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DYNAMIC MODEL OF BEAM DEFORMATION WITH CONSIDER NONLOCAL IN TIME ELASTIC PROPERTIES OF THE MATERIAL

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Abstract: In this paper, the problem of numerical dynamic calculation of a beam made of composite material with a developed internal structure is considered. The elastic properties are assumed to be nonlocal in time. A short review of the existing methods for mathematical modeling of the dynamic behavior of elements with a developed internal structure was carried out. A non-local in time model of dynamic deformation of a bending beam is constructed. Since the finite element analysis (FEA) is the most demanded numerical method for mechanical systems analysis, a non-local dynamic deformation model is integrated into the algorithm of this method. The equilibrium equation of the structure in motion is solved by an explicit scheme. The damping matrix is obtained from the condition of stationarity of the total deformation energy of a moving mechanical system. A one-dimensional non-local in time model was implemented in the MATLAB software package.

Keywords: Nonlocal mechanics, nonlocal damping, numerical simulation, finite element method

МОДЕЛЬ ДИНАМИЧЕСКОГО ДЕФОРМИРОВАНИЯ ИЗГИБАЕМОЙ БАЛКИ С УЧЕТОМ НЕЛОКАЛЬНЫХ ВО ВРЕМЕНИ УПРУГИХ СВОЙСТВ МАТЕРИАЛА

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Аннотация: В настоящей работе рассматривается задача численного динамического расчета изгибаемой балки из композитного материала с развитой внутренней структурой с учетом нелокальных во времени упругих свойств. Был проведен краткий обзор существующих методов математического моделирования динамического поведения элементов с развитой внутренней структурой. Построена нелокальная модель деформирования изгибаемой балки под действием динамической нагрузки. Поскольку метод конечных элементов (МКЭ) является наиболее востребованным численным методом анализа механических систем, нелокальная модель динамического деформирования интегрирована в алгоритм этого метода. Уравнение равновесия конструкции в движении решается по явной схеме. Матрица демпфирования получена из условия стационарности полной энергии деформирования движущейся механической системы. Одномерная нелокальная во времени модель была реализована в программном комплексе MATLAB.

Ключевые слова: Нелокальная механика, нелокальное демпфирование, численное моделирование, метод конечных элементов

INTRODUCTION

The development of construction technologies and the gradual implementation of new composite and nano-materials with "controllable" physical characteristics in construction require the creation of appropriate mathematical models that allow to reliably describe the behavior of such materials, in particular under the dynamic load.

Generally, in order to obtain sufficient accuracy of the numerical calculation, three-dimensional finite element models are used to take into account the orthotropic properties of the material. However, these models are resource-intensive and difficult to form and analyze. For instance, as a result of computation, only the fields of the stress-strain state in elements and their nodes can be obtained as a result of 3D finite element modelling, which is not always sufficient for engineering analysis of the calculation results. As an alternative to 3D modeling the one dimensional elements can be used, constructed using reasonable mathematical hypotheses that allow to describe the characteristic properties of the material. The special hypotheses turn out to be necessary, since the frequently used classical viscoelastic models proposed in the works of W. Kelvin [1], J. Maxwell [2], J. Rayleigh [3], Voigt [4] no longer allow accurately model the behavior of materials with a complex internal structure.

To simulate the dynamic behavior of structural elements made of composite materials, models based on the principles of non-local mechanics are applicable. Such models may include the models proposed in the works of A. Eringen and D. Edelen [5], D. Russell [6], Banks and Inman. [7], Lei [8] and V. D. Potapov [9].

NON-LOCAL DAMPING MODELS

A wide class of non-local models applicable to describe the dynamic behavior of composite materials are non-local damping models.

In the article [8] Y. Lei proposed a non-local damping model that takes into account the effects of time and spatial hysteresis. This model is used for dynamic analysis of structures consisting of Euler–Bernoulli beams and Kirchhoff plates. Unlike classic local damping models, the damping force in the non-local model is defined as a weighted average of the velocity field in the spatial domain determined by a kernel function based on distance measures. Also, the resulting equation of motion for beam or plate structures is a partial integro-differential equation, in contrast to the partial differential equation for the local damping model. Approximate solutions for complex eigenvalues and modes with nonlocal damping are obtained using the Galerkin method. Numerical examples demonstrate the effectiveness of the proposed method for beam and plate structures with simple boundary conditions, for non-local and inviscid damping models and various core functions.

