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# CHARACTERISTIC METHOD FOR SOLVING FILTRATION PROBLEM

***Liudmila I. Kuzmina*** [1]***, Yuri V. Osipov*** [2]***, Artem R. Pesterev*** [2]

[1] HSE University, Moscow, RUSSIA

[2] Moscow State University of Civil Engineering, Moscow, RUSSIA

**Abstract:** During construction, a liquid solution of a grout or waterproof filler is pumped into porous rock to improve its properties. The filtration of a suspension moving at a variable speed in a porous medium is simulated. A one-dimensional problem of filtration in a homogeneous porous medium with a curvilinear concentration front of suspended and retained particles is considered. For the numerical solution of the problem by the method of finite differences, the method of characteristics is used. The transition to characteristic variables allows one to straighten the front and construct a discrete grid with a constant step. When calculating the solution using an explicit difference scheme, additional points are used that do not coincide with the grid nodes. A detailed description of the algorithm for calculating a solution at the grid nodes and an example of a numerical solution of the problem are given.

**Keywords:** deep bed filtration, porous medium, suspended and retained particles, finite difference method, method of characteristics, numerical solution

# ХАРАКТЕРИСТИЧЕСКИЙ МЕТОД ДЛЯ РЕШЕНИЯ ЗАДАЧИ ФИЛЬТРАЦИИ

***Л.И. Кузьмина*** [1]***, Ю.В. Осипов*** [2]***, А.Р. Пестерев*** [2]

[1] Национальный исследовательский университет «Высшая школа экономики», г. Москва, РОССИЯ

[2] Национальный исследовательский Московский государственный строительный университет, г. Москва, РОССИЯ

**Аннотация:** При строительстве для улучшения свойств пористой породы в нее закачивается жидкий раствор укрепителя или водонепроницаемого наполнителя. Моделируется фильтрация суспензии, движущейся с переменной скоростью в пористой среде. Рассматривается одномерная задача фильтрации в однородной пористой среде с криволинейным фронтом концентраций взвешенных и осажденных частиц. Для численного решения задачи методом конечных разностей используется метод характеристик. Переход к характеристическим переменным позволяет выпрямить фронт и построить дискретную сетку с постоянным шагом. При вычислении решения по явной разностной схеме используются дополнительные точки, не совпадающие с узлами сетки. Приведено подробное описание алгоритма построения решения в узлах сетки и пример численного решения задачи.

**Ключевые слова:** глубинная фильтрация, пористая среда, взвешенные и осажденные частицы, метод конечных разностей, метод характеристик, численное решение

## 1. INTRODUCTION

Filtration of suspensions in porous media occurs in many construction technologies. To strengthen the foundation, liquid concrete is injected into loose soil, which, when solidified, strengthens the rock. During the construction of underground storage facilities for hazardous liquid waste, watertight partitions are constructed. To do this, a solution of a polymer or bentonite is pumped into the cavity, which, when wetted, expand and fill empty pores. During the reconstruction of roads, a grout is pumped into the eroded cavities under the roadbed, which is filtered in sand and fine gravel [1-6].

Filtration models describe the transport of suspended particles by a carrier fluid through pores. Under the influence of various forces (electrostatic, gravitational, hydrodynamic, mechanical and others), some particles are deposited on the framework of a porous medium and form an immobile deposit. Depending on the physical and chemical properties of the particles, of the liquid, of the porous medium frame and on the geometry of the pores, one or another mechanism of particle capture predominates. If the sizes of particles and pores are close, then the size-exclusion mechanism is the main one: suspended particles freely pass through large pores and get stuck in the neck of narrow pores [7–11].

Mathematical models of filtration include quasi-linear partial differential equations with various conditions which determine the unique solution. Analytical solutions of such problems are available only in the simplest cases. As a rule, numerical methods are used to obtain the solution [12–16].

The model of pure water displacement by a suspension in a porous medium is given by inconsistent initial and boundary conditions. The solution – the concentration of suspended particles – is discontinuous at the concentration front. It is zero before the front and positive behind the concentration front [17].

For the numerical solution of filtration problems, the finite difference method with a constant step is mainly used [18–20]. This method works well on a rectangular grid. It can be used also on a wedge obtained by cutting off from the rectangle the area before the rectilinear concentration front, where the solution is zero. The ratio between the time and coordinate steps is selected so that the grid nodes lie on the concentration front.

