_1}^{m_1(i_{e,1})}(t_1) dt_1 \times \\ \times \int_0^1 \varphi_{j_2}^{m_2(i_{e,2})}(t_2) \cdot \frac{d\varphi_{i_2}^{m_2(i_{e,2})}}{dt_2}(t_2) dt_2; \qquad (2.15)$$

$$K_{uu,4}^{i_e} = P^T \begin{bmatrix} \mu G_4 & 0 & 0 \\ 0 & (2\mu+\lambda)G_4 & 0 \\ 0 & 0 & \mu G_4 \end{bmatrix} P;$$

$$G_4 = (T^{-1})^T \Gamma_4 T^{-1}; \quad \Gamma_4 = \{\gamma_{4,i,j}\}_{i,j=1,\dots,N_p(i_e)};$$

$$\gamma_{4,i,j} = \frac{1}{h_{e,1}} \int_0^1 \varphi_{j_1}^{m_1(i_{e,1})}(t_1) \cdot \varphi_{i_1}^{m_1(i_{e,1})}(t_1) dt_1 \times \\ \times h_{e,2} \int_0^1 \frac{d\varphi_{j_2}^{m_2(i_{e,2})}}{dt_2}(t_2) \cdot \frac{d\varphi_{i_2}^{m_2(i_{e,2})}}{dt_2}(t_2) dt_2. \qquad (2.16)$$

The operator L_{uv} corresponds to

$$K_{uv}^{i_e} = K_{uv,1}^{i_e} + K_{uv,2}^{i_e} - K_{uv,3}^{i_e} - K_{uv,4}^{i_e}, \qquad (2.17)$$

where we have

$$K_{uv,1}^{i_e} = P^T \begin{bmatrix} 0 & 0 & \lambda D_1 \\ 0 & 0 & 0 \\ \mu D_1 & 0 & 0 \end{bmatrix} P;$$

$$D_1 = (T^{-1})^T \Delta_1 T^{-1}; \quad \Delta_1 = \{\delta_{1,i,j}\}_{i,j=1,\dots,N_p(i_e)};$$

$$\delta_{1,i,j} = \int_0^1 \varphi_{j_1}^{m_1(i_{e,1})}(t_1) \cdot \frac{d\varphi_{i_1}^{m_1(i_{e,1})}}{dt_1}(t_1) dt_1 \times \\ \times h_{e,2} \int_0^1 \varphi_{j_2}^{m_2(i_{e,2})}(t_2) \cdot \varphi_{i_2}^{m_2(i_{e,2})}(t_2) dt_2; \qquad (2.18)$$

$$K_{uv,2}^{i_e} = P^T \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \lambda D_2 \\ 0 & \mu D_2 & 0 \end{bmatrix} P;$$

$$D_2 = (T^{-1})^T \Delta_2 T^{-1}; \quad \Delta_2 = \{\delta_{2,i,j}\}_{i,j=1,\dots,N_p(i_e)};$$

$$\delta_{2,i,j} = h_{e,1} \int_0^1 \varphi_{j_1}^{m_1(i_{e,1})}(t_1) \cdot \varphi_{i_1}^{m_1(i_{e,1})}(t_1) dt_1 \times \\ \times \int_0^1 \varphi_{j_2}^{m_2(i_{e,2})}(t_2) \cdot \frac{d\varphi_{i_2}^{m_2(i_{e,2})}}{dt_2}(t_2) dt_2; \qquad (2.19)$$

$$K_{uv,3}^{i_e} = P^T \begin{bmatrix} 0 & 0 & \mu D_3 \\ 0 & 0 & 0 \\ \lambda D_3 & 0 & 0 \end{bmatrix} P;$$

$$D_3 = (T^{-1})^T \Delta_3 T^{-1}; \quad \Delta_3 = \{\delta_{3,i,j}\}_{i,j=1,\dots,N_p(i_e)};$$

$$\delta_{3,i,j} = \int_0^1 \frac{d\varphi_{j_1}^{m_1(i_{e,1})}}{dt_1}(t_1) \cdot \varphi_{i_1}^{m_1(i_{e,1})}(t_1) dt_1 \times \\ \times h_{e,2} \int_0^1 \varphi_{j_2}^{m_2(i_{e,2})}(t_2) \cdot \varphi_{i_2}^{m_2(i_{e,2})}(t_2) dt_2; \qquad (2.20)$$