$$L_e\dot{u}(r,t) = \int_{\Omega}\int_0^t C_e(r,\xi,t-\tau)\dot{u}(\xi,\tau)d\tau d\xi \quad (1)$$

$$L_i\dot{u}(r,t) = \int_{\Omega}\int_0^t C_i(r,\xi,t-\tau)L_s\dot{u}(\xi,\tau)\quad (2)$$

The equation of motion in this case is expressed as the following integro-differential equation in partial derivatives

$$\frac{\partial^2}{\partial x^2}\left(EI(x)\frac{\partial^2 w(x,t)}{\partial x^2}\right) + \rho A(x)\frac{\partial^2 w(x,t)}{\partial t^2} + \int_{x_1}^{x_2}\int_{-\infty}^{t} C_e(x,\xi,t-\tau)\frac{\partial w(\xi,\tau}{\partial t}d\tau d\xi + \int_{x_1}^{x_2}\int_{-\infty}^{t} C_i(x,\xi,t-\tau)\frac{\partial^2}{\partial \xi^2}\left(\gamma(\xi)\frac{\partial^3 w(\xi,t)}{\partial \xi^2 \partial \tau}\right)d = f(x,t) \quad (3)$$

Another solution of the problem of non-local damping was proposed in the article by Banks H.T., Inman D.J. [7], where various damping mechanisms are considered for a quasi-isotropic pultruded composite beam. The approach used here is physical. The partial differential equation describes the transverse vibrations of a beam with a mass at the free end. All damping mechanisms considered in the article have an explicit physical basis, in contrast to the usual modal model. Four possible damping mechanisms are considered: one external and three internal. These include: viscous damping (air damping); internal damping depending on strain rates; spatial hysteresis; and time hysteresis. In addition, various combinations of these mechanisms are considered.

These physical damping models are incorporated into the Euler-Bernoulli beam equation, with boundary conditions carefully formulated to be compatible with different damping models. The resulting partial differential equation (in case of decaying time hysteresis) is approximated using cubic splines. Time histories of the measured experimental responses are then used to estimate the parameters of the models corresponding to the data using the method of least squares. The resulting least squares evaluations of various damping parameters are then used in a partial (integro-partial) differential equation for numerical simulation of the system response. This numerically generated time response of the system being evaluated is then compared with the actual data obtained experimentally. These comparisons allow several conclusions to be drawn regarding the physical damping mechanisms present in a composite beam. In particular, it is shown that the spatial hysteresis model in combination with the external damping model leads to the best fit to the experimental data. The article also notes that the proposed damping models cannot be successfully built using standard damping coefficients in fractions of the critical, since the traditional approach to modal analysis completely masks the physics of damping mechanisms.

The solution for the non-local in space model of damping was made in the article [10]. It is shown that calibrated nonlocal model applied to one-dimensional beam adequately approximate the results of the 3D numerical simulation. An alternative model with damping non-local in time was shown in the article [11]. In comparison to the model from [10] the nonlocal in time model is integrated into the FEA algorithm.

MODELS OF ELASTIC MATERIAL PROPERTIES NON-LOCAL IN SPACE

Historically, one of the first models of an elastic medium that cannot be described within the framework of the classical theory of elasticity is the Cosserat continuum (1909). However, for a long time the work of F. Cosserat remained unnoticed, and only starting from about 1958-60 rr. generalized models of the Cosserat continuum began to be intensively developed. the theory of oriented media, asymmetric, moment, multipolar, micromorphic, etc. theories of elasticity have arisen. The equations of motion for the Cosserat model coincide with the equations of motion for a diatomic chain, and, therefore, in the (x, t)-representation they have the form

$$\begin{gathered} \rho\ddot{u} - \partial y \partial u - \partial x + \eta = q \\ I\ddot{\eta} - \chi\partial u + \Gamma\eta = \mu \end{gathered} \tag{4}$$

here u is the transverse displacement, n is the microrotation, μ is the density of the corresponding micromoments, l is the density of the moments of inertia of the particles. The remaining quantities have the same meaning as in the case of a diatomic chain

Early ideas of non-local elasticity go back to the pioneering work of Kroner [12], Kunin [13], Krumhansl [14]. Improved formulations were then presented in Edelen and Laws [15] and Eringen [16, 17, 18]. Eringen model of a nonlocal elasticity is constructed by integration of an integral member to the Hook`s law:

$$\sigma(t,x) = E\int_{-\infty}^{\infty} C(|x - x`|)\varepsilon(x`)dx`, \tag{5}$$

The stress field at a point x in an elastic continuum not only depends on the strain field at the point but also on strains at all other points of the body.

