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BULK THEORY ELASTICITY FINITE ELEMENT BASED ON PIECEWISE CONSTANT APPROXIMATIONS OF STRESSES

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Abstract. The solution of the volume theory elasticity problem was obtained on the basis of the additional energy functional and the possible displacements principle. On the basis of the possible displacements' principle, equilibrium equations for grid nodes are compiled, which are added to the additional energy functional using Lagrange multipliers. Linear functions are taken as possible displacements. The volumetric finite element based on piecewise constant approximations of stresses is presented. The stress fields are continuous along finite element boundaries and discontinuous inside ones. The calculation results of a cantilever beam and a bending plate are presented. The obtained solutions are compared with the solutions by the finite element method in displacements. The proposed finite element makes it possible to obtain more accurate stress values.

Keywords: finite element method, piecewise-constant stress approximation, bulk theory elasticity

КОНЕЧНЫЙ ЭЛЕМЕНТ ДЛЯ ОБЪЕМНОЙ ТЕОРИИ УПРУГОСТИ НА ОСНОВЕ КУСОЧНО ПОСТОЯННЫХ АППРОКСИМАЦИЙ НАПРЯЖЕНИЙ.

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Аннотация. Представлен объемный конечный элемент, основанный на кусочно-постоянных аппроксимациях напряжений. Решение строится на основе функционала дополнительной энергии и принципа возможных перемещений. На основе принципа возможных перемещений составлены уравнения равновесия узлов сетки, которые добавляются к функционалу дополнительной энергии при помощи множителей Лагранжа. В качестве возможных перемещений принимаются линейные функции. Поля напряжений непрерывны вдоль границ конечных элементов и разрывны внутри элементов. Приведены результаты расчета консольной балки и изгибаемой пластины. Полученные решения сравниваются с решениями методом конечных элементов в перемещениях. Предложенный конечный элемент позволяет получить более точные значения напряжений.

Ключевые слова: метод конечных элементов, приближение кусочно-постоянных напряжений, объемная теория упругости

INTRODUCTION

A large number of papers are devoted to the development of the finite element method in displacements. They present the variational principles of finite element method and solution algorithms [1]. Also, other numerical methods are being developed. In particular, the method of boundary elements [2] and collocation method [3]. The finite element method in displacements has been widely used to calculate structures for various purposes due to its versatility. This method is well researched in terms of convergence and accuracy of solutions. An alternative approach based on stress approximations is much less explored [4-5]. The papers [6-16] propose finite elements based on the approximation of stresses for solving problems of the plane elasticity theory, plates bending and rod systems. Currently, volumetric finite elements are increasingly used to calculate such structures as slabs, shells and composite structures [17-19], [20-22]. The development of new volumetric finite elements is still relevant. The purpose of this work is to develop a volumetric finite element based on stress approximation. Obtaining another approximate solution, which alternative to the existing approximate solutions, will allow one to obtain a two-sided estimate of the exact solution.

METHODS

The solution to the bulk theory elasticity problem will be built based on the additional energy functional:

$$\Pi = \frac{1}{2}\int_V \begin{pmatrix} \frac{1}{E}\left(\sigma_x^2+\sigma_y^2+\sigma_z^2\right) \\ -\frac{\nu}{E}\left(\sigma_x\sigma_y+\sigma_x\sigma_z+\sigma_y\sigma_z\right) \\ +\frac{1}{G}\left(\tau_{xy}^2+\tau_{xz}^2+\tau_{yz}^2\right) \end{pmatrix} dV. \quad (1)$$

E is the material elastic modulus; ν is Poisson's ratio, $G = 2(1+\mu)/E$. We write the functional (1) in matrix form:

$$\Pi = \frac{1}{2}\int_V \boldsymbol{\sigma}^{\mathbf{T}}\mathbf{E}^{\mathbf{-1}}\boldsymbol{\sigma} dV,$$

$$\boldsymbol{\sigma}^{\mathbf{T}} = \begin{pmatrix} \sigma_x & \sigma_y & \sigma_z & \tau_{xy} & \tau_{xz} & \tau_{yz} \end{pmatrix},$$

$$\mathbf{E}^{\mathbf{-1}} = \begin{bmatrix} 1/E & -\nu/E & -\nu/E & 0 & 0 & 0 \\ -\nu/E & 1/E & -\nu/E & 0 & 0 & 0 \\ -\nu/E & -\nu/E & 1/E & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/G & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/G & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/G \end{bmatrix}. \quad (2)$$

