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INFINITESIMAL AND FINITE DEFORMATIONS IN THE POLAR COORDINATE SYSTEM

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Abstract: The deformation problem of elasticity theory with regard to nonlinear deformations is examined. The expressions of deformations through displacements in the orthogonal curvilinear coordinate system are recorded. The relations for finite deformations in cylindrical and polar coordinate systems are derived. Physical relations for finite deformations and corresponding generalized stresses are recorded.

Keywords: small deformations, finite deformations, nonlinear continuum mechanics problem, curvilinear orthogonal coordinate system, cylindrical coordinate system, polar coordinate system, plane deformation, generalized stresses

БЕСКОНЕЧНО МАЛЫЕ И КОНЕЧНЫЕ ДЕФОРМАЦИИ В ПОЛЯРНОЙ СИСТЕМЕ КООРДИНАТ

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Аннотация: Рассматривается деформационная задача теории упругости с учетом нелинейных деформаций. Записываются выражения деформаций через перемещения в ортогональной криволинейной системе координат. Выводятся соотношения для конечных деформаций в цилиндрической и полярной системах координат. Записываются физические соотношения для конечных деформаций и соответствующих обобщенных напряжений.

Ключевые слова: малые деформации, конечные деформации, нелинейная задача механики сплошной среды, криволинейная ортогональная система координат, цилиндрическая система координат, полярная система координат, плоская деформация, обобщенные напряжения

1. INTRODUCTION

This article examines the deformation problem. The positions of medium points before and after deformation are given. The change in the position of the vector connecting two arbitrary points of the medium, caused by the deformation of the medium, is determined. The solution of the problem does not depend on the assumptions regarding the properties of the medium and is solved geometrically, typically in the Cartesian coordinate system [1-7]. The initial and final positions of the medium points are defined by projections on the Cartesian coordinate system axes.

Problems with angular cutouts of the boundary, with angular lines of the boundary at surface intersections, with holes in the area for the case of small deformations are considered in polar, spherical or cylindrical coordinate systems. The choice of some curvilinear orthogonal coordinate system is determined by the formulation of the problem. At the same time, the curvilinear coordinate system is chosen so that the body boundary is defined as a coordinate surface or line.

The paper [8, 9] examines a surface with an angular line formed by irregular points. In order to study the peculiarities of the solution in the vicinity of the boundary, a curvilinear coordinate system (ρ, φ, s) is introduced. The elastic problem in the vicinity of a point on the angle line of the boundary of the area is reduced to the solution of two problems: plane and antiplane deformation.

Analytical and experimental calculation methods [10-15] show that the investigation of the solution of the general elliptic boundary value problem in the vicinity of irregular boundary points is reduced to the consideration of boundary value problems for model regions: wedge or cone.

In order to solve the elastic problem for regions with an angular boundary cutout, it is convenient for wedges and regions with a hole to apply a particular kind of a curvilinear orthogonal coordinate system, the polar system [12-15].

The derivation of relations for finite deformations in the Cartesian coordinate system is discussed in detail in [1-7], whereas it is not given for the polar coordinate system.

Purpose of the work: to derive expressions for finite deformations in cylindrical and polar coordinate systems.

Objectives of the work: to obtain expressions for finite deformations through displacements, to record physical relations for finite deformations and generalized stresses in the polar coordinate system.

2. MODELING METHODS

2.1. Curvilinear orthogonal coordinate system

We consider the curvilinear orthogonal coordinate system α_1, α_2, α_3 with the origin point O, in which the unit vectors $\vec{k}_1, \vec{k}_2, \vec{k}_3$, lines tangent to coordinate lines $\alpha_1 = const$, $\alpha_2 = const, \alpha_3 = const$, are directed toward increasing parameters $\alpha_1, \alpha_2, \alpha_3$. The directions of vectors $\vec{k}_1, \vec{k}_2, \vec{k}_3$ change when passing from one point to another. The $\alpha_1, \alpha_2, \alpha_3$ parameters are related to the Cartesian coordinates x, y, z of an arbitrary point by the relations:

$$x = f_1(\alpha_1, \alpha_2, \alpha_3),\ y = f_2(\alpha_1, \alpha_2, \alpha_3),$$
$$z = f_3(\alpha_1, \alpha_2, \alpha_3) \tag{1}$$

The relations (1) are equivalent to the definition of the vector $\vec{R} = \vec{R}(\alpha_1, \alpha_2, \alpha_3)$ (fig. 1)