If the suspension velocity changes, then the concentration front becomes curvilinear [21]. In this case, to get the nodes to the concentration front, a variable grid spacing is usually chosen. It is also possible to keep the grid step constant and check each node whether it lies behind the front [22].

The proposed algorithm uses the characteristic variables of the hyperbolic system [23–25]. In these variables, the front becomes rectilinear, which makes it possible to apply the finite difference method with a constant step. The solution in Cartesian variables is obtained by reverse substitution. The algorithm is used to solve the filtration problem with deposit rise. For a variable suspension velocity, this problem has no analytical solution [26, 27].

## 2. FILTRATION MODEL WITH DEPOSIT EROSION

In the domain $\Omega = \{(x,t):\ x \ge 0,\ \ t \ge 0\}$, a one-dimensional filtration model of a monodisperse suspension moving in a porous medium with a variable speed $v(t) > 0$ is considered

$$\frac{\partial C}{\partial t} + v(t)\frac{\partial C}{\partial x} + \frac{\partial S}{\partial t} = 0\,, \qquad (1)$$

$$\frac{\partial S}{\partial t} = F(S)C - \varphi(S) \qquad (2)$$

with boundary and initial conditions

$$x = 0: C = 1, \quad t = 0: C = 0,\ S = 0\,. \qquad (3)$$

Consider (1) as an equation with respect to the unknown $C(x,t)$ for a given solution $S(x,t)$. Let us introduce a characteristic variable $\tau$. The characteristics of equation (1) are given by a system of ordinary differential equations [28]

$$\frac{dt}{d\tau} = 1, \quad \frac{dx}{d\tau} = v(t)\,. \qquad (4)$$

The initial conditions for characteristics (4) depend on their location. For characteristics coming out of the $OX$ axis

$$t(0) = 0, \quad x(0) = x_0,\ \ x_0 > 0\,, \qquad (5)$$

for characteristics coming out of the $OT$ axis

$$t(0)=t_0,\quad x(0)=0,\ t_0>0. \tag{6}$$

Integrating equations (4) with conditions (5) and (6), we obtain two families of characteristics:

$$\begin{aligned} OX:\ t=\tau,\quad x=\int_0^\tau v(t)d\tau+x_0; \\ OT:\ t=\tau+t_0,\quad x=\int_0^\tau v(t)d\tau. \end{aligned} \tag{7}$$

Since the initial and boundary conditions (3) are not consistent at the origin, the solution $C$ has a discontinuity. The discontinuity line $\Gamma$ – the concentrations front of the suspended and deposited particles is a characteristic emerging from the origin of coordinates. It is given by system (4) with the conditions $t(0)=0,\ x(0)=0$.

In this case $t=\tau$ and the concentration front is determined by the formula

$$x_\Gamma(t)=\int_0^t v(\varsigma)d\varsigma. \tag{8}$$

Using the method of characteristics, it can be shown that problem (1)-(3) has a zero solution in the domain $\Omega^0=\{(x,t):\ x\ge x_\Gamma(t),\ \ t\ge 0\}$ (before the front), behind the front in the domain $\Omega^+=\{(x,t):\ x<x_\Gamma(t),\ \ t\ge 0\}$ the solution is positive.

Substitute equation (2) into (1). In characteristic variables, equation (1) takes the form

$$\frac{dC}{d\tau}=-\left(F(S)C-\varphi(S)\right). \tag{9}$$

For the uniqueness of a positive solution, the conditions are set on the boundaries of the region $\Omega^+$:

$$C\big|_{\tau=0}=1,\qquad S\big|_{\Gamma}=0. \tag{10}$$

The limit value of the retained concentration at $t\to\infty$ is determined from the ratio

$$F(S)-\varphi(S)=0.$$

## 3. ALGORITHM FOR NUMERICAL SOLUTION

To effectively use the finite difference method in the domain $\Omega^+$, let's pass to variables $t,\ t_0\ (t\ge t_0)$:

$$\tau=t-t_0,\quad x(t,t_0)=\int_{t_0}^t v(\varsigma)d\varsigma. \tag{11}$$

From formulae (11) it follows

$$\frac{dC}{d\tau}=\frac{dC}{dt}.$$

We integrate equation (9) with respect to the variable $t$ from $t$ to $t+dt$:

$$\begin{aligned} &C\left(x(t+dt,t_0),t+dt\right)-C\left(x(t,t_0),t\right)= \\ &=-\int_t^{t+dt}\left(F(S)C-\varphi(S)\right)dt \end{aligned} \tag{12}$$