To discrete the volume, we use the rectangular parallelepiped finite elements. Let's denote the vector nodal unknown stresses vector as

$$\boldsymbol{\sigma}_{node,k}^T = \begin{pmatrix} \sigma_{x,k} & \sigma_{y,k} & \sigma_{z,k} & \tau_{xy,k} & \tau_{xz,k} & \tau_{yz,k} \end{pmatrix}. \quad (3)$$

Vector of finite element unknown stresses is

$$\boldsymbol{\sigma}_{el,k}^T = \begin{pmatrix} \boldsymbol{\sigma}_{node,1}^T & \boldsymbol{\sigma}_{node,2}^T & \ldots & \boldsymbol{\sigma}_{node,8}^T \end{pmatrix}. \quad (4)$$

To simplify the notation, let's introduce unit step functions and diagonal matrices

$$h_i(x,y,z) = \begin{Bmatrix} 1, & (x,y,z)\in V_i \\ 0, & (x,y,z)\notin V_i \end{Bmatrix}, \quad \mathbf{H}_i = \begin{bmatrix} h_i & & \\ & h_i & \\ & & h_i \end{bmatrix}. \quad (5)$$

Then the approximation matrix of stresses in the finite element volume will have the simple form (fig. 1b):

$$\mathbf{Z}_k = \begin{bmatrix} \mathbf{H}_1 & \mathbf{H}_2 & \ldots & \mathbf{H}_8 \end{bmatrix}, \quad \boldsymbol{\sigma} = \mathbf{Z}_k \boldsymbol{\sigma}_{el,k}. \quad (6)$$

Using (4-6), we express the finite element additional energy in the following form:

$$\Pi_k = \int_{V_k} \sigma_{el,k}^T (Z_k^T E_k^{-1} Z_k)\, \sigma_{el,k} \mathrm{dV}. \quad (7)$$

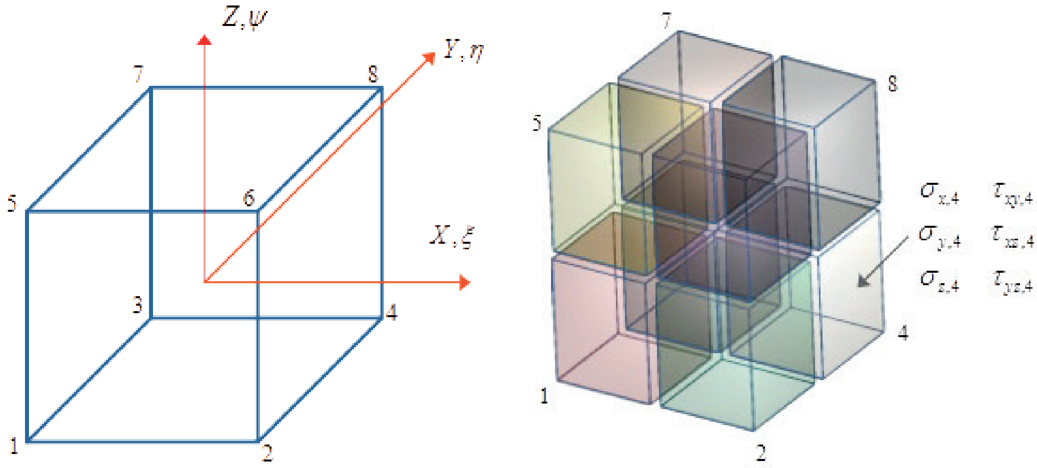
The pliability matrix of the finite element is

$$D_k = \int_{V_k} Z_k^T E_k^{-1} Z_k \,\mathrm{dV}. \quad (8)$$

The matrix has a simple block-diagonal form [24]. From local matrices of finite elements are formed global matrix D of whole system. The functional (1) of whole system:

$$\Pi = \frac{1}{2}\boldsymbol{\sigma}_V^T \mathbf{D} \boldsymbol{\sigma}_V. \quad (9)$$

$\boldsymbol{\sigma}_V$ is global vector of the system unknown stresses.



a b

Figure 1. a) global and local coordinate systems; b) division of a finite element into volumes with constant stresses

Based on the possible displacements' principle, we can compose the necessary equilibrium equations of the grid nodes. Each node of finite elements grid has three possible displacements along the OX, OY, OZ axes of the global coordinate system (Fig. 1). Approximations of the possible displacements over the finite element volume are the linear basis functions.

