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RECIPROCITY LAWS FOR OSCILLATIONS OF DISSIPATIVE SYSTEMS

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Annotation: A general method for proving reciprocity relations in an arbitrary elastic discrete dissipative system (DDS) is presented. The method is based on the use of the algebraic theorem of P.L. Pasternak and on the new properties of the Duhamel integral, which are obtained for a dissipative system with internal friction of the material, which is taken into account on the basis of the non-proportional damping model. For displacements, velocities and accelerations, the dynamic reaction equations are written in the form of systems of linear equations and their symmetrical structure is shown. The functional dependence of the force parameters of the calculation model and the corresponding kinematic parameters of the reaction is determined by an arbitrary scalar function of time. An extended interpretation of the reciprocity theorems is given and sufficient conditions for their fulfillment are formulated, which consist in the requirement that the matrix differential operator of the equation of motion be symmetrical. New laws of reciprocity in dissipative systems are formulated and proved. The reciprocity of the product between the velocities / accelerations of masses and nodal forces is established. In contrast to the well-known theorem on the reciprocity of possible work, these laws are theorems on the 1st / 2nd derivative of possible work with respect to time and therefore go beyond the Betti principle. For particular cases of these theorems, the reciprocity of velocities and reciprocity of accelerations is shown. Expressions of general and particular theorems have a fairly simple mathematical form that does not require recourse to integral transformations, and are presented in an analytical form.

Keywords: Duhamel integral, dissipative system, reaction, displacement, velocity, acceleration

СООТНОШЕНИЯ ВЗАИМНОСТИ В ДИНАМИКЕ ДИССИПАТИВНЫХ СИСТЕМ

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Аннотация: Приведен общий метод доказательства соотношений взаимности в произвольной упругой дискретной диссипативной системе (ДДС), основанный на использовании алгебраической теоремы П.Л. Пастернака и новых свойствах интеграла Дюамеля, которые получены для диссипативной системы с внутренним трением материала, учитываемым на основе модели непропорционального демпфирования. Для перемещений, скоростей и ускорений уравнения динамической реакции записаны в форме систем линейных уравнений и показана их симметричная структура. Функциональная зависимость силовых параметров расчетной модели и соответствующих кинематических параметров реакции определяется произвольной скалярной функцией времени. Дана расширенная трактовка теорем взаимности и сформулированы достаточные условия их выполнения, заключающиеся в требовании симметрии матричного дифференциального оператора уравнения движения. Сформулированы и доказаны новые законы взаимности в диссипативных системах. Установлена взаимность произведения между скоростями / ускорениями масс и узловыми силами. В отличие от известной теоремы о взаимности возможных работ данные законы представляет собой теоремы о 1-й / 2-й производной возможной работы по времени и поэтому выходят за рамки принципа Бетти. Для частных случаев этих теорем показана взаимность скоростей и взаимность ускорений. Выражения общих и частных теорем имеют достаточно простую математическую форму записи, не требующую обращения к интегральным преобразованиям, и представляются в аналитическом виде.

Ключевые слова: интеграл Дюамеля, диссипативная система, реакция, перемещение, скорость, ускорение

INTRODUCTION

The main provisions on the issues of reciprocity in elastic systems were formulated in the works of Maxwell, Betty, Rayleigh and other scientists. These problems are most fully elucidated in the works of Rayleigh, who for the first time extended the results of his research to dissipative systems, introducing the scattering function for this purpose [1].

In deriving reciprocity relations, Rayleigh considered systems with a finite number of degrees of freedom, referred to below as discrete systems. As sources of perturbation of a discrete dissipative system (DDS), he took periodic forces of a harmonic type and given mass displacements, which also obey the harmonic law.

In his proofs, Rayleigh relied on the analysis of the potential and kinetic energies of the DDS, which are homogeneous quadratic functions of generalized coordinates and velocities. As a result, a general law of reciprocity was obtained (the theorem of reciprocity of work in the form of a product of forces and displacements), regardless of the Italian scientist Betty, who formulated this law a year earlier (1872). Rayleigh also proved all particular reciprocity theorems: displacements, reactions, reactions and displacements. In addition, for a conservative system that has only kinetic energy, a work reciprocity theorem was obtained in the form of a product of impulses of forces and mass velocities and a particular theorem on the reciprocity of impulses.