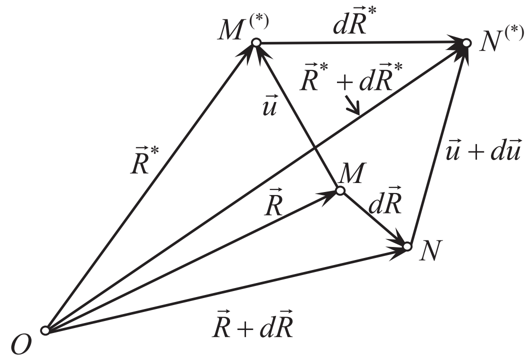

Figure 1. Displacements of points M and N to points M and N* during deformation*

The Lame parameters are described in the following form:

$$H_i = \left|\frac{\partial R}{\partial \alpha_i}\right| = \sqrt{\left(\frac{\partial x}{\partial \alpha_i}\right)^2 + \left(\frac{\partial y}{\partial \alpha_i}\right)^2 + \left(\frac{\partial z}{\partial \alpha_i}\right)^2}, \tag{2}$$

$i = 1,2,3$

The position of the arbitrary point M before deformation is determined by the radius vector $\vec{R} = \vec{R}(\alpha_1, \alpha_2, \alpha_3)$, where α_i – the orthogonal curvilinear coordinates of the vector $\vec{R}$. After the deformation, the point M transitions to the point M^* (fig. 1), which is defined by the radius vector;

$$\vec{R}^* = \vec{R} + \vec{u} = \vec{R} + u_1\overline{k_1} + u_2\overline{k_2} + u_1\overline{k_3}$$

where $u_i, i = 1,2,3$ are the projections of the displacement vector $\vec{u}$ on the axes of the local trihedron plotted at point *M*.

The point $N(\alpha_1 + d\alpha_1, \alpha_2 + d\alpha_2, \alpha_3 + d\alpha_3)$ prior to the deformation is defined by the vector

$$\vec{R} + \overrightarrow{dR} = \vec{R} + \sum_i \frac{\partial \vec{R}}{\partial \alpha_i} d\alpha_i = \vec{R} + \sum_i \mathrm{H}_i \vec{\kappa_i}\, d\alpha_i$$

Following the deformation, the point $N(\alpha_1 + d\alpha_1, \alpha_2 + d\alpha_2, \alpha_3 + d\alpha_3)$ transitions to the point N^* (fig. 1), which is defined by the radius vector in the following form:

$$\begin{aligned} &\vec{R}^* + d\vec{R}^* = \vec{R} + \vec{u} + d(\vec{R} + \vec{u}) = \\ &= \vec{R} + \vec{u} + \sum_i \frac{\partial \vec{R}}{\partial \alpha_i} d\alpha_i + \sum_i \frac{\partial \vec{u}}{\partial \alpha_i} d\alpha_i, \end{aligned}$$

where $\vec{u} = u_1\vec{k_1} + u_2\vec{k_2} + u_1\vec{k_3}$.

Given the differentiation of unit vectors $\vec{k_i}$, we obtain

$$\begin{aligned} d\overrightarrow{R^*} &= \left[(1+e_{11})H_1 d\alpha_1 + (\tfrac{1}{2}e_{12} - \omega_3)H_2 d\alpha_2 + (\tfrac{1}{2}e_{13} + \omega_2)H_3 d\alpha_3 \right] \vec{\kappa_1} + \\ &+ \left[(\tfrac{1}{2}e_{12} + \omega_3)H_1 d\alpha_1 + (1+e_{22})H_2 d\alpha_2 + (\tfrac{1}{2}e_{23} - \omega_1)H_3 d\alpha_3 \right] \vec{\kappa_2} + \\ &+ \left[(\tfrac{1}{2}e_{13} - \omega_2)H_1 d\alpha_1 + (\tfrac{1}{2}e_{23} + \omega_1)H_2 d\alpha_2 + (1+e_{22})H_3 d\alpha_3 \right] \vec{\kappa_3} \end{aligned}$$

The expressions for e_{ij}, ω_i have the following form:

$$\begin{aligned} e_{11} &= \frac{1}{H_1}\frac{\partial u_1}{\partial \alpha_1} + \frac{1}{H_1 H_2}\frac{\partial H_1}{\partial \alpha_2}u_2 + \frac{1}{H_1 H_3}\frac{\partial H_1}{\partial \alpha_3}u_3, \\ e_{22} &= \frac{1}{H_2}\frac{\partial u_2}{\partial \alpha_2} + \frac{1}{H_2 H_3}\frac{\partial H_2}{\partial \alpha_3}u_3 + \frac{1}{H_2 H_1}\frac{\partial H_2}{\partial \alpha_1}u_1, \\ e_{33} &= \frac{1}{H_3}\frac{\partial u_3}{\partial \alpha_3} + \frac{1}{H_1 H_3}\frac{\partial H_3}{\partial \alpha_1}u_1 + \frac{1}{H_3 H_2}\frac{\partial H_3}{\partial \alpha_2}u_2, \\ e_{12} = e_{21} &= \frac{H_2}{H_1}\frac{\partial}{\partial \alpha_1}\left(\frac{u_2}{H_2}\right) + \frac{H_1}{H_2}\frac{\partial}{\partial \alpha_2}\left(\frac{u_1}{H_1}\right) \\ e_{13} = e_{31} &= \frac{H_3}{H_1}\frac{\partial}{\partial \alpha_1}\left(\frac{u_3}{H_3}\right) + \frac{H_1}{H_3}\frac{\partial}{\partial \alpha_3}\left(\frac{u_1}{H_1}\right), \end{aligned} \tag{2}$$