$$\begin{aligned}\delta u_i &= \frac{1}{8}\left(1+\xi\xi_i\right)\left(1+\eta\eta_i\right)\left(1+\psi\psi_i\right),\\ \delta v_i &= \frac{1}{8}\left(1+\xi\xi_i\right)\left(1+\eta\eta_i\right)\left(1+\psi\psi_i\right),\\ \delta w_i &= \frac{1}{8}\left(1+\xi\xi_i\right)\left(1+\eta\eta_i\right)\left(1+\psi\psi_i\right).\end{aligned} \tag{10}$$

In (10), the local dimensionless node i coordinates of the finite element, taking the values -1 or 1. Dimensionless coordinates are related to local coordinates x, y, z by known expressions:

$$\xi = \frac{2x}{a}, \eta = \frac{2y}{b}, \psi = \frac{2z}{c}. \tag{11}$$

Expressions (10) correspond to unit displacement of node i and zero displacements of other nodes of finite elements adjacent to the considered node. For each of these finite elements, the considered node will have different index values i in the range from 1 to 8. The index value corresponds to the ordinal node number in the description of the finite element (Fig. 1). The node i possible displacement along the X axis causes the following deformations in the finite elements adjacent to the node:

$$\begin{aligned}\delta\varepsilon_x &= \frac{\partial\left(\delta u_i\right)}{\partial x} = \frac{\xi_i\left(1+\eta\eta_i\right)\left(1+\psi\psi_i\right)}{4a},\\ \delta\gamma_{xy} &= \frac{\partial\left(\delta u_i\right)}{\partial y} = \frac{\eta_i\left(1+\xi\xi_i\right)\left(1+\psi\psi_i\right)}{4b},\\ \delta\gamma_{xz} &= \frac{\partial\left(\delta u_i\right)}{\partial z} = \frac{\psi_i\left(1+\xi\xi_i\right)\left(1+\eta\eta_i\right)}{4c}.\end{aligned} \tag{12}$$

Other strains $\varepsilon_y = \varepsilon_z = \gamma_{yz} = 0$. The possible strain energy of finite element k is the integral over its volume.

$$\begin{aligned}\delta U_{i,x}^k = \iiint_{V^e} (&\sigma_x\delta\varepsilon_x + \sigma_y\delta\varepsilon_y + \sigma_x\delta\varepsilon_x + \sigma_z\delta\varepsilon_z\\ &+ \tau_{xy}\delta\gamma_{xy} + \tau_{yz}\delta\gamma_{yz} + \tau_{xz}\delta\gamma_{xz})dxdydz.\end{aligned} \tag{13}$$

Taking into account expressions (8) and (12), (13) we obtain

$$\delta U_{i,x}^{k}=\frac{1}{32}\sum_{j=1}^{8}\left(\xi_i bc\left(1+\frac{\eta_i\eta_j}{2}\right)\left(1+\frac{\psi_i\psi_j}{2}\right)\sigma_{x,j}\right.$$
$$+\eta_i ac\left(1+\frac{\xi_i\xi_j}{2}\right)\left(1+\frac{\psi_i\psi_j}{2}\right)\tau_{xy,j} \quad (14)$$
$$\left.+\psi_i ab\left(1+\frac{\eta_i\eta_j}{2}\right)\left(1+\frac{\xi_i\xi_j}{2}\right)\tau_{xz,j}\right).$$

Expression (14) can be represented in vector form

$$\delta U_{i,x}^{k}=\mathbf{C}_{i,x}^{k}\boldsymbol{\sigma}_{el,k}. \quad (15)$$

Similar expressions can be written for possible displacements of nodes along the axes Y and Z:

$$\delta U_{i,y}^{k}=\mathbf{C}_{i,y}^{k}\boldsymbol{\sigma}_{el,k},\ \delta U_{i,z}^{k}=\mathbf{C}_{i,z}^{k}\boldsymbol{\sigma}_{el,k}. \quad (16)$$

From vectors $\mathbf{C}_{i,x}^{k}$, $\mathbf{C}_{i,y}^{k}$, $\mathbf{C}_{i,z}^{k}$ of all finite elements the "equilibrium" matrix $\mathbf{C}$ for all system is formed. The matrix $\mathbf{C}$ rows number is equal number of system nodes possible displacements, and the columns number is equal the total number of nodal unknown displacements. The matrix has a band structure of non-zero elements.