In [2], a purely algebraic interpretation of the principle of reciprocity was given, according to which reciprocity was considered as a property characteristic of any system of n linear equations with a symmetric structure of coefficients.

Reciprocity theorems are used in various fields of science and technology: acoustics, electrical engineering, geophysics, etc. In the field of practical applications of elastodynamics [3], reciprocity theorems are used in microelectromechanical devices, which are typical structural elements of building mechanics. In miniature, they are thin bodies such as beams, plates, shells and membranes. In the field of nanotechnologies (nano- and metamaterials) [4], an approach is used that makes it possible to apply the Betti–Rayleigh reciprocity theorem to calculate scattered fields in the form of plane waves at the interface between a homogeneous medium and a metamaterial.

The principle of reciprocity assumes the invariance of the characteristics of the frequency functions between any two material points during the replacement between the source and the receiver. This fact laid the foundation for many practical studies on ground vibrations caused by traffic [5], in seismic exploration [6], and problems of soil dynamics [7–10].

In [5], a problem related to soil dynamics was solved by checking a numerical model of vibration caused by road roughness during the movement of a vehicle such as a car. Rayleigh surface waves, generated by vertical vibrations from dynamic loads on the axle and propagating in the soil, are studied. Soil response in the free field was calculated using the dynamic Betty-Rayleigh reciprocity theorem using the transfer function between road and receiver. In the field of geophysics, in particular, seismic exploration [6], a method has been developed for detecting underground cavities using special dynamic test procedures designed to determine the characteristics of underground anomalies. The studies were carried out using active or passive sources and receivers placed on the earth's surface or inside wells for Rayleigh surface waves scattered from inhomogeneities in layered half-spaces. Discrete analytic expressions for the Green's functions are obtained on cavities using the thin layer method based on the Betti–Rayleigh reciprocity theorem.

Estimates of the levels of vibrations of the surface of the soil medium from a point source located inside the half-space were carried out in [7] based on the theorem on the reciprocity of displacements. The same theorem was used in studies on assessing the risk of damage to buildings due to sedimentation of the soil surface during shield penetration [8], when moving high-speed vehicles (trains, cars) [9]. The article

[10] considers the case of the impact of a seismic wave on a rigid foundation located in difficult soil conditions. The assessment of seismic load on a rigid foundation was carried out in the frequency range using the reciprocity theorem.

The papers [11, 12] provide proofs of reciprocity theorems for an arbitrary dynamic load without taking into account [11] and taking into account internal friction [12]. In the latter case, the damping matrix was adopted on the basis of the proportional damping model, which made it possible to write the differential equations of motion in a separated form.

For dynamic problems of the theory of elasticity, an approach [13] to the derivation of the reciprocity theorem based on the variational principle was proposed, in contrast to the generally accepted method of deriving reciprocity relations, which used the integral Laplace transform. When writing the equation of motion, the internal friction of the material is not taken into account. In [14], reciprocity relations were obtained in differential form for differential operators of elasticity theory and, for a particular discrete case, transitions to classical reciprocity relations between the coefficients of the force method, displacement method, and the mixed method were shown.

A number of works on reciprocity relations were carried out in the field of nonlinear problems of building mechanics, in particular, for nonlinear elastic systems it was shown [15] that reciprocity relations follow from the conditions of conservative properties of the system. The article [16] developed a technique applicable to discrete or continuous systems with smooth or non-smooth nonlinearities using the Betti-Maxwell reciprocity theorem.

A large cycle of works relates to the elastodynamics of Rayleigh surface waves in solids, where an effect associated with the violation of reciprocity in an elastic medium was discovered. In works [17, 18], devoted to the study of the propagation of Rayleigh waves with masses arranged in the form of a hexagonal and triangular lattice, this effect is explained by the lack of symmetry in the modes of natural oscillations, which manifested itself in the asymmetry of responses in the form of displacement fields in the direction of the medium. Also, the effect of the lack of symmetry in the modes of natural oscillations was noted in [19]. In [20], the violation of reciprocity in the elastodynamics of linear time-invariant acoustic waveguides was studied, which, according to the authors, is possible only due to violation of the time reversal symmetry at the microlevel.