The Eringen model of non-local elastic material was further developed in works of A.A. Pisano [19]. The article solved the problem of stretching of a bar of finite length, to both ends of which longitudinal forces are applied. This integral-type relation contains a non-local damping function designed to capture the process of diffusion of non-local effects. This article is devoted to finding an exact solution to a simple mechanical problem. The solution of this problem is obtained through the transformation of the second kind Fredholm integral equation, which determines the problem, into two second kind Volterra integral equations. As a result, an exact solution in terms of strains for a non-local elastic rod is obtained.

The article [9] of V.D. Potapov is devoted to the study of the stability of an infinitely long rod lying on an elastic foundation and under the action of a constant or periodically changing longitudinal force. The rod material used in the calculations is characterized by non-local viscoelastic properties. The influence of the parameters characterizing the nonlocality of the viscoelastic properties of the material, as well as the parameters of the load on the buckling and stability of the rod, is analysed. The article assumes that the relationship between stress σ and strain ε for the rod material has the form:

$$\sigma(t,x) = E\int_{-\infty}^{\infty} C(|x - x`|)(1 - \mathrm{R})\varepsilon(\tau, x`)dx`, \quad (6)$$

where $C(|x - x`|)$ is the kernel of non-local viscosity along the coordinate, E is the modulus of elasticity, R is the integral viscoelasticity operator:

$$\mathrm{R}\varepsilon(\tau, x`) = \int_{-\infty}^{t} R(t-\tau)\varepsilon(\tau, x`)\, d\tau \qquad (7)$$

$R(t,\tau)$ – viscoelasticity kernel, t, τ – time, x, x` – coordinates measured along the rod axis.

It is obvious that, in addition to nonlocality in space, the material considered in the article has the property of memory, since the kernel $R(t,\tau)$ makes it possible to take into account the deformed states of the system over the entire loading history.

THE MODEL OF ELASTIC PROPERTIES OF A MATERIAL IS NON-LOCAL IN TIME

The article [11] shows that a nonlocal in time model of the dissipative properties of the material can be relatively easily integrated into the FEA algorithm. Therefore, the nonlocal model of the elastic properties of the material considered in this work was also implemented in relation to the FEA. Hence, after the calibration such model can be used in solving of applied dynamic problems.

For the model presented in [9], the lower limit of integration over the time domain was taken equal to minus infinity. Strictly speaking, this is mathematically correct, but physically, any system manifests itself for a finite period of time. Yu.N. Rabotnov in [20] notes that the beginning of the real existence of the system has to be chosen as a lower limit. Therefore, in the following equations the initial moment of the oscillatory process *t=0* was used as the lower limit of integration.

DYNAMIC MODEL OF BENDING BEAM DEFORMATION WITH CONSIDERING NON-LOCAL IN TIME ELASTIC PROPERTIES OF THE MATERIAL

Considering the above a nonlocal in time model of the elastic properties of the material can be effectively used for modeling of the composite elements dynamic behavior. In this paper it is assumed that the elastic forces in the structure depend not only on the displacement values at the current time, but also on the previous time history of deformation of the structure. Moreover, the greater the time interval between two moments of time, the less is the influence that one of them has on the other. In the other words, the memory is considered to be fading.

Further calculations were carried out in the MATLAB software package. As a numerical example a 10-meter beam with fixed ends was modeled. The beam material is pultruded fiberglass. The Young's modulus of this material in the longitudinal direction is equal to 28 GPa. The beam cross section is rectangular: 0.3m high and 0.2m wide. The coefficient of relative damping of the material is assumed to be 0.015. The beam is loaded with an instantaneously applied and uniformly distributed load.

In the FEA algorithm, the equilibrium equation for a structure deformed in motion is represented as [21]:

$$M \cdot \ddot{\bar{V}}(t) + D \cdot \dot{\bar{V}}(\tau) + K \cdot \bar{V}(t) = \bar{F}(t) \qquad (8)$$

Taking into account the elastic properties nonlocal in time, this expression takes the form:

$$M \cdot \ddot{\bar{V}}(t) + D \cdot \dot{\bar{V}}(\tau) + \\ +K \cdot \int_0^t R(t-\tau) \cdot \bar{V}(t) d\tau = \bar{F}(t) \tag{9}$$

Here $R(t-\tau)$ is the kernel of the elasticity operator used in this work. This function describes the decrease in the influence of the past history of dynamic deformation of the element on the current value of elastic forces in the system. In this case, the normalization condition is satisfied:

$$\int_0^t R(t-\tau)\, d\tau = 1 \tag{10}$$

For numerical calculations, the error function was used as the memory core, which, subject to the normalization condition, takes the form:

$$R(t-\tau) = \frac{2\eta}{\sqrt{\pi}} \cdot e^{-\eta^2 (t-\tau)^2} v \tag{11}$$

Here η is a parameter characterizing the scale of nonlocality of elastic forces in time. The small η parameter conforms to the highly nonlocal properties of the material.