Approximate calculation of the integral on the right side of equation (12) is performed using the trapezoid method:

$$\begin{aligned} &\int_t^{t+dt}\left(F(S)C-\varphi(S)\right)dt=\frac{dt}{2}\Big(F\left(S(x(t+dt,t_0),t+dt)\right)C(x(t+dt,t_0),t+dt)- \\ &-\varphi\left(S(x(t+dt,t_0),t+dt)\right)+F\left(S(x(t,t_0),t)\right)C(x(t,t_0),t)-\varphi\left(S(x(t,t_0),t)\right)\Big). \end{aligned} \tag{13}$$

Substitute the integral (13) into equation (12):

$$C\left(x(t+dt,t_0),t+dt\right)= \\ =\frac{\left(2-F\left(S(x(t,t_0),t)\right)dt\right)C\left(x(t,t_0),t\right)+dt\left(\varphi\left(S(x(t+dt,t_0),t+dt)\right)+\varphi\left(S(x(t,t_0),t)\right)\right)}{2+F\left(S(x(t+dt,t_0),t+dt)\right)dt}. \qquad (14)$$

To find the value $S(x(t+dt,t_0),t+dt)$, integrate equation (2) over time for a fixed $x=x(t,t_0)$:

$$S(x(t,t_0),t+dt)-S(x(t,t_0),t)=\int_t^{t+dt}\left(F(S)C-\varphi(S)\right)dt. \qquad (15)$$

Calculate the integral on the right side of (15) using the left rectangle method:

$$S(x(t,t_0),t+dt)=S(x(t,t_0),t)+\left(F\left(S(x(t,t_0),t)\right)C\left(x(t,t_0),t\right)-\varphi\left(S(x(t,t_0),t)\right)\right)dt. \qquad (16)$$

Formulae (14), (16) make it possible to sequentially calculate the discrete values of the solution.

First, consider an example of how the algorithm works on the first few iterations. Denote by $dt$ a constant grid step in variables $t$, $t_0$ and let $C_{j,i}$, $S_{j,i}$ be the solution at the node $(t,t_0)=(jdt,idt),\ 0\le j\le N,\ 0\le i\le j$. We calculate solutions at the grid nodes by time layers starting from $t=0$.

The layer $t=0$ contains one grid node (0,0), in which the solution values are known from condition (3): $C_{0,0}=1,\ S_{0,0}=0$.

The layer $t=dt$ contains two mesh nodes: $(dt,0)$ and $(dt,dt)$. For $t_0=0$ the point $(x,t)\in\Gamma$ and according to (10) $S_{1,0}=0$. The value of $C_{1,0}$ is obtained from formula (14):

$$C_{1,0}=\frac{\left(2-F(0)dt\right)\cdot 1+2\varphi(0)dt}{2+F(0)dt}.$$

At the point $(dt,dt)$, the coordinate $x=0$ and according to (3) $C_{1,1}=1$. The value $S_{1,1}=S(0,dt)$ is determined by the formula (16):

$$S_{1,1}=S_{0,0}+\left(F(S_{0,0})\cdot 1-\varphi(S_{0,0})\right)dt.$$

The layer $t=2dt$ includes three mesh nodes: $(2dt,0)$, $(2dt,dt)$, $(2dt,2dt)$. For $t_0=0$ the point $(x,t)\in\Gamma$ and $S_{2,0}=0$. The value $C_{2,0}$ is determined by formula (14). At the node $(2dt,2dt)$, the coordinate $x=0$ and $C_{2,2}=1$. The value $S_{2,2}=S(0,2dt)$ is calculated by formula (16):

$$S_{2,2}=S_{1,1}+\left(F(S_{1,1})\cdot 1-\varphi(S_{1,1})\right)dt.$$

Let's define the value $S_{2,1}$ in the node $(2dt,dt)$. First, we find $S\left(x(dt,0),2dt\right)$ by formula (16):

$$S\left(x(dt,0),2dt\right)=S_{1,0}+\left(F(S_{1,0})C_{1,0}-\varphi(S_{1,0})\right)dt.$$

Note that the point $(t,t_0)$ corresponding to the Cartesian coordinates $x=x(dt,0),\ t=2dt$ lies on the layer $t=2dt$, but generally speaking, does not coincide with the grid nodes. It follows from formula (11) that the point $(t,t_0)$ cannot coincide with the node $(2dt,0)$.