1. Research method

A more general and at the same time simpler method for proving reciprocity theorems, which does not require the use of integral transformations, is described in the articles [21, 22]. The approach is based on new properties of the Duhamel integral, obtained by analyzing the equation of motion of the DDS and its characteristic equation. The interpretation of well-known theorems is extended, proofs of new reciprocity theorems are obtained, and conditions are formulated that ensure reciprocity properties in DDS.

The differential equation of motion of the DDS and its characteristic equation corresponding to the homogeneous equation have the form:

$$M\ddot{Y}(t) + C\dot{Y}(t) + KY(t) = P(t) \tag{1}$$

$$MS^2 + CS + K = 0, \tag{2}$$

where $M = \mathrm{diag}\,(m_1,\ldots, m_n)$, C, K – are real and symmetric matrices of masses, damping and stiffness; $Y(t) = [y_j(t)]$, $P(t) = [p_j(t)]$ ($j = 1, \ldots , n$) – displacement and external load vectors; $S \in M_n(\mathbf{C})$ – the required matrix of dynamic characteristics of DDS; n is the number of degrees of freedom. The transition from equation (1) to (2) is performed using the fundamental matrix $\Phi(t) = e^{St}$.

When analyzing the characteristic equation (2), an arbitrary type of damping is considered, which is determined by the matrix C. Here, the only mandatory condition is the requirement for the symmetry of the matrix $C = C^{\mathrm{T}}$. In the presence of symmetry of the matrix C, the type of damping can be adopted based on any model (proportional or non-proportional) damping, and then the solution of equation (2) is written as:

$$S = M^{-1}(-C + V + U)/2 \in M_n(\mathrm{C}), \qquad (3)$$

where: $V = -V^{\mathrm{T}}$, $U = U^{\mathrm{T}}$. With a small dissipation characteristic of conventional building structures, additional properties are valid: V, $iU \in M_n(\mathrm{R})$ (i – imaginary unit).

The vector $P(t)$ is represented as a product of an arbitrary scalar function of time $\xi(t)$, which specifies the shape of the dynamic load, and the amplitude vector $P_0 = [p_{0j}]$ ($j = 1, \dots, n$):

$$P(t) = \xi(t)P_0. \qquad (4)$$

In this case, the reaction equations of the system - displacement vectors (Duhamel integral), velocities and accelerations, take the form:

$$\left.\begin{aligned} Y(t) &= 2\mathrm{Re}\,\{I(t)\}P_0, \\ \dot{Y}(t) &= 2\mathrm{Re}\{SI(t)\}P_0, \\ \ddot{Y}(t) &= 2\mathrm{Re}\{S^2 I(t)\}P_0 + M^{-1}P(t), \end{aligned}\right\} \qquad (5)$$

where

$$I(t) = \{U^{-1}\int_{t_0}^{t} \Phi\,(t-\tau)^{\mathrm{T}}\xi(\tau)\,\mathrm{d}\tau\}. \qquad (6)$$

It follows from the analysis of equation (2) in [23] that solution (3) satisfies the equalities:

$$S^{\mathrm{T}}U = US, \quad \Phi(t)^{\mathrm{T}}U = U\Phi(t), \qquad (7)$$

which establish the fact of symmetry of matrices US, $U\Phi(t)$. This fact regarding the second equation in (7) is based on the property of the exponential matrix $\Phi(t) = e^{St}$ to be expanded into an infinite series in powers of the matrix argument S:

$$\Phi(t) = \sum_{k=0}^{\infty}\frac{(St)^k}{k!}.$$

Indeed, the equality $US = S^{\mathrm{T}}U$ implies a whole class of equalities related to any integer power of the matrix S: $US^k = (S^k)^{\mathrm{T}}U$ ($k = 0, \pm1, \pm2, \dots$). From here one can obtain any linear combination of such equalities, and hence a combination containing the coefficients of the expansion series of the fundamental matrix $\Phi(t)$. This leads to an important symmetry property of the auxiliary matrix functions $S^kU^{-1}\Phi(t-\tau)^{\mathrm{T}}$ ($k = 0, \pm1, \pm2, \dots$), from which it is sufficient to choose only the dependencies for $k = 0, 1, 2$:

$$U^{-1}\Phi(t-\tau)^{\mathrm{T}},\ SU^{-1}\Phi(t-\tau)^{\mathrm{T}},$$
$$S^2U^{-1}\Phi(t-\tau)^{\mathrm{T}}.$$