The work of concentrated external forces and volume-distributed loads directed along the X axis with a possible displacement δu_i is determined by the following expression:

$$\delta V_{i,x}=P_{i,x}\delta u_i+\sum_k\int_0^a\int_0^c\int_0^b q_x^{gr}\delta u_i dxdydz. \quad (17)$$

For the case of a uniformly distributed load

$$\delta V_{i,x}=P_{i,x}+\sum_k\frac{1}{8}q_x^V abc=\overline{P}_{i,x}. \quad (18)$$

Similarly, let's get

$$\delta V_{i,y}=P_{i,y}+\sum_k\frac{1}{8}q_y^V abc=\overline{P}_{i,y},$$
$$\delta V_{i,z}=P_{i,z}+\sum_k\frac{1}{8}q_z^V abc=\overline{P}_{i,z}. \quad (19)$$

The global load vector $\mathbf{P}$ is formed from $\overline{P}_{i,x}$, $\overline{P}_{i,y}$, $\overline{P}_{i,z}$. Let's the algebraic equations system for the equilibrium of nodes in matrix form:

$$\mathbf{C}\boldsymbol{\sigma}_V+\mathbf{P}=0. \quad (20)$$

Using the Lagrange multipliers method, we add the equilibrium equations to the functional (9):

$$\Pi=\frac{1}{2}\boldsymbol{\sigma}_V^T\mathbf{D}\boldsymbol{\sigma}_V+\mathbf{w}^T(\mathbf{C}\boldsymbol{\sigma}_V+\mathbf{P})\to\min. \quad (21)$$

$\mathbf{w}$ is the nodes displacements vector. Equating to zero the functional derivatives with respect to $\boldsymbol{\sigma}_V$ and $\mathbf{w}$ we obtain the system of equations:

$$\begin{gathered}\mathbf{D}\boldsymbol{\sigma}_V+\mathbf{w}^T\mathbf{C}=0,\\ \mathbf{C}\boldsymbol{\sigma}_V+\mathbf{P}=0.\end{gathered} \quad (22)$$

We express the vector σ_V from the first matrix equation and put it into the second one. Then we get

$$\begin{gathered}\sigma_V=-C^T D^{-1}w,\\ K=-C^T D^{-1}C,\\ Kw=P.\end{gathered} \quad (24)$$

$\mathbf{K}$ is the whole system stiffness matrix. That matrix also has a band structure of non-zero elements. The matrix $\mathbf{D}$ has a block-diagonal structure and is inversed analytically. When calculating the product $\mathbf{C}^T\mathbf{D}^{-1}\mathbf{C}$, the band structure of non-zero elements of the matrix $\mathbf{C}$ is taken into account. Solving the equations system, we determine the nodal displacements, and then calculate the nodal stresses (24). Thus, the stress fields are continuous along element boundaries and discontinuous inside ones. On the contrary, when using the finite element method based on the displacement approximations, the stress fields are continuous inside the finite elements and discontinuous along their boundaries.

RESULTS AND DISCUSSIONS

To assess the accuracy of the proposed finite elements in stresses, the calculations of the cantilever beam and the hinged plate were performed (Fig. 2). The beam length is 6*m*, the section height and width are 1*m*. The plate length and width are 2*m*. Plate thickness is 0.1*m*. The material elasticity modulus is 1000000kN/m^2. Poisson's ratio is 0.25. The beam was calculated for unit force, which applied as uniformly distributed tangential stress to its end. The plate was designed for a uniformly distributed load of 10kN/m^2. Due to the symmetry of the plate, its quarter was calculated under the appropriate boundary conditions. For comparison, the beam and plate were also calculated by the classical finite element method based on displacement approximations. The calculation results for various finite element meshes are shown in Tables 1 and 2.

For the beam, Table 1 shows the free end displacements and the stresses of the outermost fiber in the pinch. For the plate, Table 2 shows the center displacements and the stresses in the outermost fiber in the center.

a) b)