$$K_{uv,4}^{i_e} = P^T \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \mu D_4 \\ 0 & \lambda D_4 & 0 \end{bmatrix} P;$$

$$D_4 = (T^{-1})^T \Delta_4 T^{-1}; \quad \Delta_4 = \{\delta_{4,i,j}\}_{i,j=1,\dots,N_p(i_e)};$$

$$\delta_{4,i,j} = h_{e,1} \int_0^1 \varphi_{j_1}^{m_1(i_{e,1})}(t_1) \cdot \varphi_{i_1}^{m_1(i_{e,1})}(t_1) dt_1 \times$$
$$\times h_{e,2} \int_0^1 \frac{d\varphi_{j_2}^{m_2(i_{e,2})}}{dt_2}(t_2) \cdot \varphi_{i_2}^{m_2(i_{e,2})}(t_2) dt_2; \qquad (2.21)$$

3. EXAMPLE OF ANALYSIS

3.1. Formulation of the problem.

As a model example, let us consider the determination of the displacements of a three-dimensional beam, fixed along the side faces in

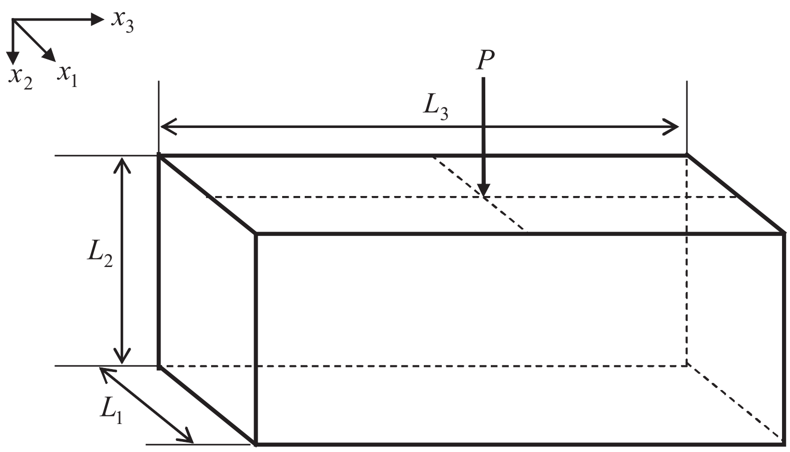


Figure 3.1. Example of analysis.

three directions, under the influence of a load concentrated in the center (Figure 3.1).

Let us consider the following geometric parameters:

$$L_1 = L_2 = 3.6 \text{ m}; \; L_3 = 8 \text{ m}.$$

Let us consider the following design parameters of material of the beam: coefficient of elasticity $E = 26500 \cdot 10^4$ kN/m^2; Poisson's ratio $\nu = 0.15$. Let external load parameter be equal to $P = 100$ kN.

3.2. Structural analysis with allowance for localization.

Let us determine the localization of the solution in the zone of application of the load. Let us set

$$N_{e,1} = N_{e,2} = 4$$

is the number of elements in each discrete direction. The lengths of the element are equal to

$$h_{e,1} = \ell_1 / N_{e,1} = 3.6 / 4 = 0.9; \quad h_{e,2} = h_{e,1} = 0.9.$$

Let us set the fulfilment by directions (Figure 3.2):

$$N_{p,1}(1) = N_{p,1}(4) = 2$$

is the fulfilment in the first direction by a spline of the first order: $m_1(1) = m_1(4) = 1$;

$$N_{p,1}(2) = N_{p,1}(3) = 4$$

is the fulfilment in the first direction by a spline of the third order: $m_1(2) = m_1(3) = 3$;

$$N_{p,2}(1) = N_{p,2}(2) = 4$$

is the fulfilment in the second direction by a spline of the third order: $m_2(1) = m_2(2) = 3$;

$$N_{p,2}(3) = N_{p,2}(4) = 2$$

is the fulfilment in the second direction by a spline of the first order: $m_1(3) = m_1(4) = 1$.

Thus, the localization area includes elements with numbers (2, 1), (3, 1), (2, 2), (3, 2).

The total number of nodes of the mesh is equal to $N_p = 9 \cdot 9 = 81$.

The total number of unknown nodal displacements is equal to $N_d = 3 \cdot 81 = 243$.

Total number of nodal unknowns is equal to $N_U = 2 \cdot 243 = 486$.