On a layer $t=2dt$ he value $S_{2,1}$ is obtained by linear interpolation in *x* using two known values $S\left(x(dt,0),2dt\right)$ and $S_{2,0}$:

$$S_{2,1} = \frac{S\left(x(dt,0),2dt\right)\left(x(2dt,dt)-x(2dt,0)\right)+S_{2,0}\left(x(dt,0)-x(2dt,dt)\right)}{x(dt,0)-x(2dt,0)}.$$

The value $C_{2,1}$ is determined by formula (14).

Consider the algorithm in the general case. Let the solution in all $j+1$ nodes of the layer $t = jdt$ is known. Let us find the solution at the nodes of the next layer $t+dt=(j+1)dt$. The node $(j+1,0)$ lies on the front $\Gamma$ and $S_{j+1,0}=0$, $C_{j+1,0}$ is calculated by the formula (14). The node $(j+1,j+1)$ lies on the axis *OT*, $C_{j+1,j+1}=1$, $S_{j+1,j+1}$ is obtained from the formula (14). Let the solutions at the nodes $(j+1,0)$, ..., $(j+1,i-1)$ be computed. To obtain the solution at the internal node $(j+1,i)$, $1\le i\le j$, calculate $S\left(x(t,t_0),t+dt\right)$, $t_0=idt$ by formula (16) and linearly interpolate $S_{j+1,i}$ over two values $S\left(x(t,t_0),t+dt\right)$ and $S_{j+1,i-1}$. The value $C_{j+1,i}$ is determined by formula (14).

Since the computational formulae of the algorithm use one time layer, an array of solutions of only one time layer is stored in the computer's RAM.

Note that when passing to Cartesian coordinates $x,t$, the grid becomes non-uniform, the time step $dt$ is constant, but in the layer $t_j = jdt$ the nodes are distributed non-uniformly due to variable speed $v(t)$:

$$x_i = x(jdt,idt) = \int_{idt}^{jdt} v(\varsigma)d\varsigma .$$

## 4. RESULTS OF NUMERICAL SIMULATION

The calculation of the filtration problem with sediment rise is performed for system (1), (2) with the functions

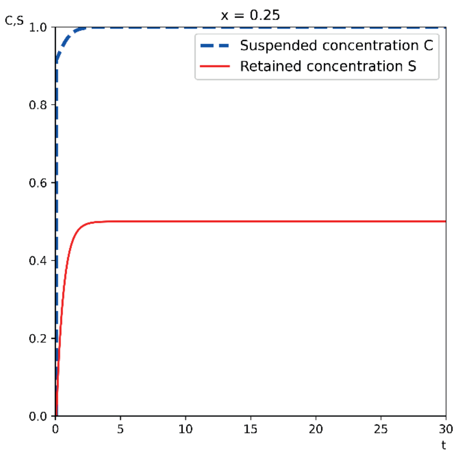
$$v(t)=1.5\left(1+e^{-2t}\right),\ F(S)=1-S,\ \varphi(S)=S .$$

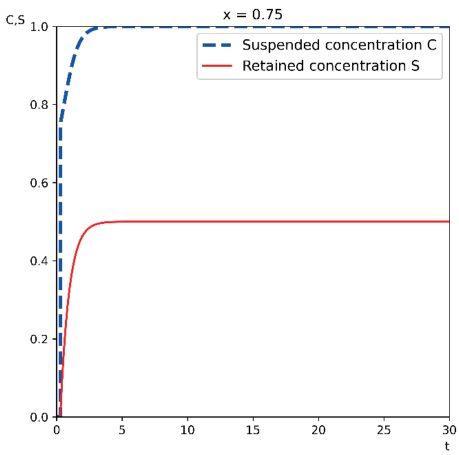
In variables $t$, $t_0$ the grid step $dt=0.01$.

Since the step in *x* is not constant in Cartesian coordinates $x,t$, the grid nodes, generally speaking, do not lie at the intersection of the time layers of the grid with the lines $x=const$. To plot two-dimensional graphs of solutions at constant *x*, it is required to interpolate solutions using values at close nodes. The Python software automatically performs interpolation and presents graphs using built-in algorithms of open-source library SciPy [29, 30].

Fig. 1 and 2 show solutions at constant time *t* and constant coordinate *x*.

According to Fig. 1 and 2 the deposit concentration is rapidly approaching the limit value $S^*=0.5$.