$$\begin{aligned} e_{23} = e_{32} &= \frac{H_2}{H_3}\frac{\partial}{\partial \alpha_3}\left(\frac{u_2}{H_3}\right) + \frac{H_3}{H_2}\frac{\partial}{\partial \alpha_2}\left(\frac{u_3}{H_3}\right), \\ 2\omega_1 &= \frac{1}{H_2 H_3}\left[\frac{\partial}{\partial \alpha_2}(H_3 u_3) - \frac{\partial}{\partial \alpha_3}(H_2 u_2)\right], \\ 2\omega_2 &= \frac{1}{H_1 H_3}\left[\frac{\partial}{\partial \alpha_3}(H_1 u_1) - \frac{\partial}{\partial \alpha_1}(H_3 u_3)\right], \\ 2\omega_3 &= \frac{1}{H_1 H_2}\left[\frac{\partial}{\partial \alpha_1}(H_2 u_2) - \frac{\partial}{\partial \alpha_2}(H_1 u_1)\right]. \end{aligned}$$

We consider the relative elongation at the point M as:

$$E_{MN} = \frac{ds^* - ds}{ds} = \frac{|M^*N^*| - |MN|}{|MN|}.$$

Given the designations of the elongation E_{MN}, the difference $\left|\overrightarrow{M^*N^*}\right|^2 - \left|\overrightarrow{MN}\right|^2$ will be recorded as:

$$\begin{aligned} E_{MN}(1 + \tfrac{1}{2}E_{MN})ds^2 &= \varepsilon_{11}H_1^2 d\alpha_1^2 + \varepsilon_{22}H_2^2 d\alpha_2^2 + \\ &+ \varepsilon_{33}H_3^2 d\alpha_3^2 + \varepsilon_{12}H_1 H_2 d\alpha_1 d\alpha_2 + \\ &+ \varepsilon_{13}H_1 H_3 d\alpha_1 d\alpha_3 + \varepsilon_{23}H_2 H_3 d\alpha_2 d\alpha_3, \end{aligned}$$

where ε_{ij} are the deformation components in the curvilinear coordinate system $\alpha_1, \alpha_2, \alpha_3$:

$$\begin{aligned} \varepsilon_{11} &= e_{11} + \frac{1}{2}\left[e_{11}^2 + (\tfrac{1}{2}e_{12} + \omega_3)^2 + (\tfrac{1}{2}e_{13} - \omega_2)^2\right], \\ \varepsilon_{12} &= e_{12} + e_{11}(\tfrac{1}{2}e_{12} - \omega_3) + e_{22}(\tfrac{1}{2}e_{12} + \omega_3) + \\ &+ (\tfrac{1}{2}e_{13} - \omega_2)(\tfrac{1}{2}e_{23} + \omega_1), \ldots \end{aligned} \tag{3}$$

(cyclic permutation)

2.2. Cylindrical coordinate system

We consider the cylindrical coordinate system: $x = r\cos\varphi,\ y = r\sin\varphi,\ z = z$. The curvilinear coordinates are rewritten as: $\alpha_1 = r, \alpha_2 = \varphi, \alpha_3 = z$. The Lame parameters are: $H_1 = 1, H_2 = r, H_3 = 1$.

Given the expressions e_{ij}, ω_i in the form (2), we obtain expressions for deformations in the cylindrical coordinate system:

$$\varepsilon_{11} = \frac{\partial u_1}{\partial r} + \frac{1}{2}\left[\left(\frac{\partial u_1}{\partial r}\right)^2 + \left(\frac{\partial u_2}{\partial r}\right)^2 + \left(\frac{\partial u_3}{\partial r}\right)^2\right],$$

$$\varepsilon_{22} = \frac{1}{r}\frac{\partial u_2}{\partial \varphi} + \frac{u_1}{r} + \frac{1}{2}[(\frac{1}{r}\frac{\partial u_2}{\partial \varphi} + \frac{u_1}{r})^2 + $$
$$+ (\frac{1}{r}\frac{\partial u_1}{\partial \varphi} - \frac{u_2}{r})^2 + (\frac{1}{r}\frac{\partial u_3}{\partial \varphi})^2],$$