These matrices are contained in the structure of equations (5). They coincide up to a scalar factor with the corresponding expressions, which are under the integral sign in (5). Note that for elastic vibrations of the system, the matrices U^{-1}, SU^{-1} и S^2U^{-1} do not depend on the integration variable and, therefore, can be introduced under the integral sign (6) in equations (5).

Generalizing these results, we represent the vector form of equations (5) in the form:

$$\left.\begin{aligned} Y(t) = D(t)P(t),\ \ \dot{Y}(t) = V(t)P(t), \\ \ddot{Y}(t) = A(t)P(t), \end{aligned}\right\} \qquad (8)$$

where

$$\left.\begin{aligned} D(t) = D(t)^{\mathrm{T}} &= 2\mathrm{Re}\,\{I(t)\}\xi(t)^{-1}, \\ V(t) = V(t)^{\mathrm{T}} &= 2\mathrm{Re}\,\{SI(t)\}\xi(t)^{-1}, \\ A(t) = A(t)^{\mathrm{T}} &= 2\mathrm{Re}\,\{S^2I(t)\}\xi(t)^{-1} + M^{-1}, \end{aligned}\right\} \qquad (9)$$

2. Results and discussion

In view of the symmetry (9), these relations express special cases of reciprocity in a dissipative system, representing the displacement reciprocity theorem (Maxwell's theorem, first equation): $D(t) = D(t)^{\mathrm{T}}$, the velocity reciprocity theorem (second equation): $V(t) = V(t)^{\mathrm{T}}$ and the acceleration reciprocity theorem (third equation): $A(t) = A(t)^{\mathrm{T}}$.

Velocity reciprocity theorem in DDS:

The speed $V_{ik}(t)$, imparted to the mass mi from the action of a unit force given by the law $P_k(t) = 1{\cdot}\xi(t)$ and applied to the mass m_k, is equal to the same speed $V_{ki}(t) = V_{ik}(t)$, that occurs when moving mass m_k, as a result of the action of a unit force $P_k(t) = 1{\cdot}\xi(t)$, applied to the mass m_i.

The theorem of reciprocity of accelerations in DDS:

The acceleration $A_{ik}(t)$, arising from the movement of the mass m_i from the action of a unit force given by the law $P_k(t) = 1{\cdot}\xi(t)$ and applied to the mass m_k, is equal to the same acceleration $A_{ki}(t) = A_{ik}(t)$, arising during the movement of the mass m_k, caused by the action of a unit force $P_i(t) = 1{\cdot}\xi(t)$, applied in the mass m_i.

The presence of reciprocity of accelerations is recognized by many experts as a fact [11], although there is no proof of the existence of this theorem in the scientific literature.

According to the physical meaning, the elements of matrices (9) are, respectively, the dynamic displacements, velocities and accelerations of the DDS nodes. The dynamic nature of these quantities and reciprocity relations is indicated by the scalar function $\xi(t)$ of the load form, which is included in all expressions of matrices (9). These matrices have a fairly simple mathematical form of notation, represented by analytical expressions and do not require recourse to integral transformations.

It should be noted that Rayleigh in his studies initially considered periodic effects of both force and kinematic types, taking as a scalar harmonic function of time $\xi(t) = \sin(\vartheta t + \varphi)$, where the angular frequency ϑ and the initial phase φ of the periodic effect were considered constant values for all nodes of the discrete system. In this case, Rayleigh considered only steady-state oscillations. Indicative, in this regard, is his statement, which has the character of insight: “If the force is not a harmonic function of time, then the types of oscillations in different parts of the system generally differ from each other and from the type of force oscillation” [1]. That is, Rayleigh well understood that with an arbitrary law of acting forces, the oscillatory process in the system becomes too complicated to study the emerging "types of oscillations" and requires more advanced methods of analysis, which at that time were not available.