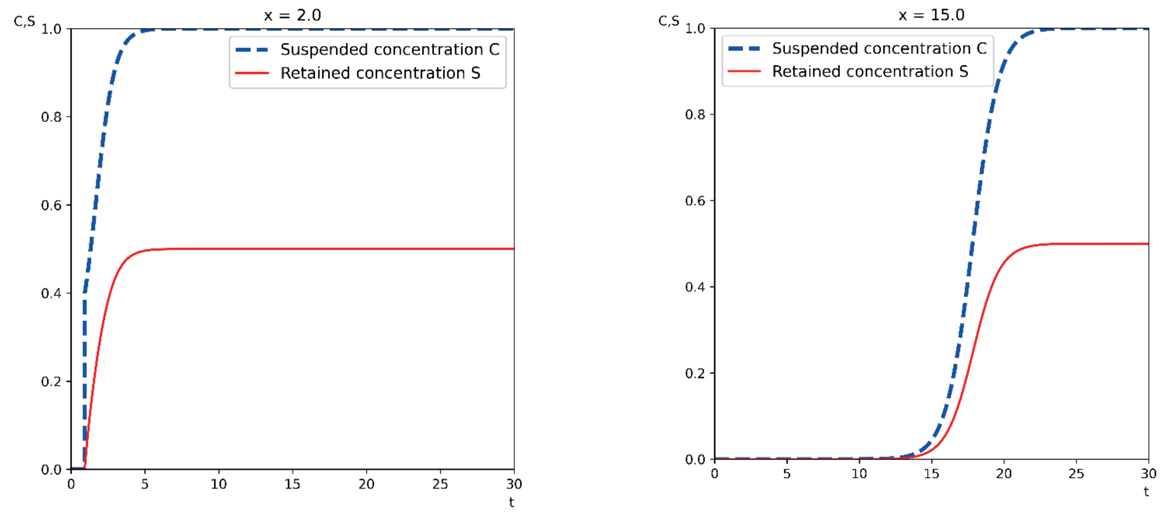



*Figure 1. Solutions C and S at constant x.*

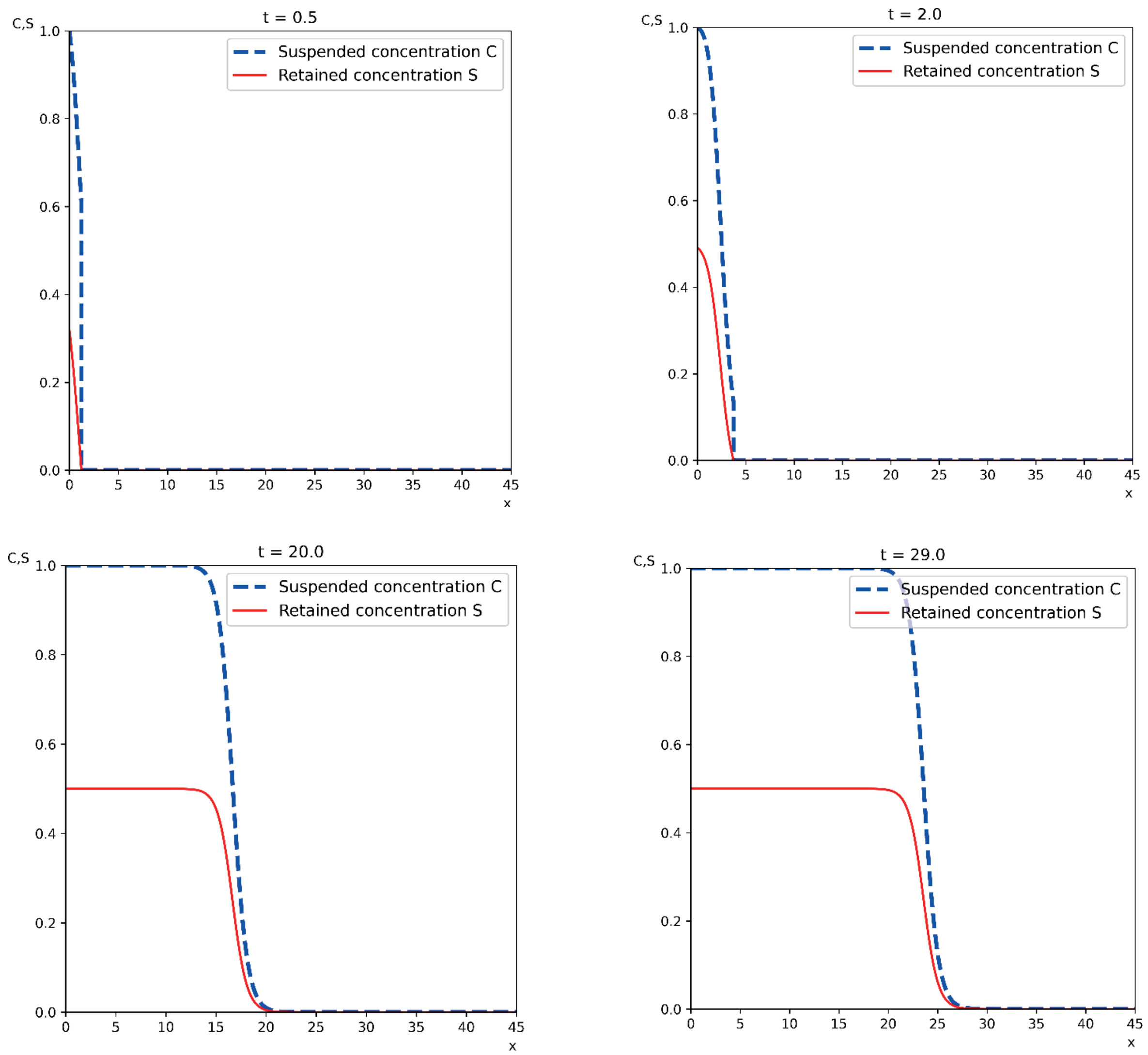



*Figure 2. Solutions C and S at constant t*

## 5. DISCUSSION

The change of variables in the system of filtration equations makes it possible to straighten the curvilinear concentration front and construct a discrete grid with a constant step in both new variables. This method is used to derive the lattice Boltzmann equations for solving hydrodynamic problems [31].

There are various options for choosing variables, in which the solution area is limited by straight lines. For example, characteristic variables $\tau, t_0$ can be used. We have chosen variables $t, t_0$ to simplify both equations of system (1), (2).

In the filtration problem without deposit lifting (in equation (2) $\varphi(S)=0$), the suspension velocity is reduced to zero. This significantly complicates the computational procedure. However, the proposed method is easily transferred to this case. This problem will be considered separately.

## 6. CONCLUSIONS

To calculate the filtration problem with a variable suspension speed

- the finite difference method is applied,
- the method of characteristics is used,
- the solution is obtained on a grid with a constant step,
- to find the solution, auxiliary points are used that do not coincide with the grid nodes.

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*Liudmila I. Kuzmina*, Candidate of Physical and Mathematical Sciences, Associate Professor, Department of Applied Mathematics, National Research University Higher School of Economics, 101000, Russia, Moscow, Myasnitskaya st., 20, tel. +7(495) 77295 90 * 15219, e-mail: lkuzmina@hse.ru.