Figures 3.3 and 3.4 graphically represent selected examples of the obtained solution results. Let us use the following notations:

$$h_2 = h_{e,2} / 3 = 0.9 / 3 = 0.3$$

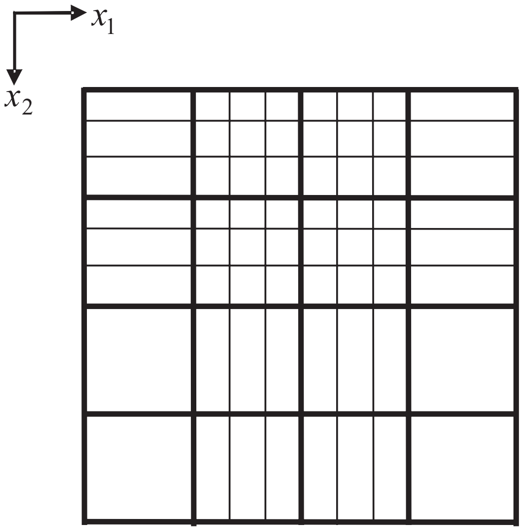


Figure 3.2. Mesh discretization with allowance for localization.

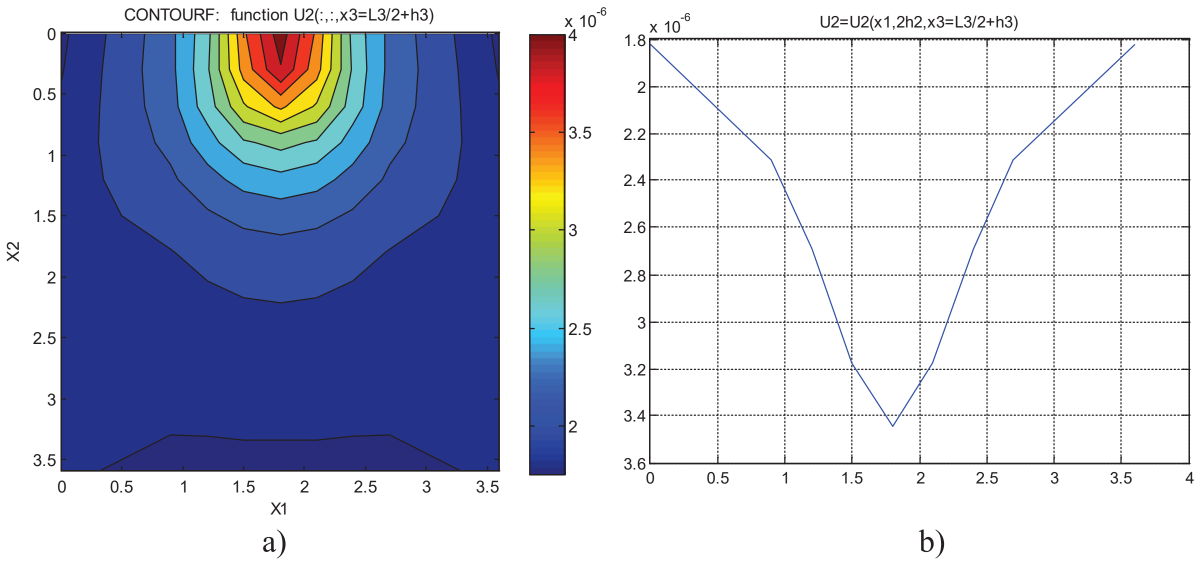


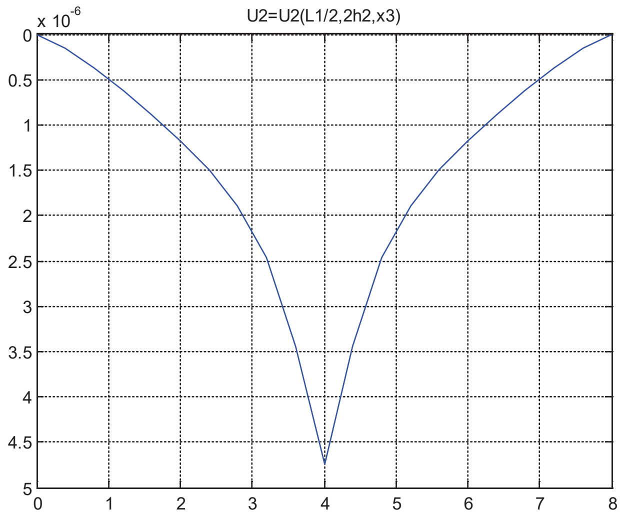
Figure 3.3. Nodal displacements u_2:

a) in section $x_3 = \ell_3 / 2 + h_3$; b) in section and at the level $x_2 = 2h_2$