Systems of equations (8) at any fixed moment of time are considered as systems of algebraic equations with a symmetric structure of coefficients, for which the property of reciprocity can be shown based on the theorem of prof. P.L. Pasternak [2]. We present a scheme for proving the law of reciprocity using the first equation in (8) as an example.

Let some other group of new forces $p_j(t)'$ ($j = 1, \dots, n$) represented by the vector $P(t)'$ act on the same n-mass dissipative system, then from formula (4) we have $P(t)' = \xi(t)P_0'$, where $P_0' = [p_{0j}']$ ($j = 1, \dots, n$). This action will cause the response of the elastic system in the form of a new displacement vector $Y(t)' = [y_j(t)']$ ($j = 1, \dots, n$). This set of forces and displacements will correspond to the equation $Y(t)' = D(t)P(t)'$, which is converted to the form $P(t)' = D(t)^{-1}Y(t)'$. After the operation of transposing the first equation in (8) $Y(t)^{\mathrm{T}} = P(t)^{\mathrm{T}}D(t)$ and term by term multiplication of the left and right parts of the obtained equalities as a row by a column, we obtain an equation. This equation expresses the equality of possible works (the Betti principle) performed in an elastic dissipative system by the force vectors $P(t)$ and $P(t)'$ on displacements represented by the corresponding vectors $Y(t)'$, $Y(t)$:

$$Y(t)^{\mathrm{T}}P(t)' = Y(t)'^{\mathrm{T}}P(t). \tag{10}$$

Similarly, for two other linear systems in (8) with symmetric matrices $V(t)$ and $A(t)$, reciprocity relations are constructed according to the above scheme:

$$\left.\begin{aligned} \dot{Y}(t)^{\mathrm{T}}P(t)' &= \dot{Y}(t)'^{\mathrm{T}}P(t), \\ \ddot{Y}(t)^{\mathrm{T}}P(t)' &= \ddot{Y}(t)'^{\mathrm{T}}P(t). \end{aligned}\right\} \tag{11}$$

Relations (11) represent new laws of reciprocity in the dynamics of dissipative systems. These laws are outside the scope of the Betty principle (10), since their expressions do not represent possible work. The first relation in (11) has the dimension "kN·m/s" and it can be considered as a theorem on the reciprocity of the 1st derivative of the possible work with respect to time, the second relation (with the dimension "kN·m/s^2") - as the reciprocity theorem 2nd derivative of possible work with respect to time.

Laws (11) establish *the reciprocity of the product of velocities / accelerations of masses in one state of the system and the nodal forces in its other state*. In this case, the proof of particular cases of reciprocity relations (9) follows from the property of the Duhamel integral based on the symmetric structure of the integrands (6). To construct general reciprocity laws (10), (11), Pasternak's theorem is used, which is based on systems of equations (8), which have a proven symmetric structure of coefficients.

Thus, the prerequisite for the existence of reciprocity relations in a dissipative system is due to the presence of symmetry of the differential operator of the equation of motion (1), which is a sufficient condition for the fulfillment of the reciprocity laws shown above. Less obvious is the necessity of a condition for the fulfillment of these laws.

Indeed, symmetry breaking in the structure of a differential operator, for example, when using a damping model with a non-symmetric matrix $C \neq C^T$, will affect the properties of the characteristic equation (2) and auxiliary equalities (7). This will affect the structure of the expression for the dynamic response of the system (5), (6). With this formulation of the problem, the laws of reciprocity will be formulated in a more complex, generalized form. This indicates that in the field of dissipative systems there are still many problematic issues that need to be resolved.

An illustration of the reciprocity theorems is given on the example of vibrations of a rigidly clamped steel I-beam (№ 50, J_x = 39727 cm^4) for its various states (Fig. 1, 2, 3). The beam length is l = 1500 cm, point masses m_i = 0.5282 kNs^2/cm at n = 7.