*Yuri V. Osipov*, Candidate of Physical and Mathematical Sciences, Professor, Department of Computer Science and Applied Mathematics, Moscow State University of Civil Engineering, 129337, Russia, Moscow, Yaroslavskoe Shosse, 26, tel. +7(499)1835994, e-mail: yuri-osipov@mail.ru.

*Artem R. Pesterev*, 3rd year bachelor's degree student of the direction of training 01.03.04 of the Institute IDTMC of the National Research Moscow State University of Civil Engineering; 129337, Russia, Moscow, Yaroslavskoe shosse, 26; tel. +7(499)1834674; e-mail: vip.pesterev_a@mail.ru.

*Кузьмина Людмила Ивановна*, доцент, кандидат физико-математических наук, Департамент прикладной математики, Национальный исследовательский университет «Высшая школа экономики»; 101000, г. Москва, ул. Мясницкая, д. 20, тел. +7(495) 77295 90 *15219; e-mail: lkuzmina@hse.ru.

*Осипов Юрий Викторович* профессор, кандидат физико-математических наук, кафедра информатики и прикладной математики Национального исследовательского Московского государственного строительного университета; 129337, Россия, г. Москва, Ярославское шоссе, д. 26; тел. +7(499)1835994; e-mail: yuri-osipov@mail.ru.

*Пестерев Артем Романович*, студент 3 курса бакалавриата направления подготовки 01.03.04 института ИЦТМС Национального исследовательского Московского государственного строительного университета; 129337, Россия, г. Москва, Ярославское шоссе, д. 26; тел. +7(499)1834674; e-mail: vip.pesterev_a@mail.ru.