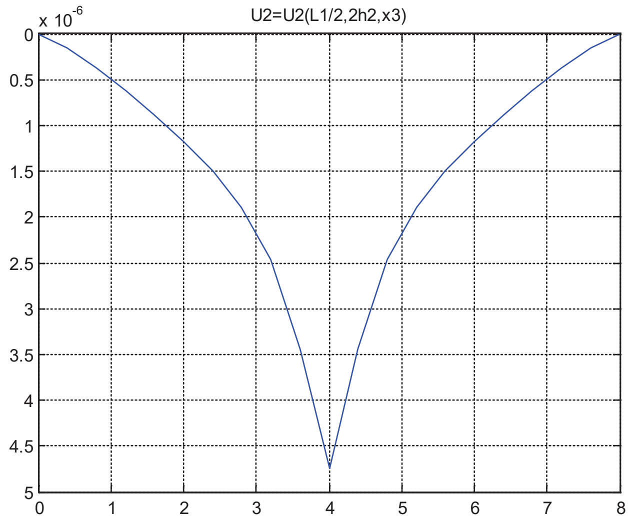
is the step of discretization of the finite element in the localization area in the second direction (x_2);

$$h_3 = \ell_3 / 20 = 8 / 20 = 0.4$$

– step of printing the result along the continual direction (x_3).



a)



b)

<u>*Figure 3.4.*</u> *Displacements along continual direction for* $x_1 = \ell_1 / 2$ *and* $x_2 = 2h_2$ *:*
a) component u_2 *; b) component* u_3 .

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Marina L. Mozgaleva, Senior Scientist Researcher, Dr.Sc.; Professor of Department of Applied Mathematics, National Research Moscow State University of Civil Engineering; 26, Yaroslavskoe Shosse, Moscow, 129337, Russia; phone/fax +7(499) 183-59-94; Fax: +7(499) 183-44-38; Email: marina.mozgaleva@gmail.com.

Pavel A. Akimov, Full Member of the Russian Academy of Architecture and Construction Sciences, Professor, Dr.Sc.; Rector of National Research Moscow State University of Civil Engineering; 26, Yaroslavskoe Shosse, Moscow, 129337, Russia; phone: +7(495) 651-81-85; Fax: +7(499) 183-44-38; E-mail: AkimovPA@mgsu.ru, rector@mgsu.ru, pavel.akimov@gmail.com.

Taymuraz B. Kaytukov, Advisor of the Russian Academy of Architecture and Construction Sciences, Associate Professor, Ph.D.; Vice-Rector, Associate Professor Department of Applied Mathematics, National Research Moscow State University of Civil Engineering; 26, Yaroslavskoe Shosse, Moscow, 129337, Russia; phone: +7(499) 929-52-29; fax: +7(499) 183-59-94; Email: KaytukovTB@mgsu.ru.

Мозгалева Марина Леонидовна, старший научный сотрудник, доктор технических наук; профессор кафедры прикладной математики Национального исследовательского Московского государственного строительного университета; 129337, Россия, г. Москва, Ярославское шоссе, дом 26; телефон/факс: +7(499) 183-59-94; Email: marina.mozgaleva@gmail.com.

Акимов Павел Алексеевич, академик РААСН, профессор, доктор технических наук; ректор Национального исследовательского Московского государственного строительного университета; 129337, Россия, г. Москва, Ярославское шоссе, дом 26; телефон: +7(495) 651-81-85; факс: +7(499) 183-44-38; Email: AkimovPA@mgsu.ru, rector@mgsu.ru, pavel.akimov@gmail.com.

Кайтуков Таймураз Батразович, советник РААСН, доцент, кандидат технических наук; проректор, профессор кафедры прикладной математики Национального исследовательского Московского государственного строительного университета; 129337, Россия, г. Москва, Ярославское шоссе, дом 26; телефон: +7(499) 929-52-29; факс: +7(499) 183-44-38; Email: KaytukovTB@mgsu.ru.