The damping matrix is adopted in accordance with the non-proportional damping model used in the author's works [23]: $C = \frac{1}{2}(KT + TK)$, where $T = \mathrm{diag}(t_1, \ldots, t_7)$; $t_i = \frac{\delta}{\pi}\sqrt{\frac{m_j}{r_{jj}}}$; m_j, r_{jj} – diagonal elements of matrices M, K (δ = 0.07 – logarithmic oscillation decrement).

Parameters of natural oscillations of a steel beam: the frequency of the fundamental tone of oscillations was ω_{min} = 52.78 rad/s, the period of oscillations was T = 0.12 s.

The reciprocity of velocities and accelerations is shown in fig. 1 for the case of action of single rectangular pulses $P_k(t) = 1 \cdot \xi(t)$ with a length of 0.1 s (pulse shape $\xi(t) = 1$). In the first state, the reaction of the beam at the 3rd node from the action of the impulse $1 \cdot \xi(t)$ applied to the mass m_6 is shown. Oscillograms of velocities and accelerations are red (Fig. 1). The scale of accelerations (dashed line) is reduced by 50 times. In the second state, the same reactions (blue color) occur at the 6th node from the action of the impulse $1 \cdot \xi(t)$ applied to the mass m_3.

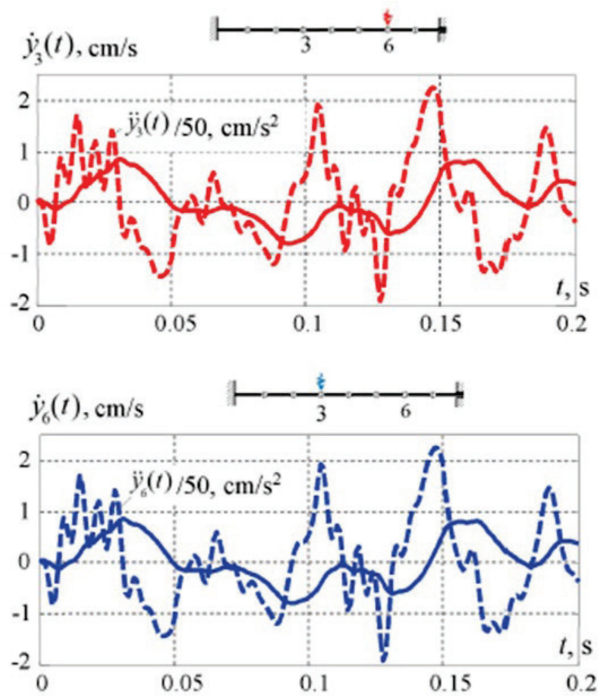

Figure 1. Oscillograms of velocities (solid line) and accelerations (dotted line) in a rigidly clamped beam under the action of a rectangular pulse equal to 1 in two states of the system: at the 6th node (response at the 3rd node - red) and at the 3rd node (response at the 6th node - blue)

On Figure 2, the action of vibrational forces with an angular frequency of 300 s^{-1} is considered: $P(t) = \xi(t)P_0$, where $\xi(t) = \sin 300t$. The reciprocity of the 1st derivative of the possible work with respect to time is shown. Flat curves on the graphs are plotted only for the peak load values shown on the right side of Figure 2. In the first state, the amplitudes of the forces p_{04}, p_{06} are multiplied by the speeds of the same nodes, caused by the amplitudes of the forces of the second state: p_{02}, p_{03}, p_{05}. The given oscillogram of the 1st derivative of the possible work (red color) coincides with a similar value (blue color) obtained by multiplying the amplitudes of the forces of the 2nd state at the speed of nodes 2, 3, 5, caused by the amplitudes of the forces of the 1st state.