For the numerical solution of the equation of dynamic equilibrium, the method of central differences was used. When converting equation (2) into a computational scheme using the method of central differences, the equation of motion takes the form:

$$\frac{1}{\Delta t^2} \cdot M \cdot (\bar{V}_{i+1} - 2\bar{V}_i + \bar{V}_{i-1}) + \\ + \frac{1}{2\Delta t} D \cdot (\bar{V}_{i+1} - \bar{V}_{i-1}) + K \cdot \bar{Z} = \bar{F}_i \tag{12}$$

where $\bar{Z}$ – is a discrete analogue of the integral kernel $\int_0^t R(t-\tau) d\tau$

$$\bar{Z} = \sum_1^i \frac{2\eta}{\sqrt{\pi}} \cdot e^{-\eta^2 \left(t - \left(\tau - \frac{\Delta t}{2}\right)\right)^2} \bar{V}_i. \tag{13}$$

The computational scheme for the sequential step-by-step calculation of $\bar{V}_{i+1}$ through the vectors $\bar{V}_{\mathrm{i}}$ and $\bar{V}_{i-1}$, which are calculated in the previous steps, is based on (12)

$$\left(\frac{1}{\Delta t^2} M + \frac{1}{2\Delta t} D\right) \bar{V}_{i+1} + \left(-\frac{2}{\Delta t^2} M\right) \bar{V}_i \\ + \left(\frac{1}{\Delta t^2} M - \frac{1}{2\Delta t} D\right) \cdot \bar{V}_{i-1} + \\ + \mathrm{K} \cdot \bar{Z} = \bar{F}_i. \tag{14}$$

We set in (14):

$$Q = \left(\frac{1}{\Delta t^2} M + \frac{1}{\Delta t} D\right)^{-1}, \\ Q_1 = Q \cdot \left(-\frac{2}{\Delta t^2} M\right), \\ Q_2 = \left(\frac{1}{\Delta t^2} M - \frac{1}{\Delta t} D\right) Q, \\ Q_3 = Q \cdot K. \tag{15}$$

The final scheme for a step-by-step solution in time of the discrete equation of motion (9) using the accepted model of deformation of a material with memory takes the form:

$$\bar{V}_{i+1} = Q \cdot \bar{F}_i - Q_1 \cdot \bar{V}_i - Q_2 \cdot \bar{V}_{i-1} - Q_3 \\ \cdot \bar{Z}. \tag{16}$$

At the first step, for i = 1, $\bar{V}_0 = 0$ and $\bar{V}_1 = 0$ are taken as initial conditions.

Further calculations were carried out in the MATLAB software package.

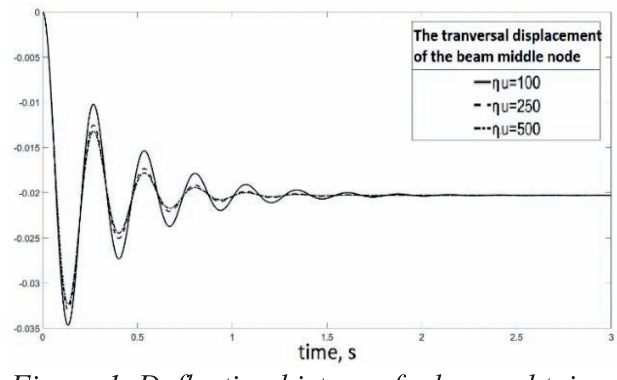

Figure 1. Deflection history of a beam obtained using a model of elastic material properties nonlocal in time for different values of η

As a result of the calculations, graphs of functions with different values of η equal to 100, 250 and 500, were obtained. Analyzing these graphs, we can say that this parameter affects the amplitude of oscillations, but does not affect the frequency of oscillations, a decrease in the parameter η leads to an increase in amplitude. Hence, higher deflection attitude corresponds to the higher level of nonlocality.

CONCLUSION

The article provides a brief overview of existing methods of mathematical modeling of the dynamic behavior of elements made of materials with a developed internal structure. A non-local model of deformation of a bending beam is constructed using the finite element method. The method of central differences was used for the numerical solution of the equation of motion. And the relationship between the scale parameter η and the amplitude of vibrations of the bent beam is shown.

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