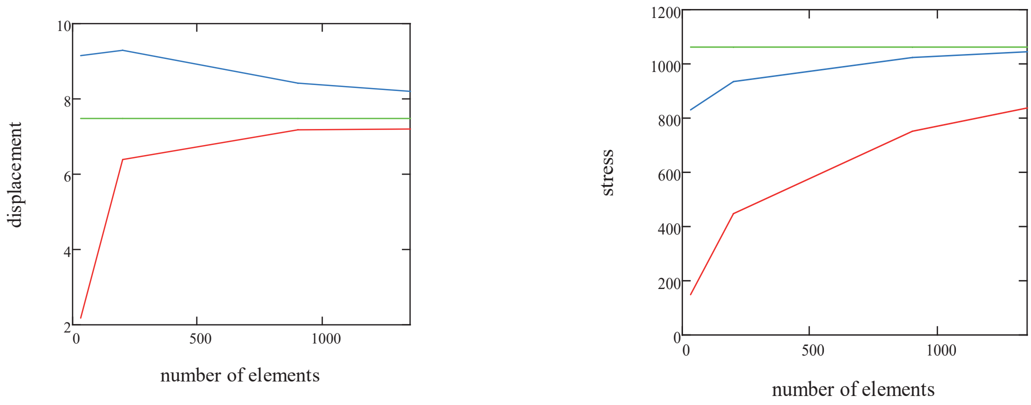
Figure 2. a) pinched beam; b) bending hinged plate

Table 1. Bending hinged plate

FEM grid		FEM in displacements		FEM in stresses	
number of elements by thickness	number of elements by length (width)	w, *mm*	Stress, kN/m^2	w, mm	Stress, kN/m^2
2	4	2.18	148.9	9.15	830.3
2	10	6.39	447.5	9.29	934.8
4	15	7.18	751.6	8.42	1023.6
6	15	7.20	837.6	8.20	1044.9
Analytical decision in series		**7.48**	**1062**	**7.48**	**1062**

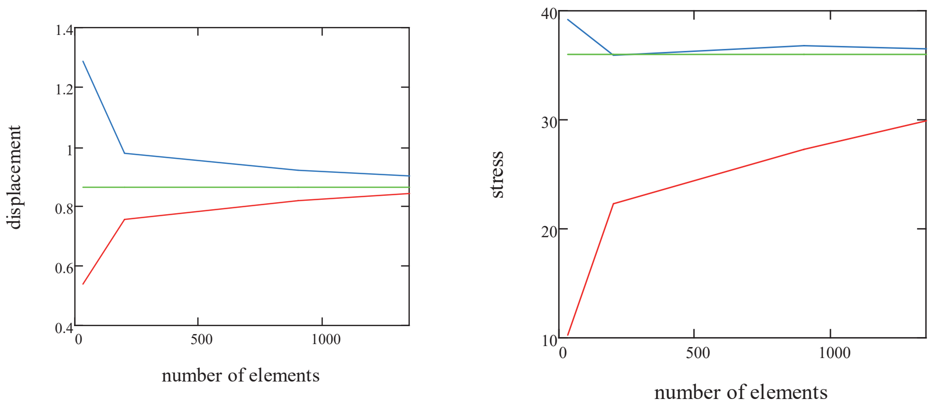
Table 2. Pinched beam

FEM grid		FEM in displacements		FEM in stresses	
number of elements by length	number of elements by high (width)	w, *mm*	Stress, kN/m^2	w, mm	Stress, kN/m^2
5	2	0.539	10.25	1.287	39.20
10	4	0.756	22.30	0.978	35.92
15	6	0.819	27.29	0.921	36.80
20	8	0.843	29.91	0.902	36.51
Analytical decision		**0.864**	**36**	**0.864**	**36**



a) b)

Figure 3. Bending hinged plate. The red line is the solution of FEM in displacements, the blue line is the solution of FEM in stresses, the green line is the exact solution



a) b)

Figure 4. Pinched beam. The red line is the solution of FEM in displacements, the blue line is the solution of FEM in stresses, the green line is the exact solution

The calculations results show that the proposed finite element, based on the stresses approximation, makes it possible to obtain, in comparison with the traditional finite element, more accurate stress values. In addition, the displacements obtained by the proposed method, when refining the finite elements mesh, tend to exact values from above. Quite often, the maximum stresses occur at the subject area boundary, so the proposed method is preferable in this case, since it allows us to calculate the stresses directly at the points of boundary. The above structural calculations confirm this. It is known that when the finite elements mesh is refined, the stresses in elements tend to constant values. The using the piecewise constant approximations of stresses makes it possible to ensure this condition directly.

CONCLUSION

The volumetric finite element based on piecewise constant approximations of stresses is presented. The stress fields are continuous along finite element boundaries and discontinuous inside ones. The solution of the volume theory elasticity problem was obtained on the basis of the additional energy functional and the possible displacements principle. The proposed finite element makes it possible to obtain, in comparison with the method of finite elements in displacements, more accurate stress values. The displacements obtained by the proposed method, when refining the finite elements mesh, tend to exact values from above.

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