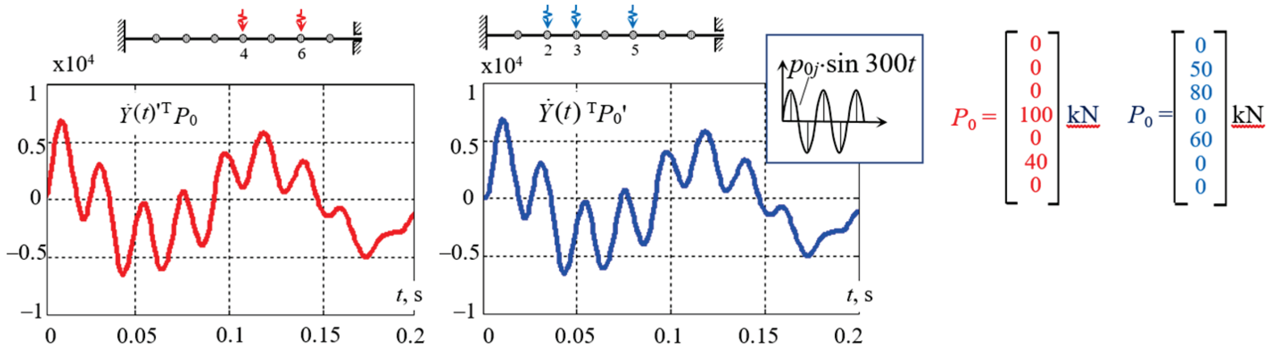

Figure 2. Reciprocity of the 1st derivative of the possible work under the action of vibrational forces in a rigidly clamped beam

On Figure 3 shows a spatial plot of the surface of the 2nd derivative of the possible work from the action of a sinusoidal impulse load $P(t) = \xi(t)P_0$, where $\xi(t) = \sin \pi t/t_1$, $t_1 = 0.1$ s is the pulse length. The pulse amplitudes have the same values as under the action of the vibration load in the previous example.

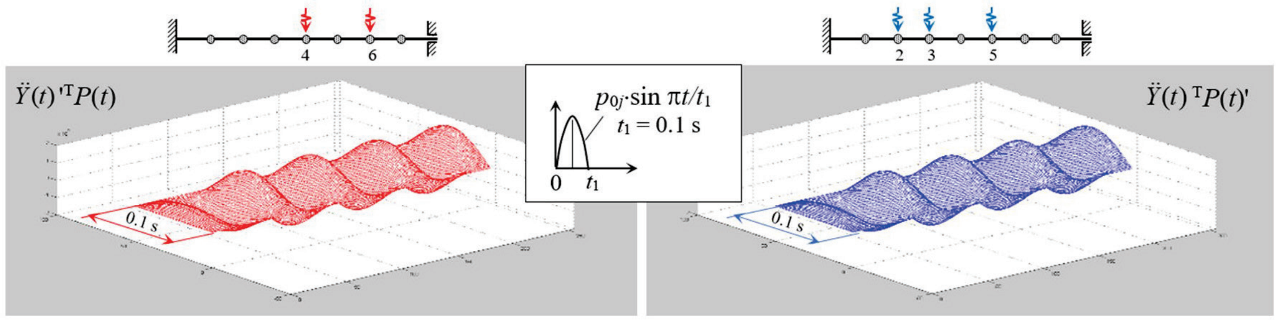

Figure 3. Surfaces of the 2nd derivative of the possible work under the action of sinusoidal pulses

CONCLUSION

1. Thus, a general method for proving reciprocity relations is given, based on the properties of the Duhamel integral and the algebraic theorem of Professor P.L. Pasternak.
2. In the formulation of the problem according to the reciprocity theorems, the following is allowed: the form of the external load is determined by an arbitrary scalar function of time; fluctuations can be both steady and unsteady; DDS damping type can also be arbitrary (proportional, non-proportional).
3. Obtained and proved particular laws of reciprocity of velocities and accelerations in dissipative systems. The expressions of the symmetric matrices of velocities and accelerations have

an analytical form of notation that does not require recourse to integral transformations.

4. The area of application of the laws of reciprocity, which go beyond the framework of the Betty principle, has been expanded. General reciprocity theorems about the 1st and 2nd derivatives of possible work with respect to time are proved.

The proposed approach indicates that the Duhamel integral is not only a tool for analyzing DDS, on the basis of which quantitative estimates of reaction parameters are built, but also an effective tool for studying such systems in order to obtain qualitative estimates. The revealed properties of symmetry in the structure of the reaction equations open up new possibilities for studying and understanding the theoretical foundations of the ongoing processes hidden in the nature of the oscillations of dissipative systems. In the field of fundamental research, which includes problems of reciprocity relations, the solution of these problems is highly topical and is of exceptional importance for practical use.

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