$$\varepsilon_{33} = \frac{\partial u_3}{\partial z} + \frac{1}{2}\left[\left(\frac{\partial u_1}{\partial z}\right)^2 + \left(\frac{\partial u_2}{\partial z}\right)^2 + \left(\frac{\partial u_3}{\partial z}\right)^2\right],$$

$$\varepsilon_{12} = \frac{\partial u_2}{\partial r} - \frac{u_2}{r} + \frac{1}{r}\frac{\partial u_1}{\partial \varphi} + \frac{\partial u_1}{\partial r}(\frac{1}{r}\frac{\partial u_1}{\partial \varphi} - \frac{u_2}{r}) + $$
$$+ \frac{\partial u_2}{\partial r}(\frac{1}{r}\frac{\partial u_2}{\partial \varphi} + + \frac{1}{r}u_1) + \frac{1}{r}\frac{\partial u_3}{\partial r}\frac{\partial u_3}{\partial \varphi}, \tag{4}$$

$$\varepsilon_{13} = \frac{\partial u_3}{\partial r} + \frac{\partial u_1}{\partial z} + \frac{\partial u_1}{\partial r}\frac{\partial u_1}{\partial z} + \frac{\partial u_2}{\partial r}\frac{\partial u_2}{\partial z} + \frac{\partial u_3}{\partial r}\frac{\partial u_3}{\partial z},$$

$$\varepsilon_{23} = r\frac{\partial u_2}{\partial z} + \frac{1}{r}\frac{\partial u_3}{\partial \varphi} + (\frac{1}{r}\frac{\partial u_1}{\partial \varphi} - \frac{u_2}{r})\frac{\partial u_1}{\partial z} + $$
$$+ (\frac{1}{r}\frac{\partial u_2}{\partial \varphi} + \frac{u_1}{r})\frac{\partial u_2}{\partial z} + \frac{1}{r}\frac{\partial u_3}{\partial \varphi}\frac{\partial u_3}{\partial z}.$$

In the case of small deformations, the relations (4) are recorded in the following form:

$$\varepsilon_{11} = \frac{\partial u_1}{\partial r},\quad \varepsilon_{22} = \frac{1}{r}\frac{\partial u_2}{\partial \varphi} + \frac{u_1}{r},\quad \varepsilon_{33} = \frac{\partial u_3}{\partial z},$$
$$\varepsilon_{12} = \frac{\partial u_2}{\partial r} - \frac{u_2}{r} + \frac{1}{r}\frac{\partial u_1}{\partial \varphi},\quad \varepsilon_{13} = \frac{\partial u_3}{\partial r} + \frac{\partial u_1}{\partial z},$$
$$\varepsilon_{23} = r\frac{\partial u_2}{\partial z} + \frac{1}{r}\frac{\partial u_3}{\partial \varphi}. \tag{5}$$

2.3. Polar coordinate system

We consider the polar orthogonal coordinate system: $\alpha_1 = r$, $\alpha_2 = \varphi$, $\alpha_3 = z$. The Lame parameters are: $H_1 = 1, H_2 = r, H_3 = 1$.

A plane deformation state is assumed, and the body points are displaced in planes perpendicular to the *OZ* axis:

$$u_1 = u_1(\alpha_1, \alpha_2, 0),\ u_2 = u_2(\alpha_1, \alpha_2, 0),$$
$$u_3(\alpha_1, \alpha_2, 0) = 0, \tag{6}$$

then, for e_{ij} in the form (2), the following holds: $e_{13} = e_{31} = 0, e_{23} = e_{32} = 0, e_{33} = 0$, $e_{33} = 0$,

The deformations and generalized stresses take the following form:

$$\varepsilon_{33} = 0,\ \varepsilon_{13} = \varepsilon_{31} = 0,\ \varepsilon_{23} = \varepsilon_{32} = 0$$
$$\sigma^*_{13} = \sigma^*_{31} = 0,\quad \sigma^*_{23} = \sigma^*_{32} = 0. \tag{7}$$

In the polar coordinate system, the deformations through displacements are recorded as:

$$\varepsilon_{11} = \frac{\partial u_1}{\partial r} + \frac{1}{2}\left[\left(\frac{\partial u_1}{\partial r}\right)^2 + \left(\frac{\partial u_2}{\partial r}\right)^2\right],$$

$$\varepsilon_{22} = \frac{1}{r}\frac{\partial u_2}{\partial \varphi} + \frac{u_1}{r} + $$
$$+ \frac{1}{2}\left[\left(\frac{1}{r}\frac{\partial \mathrm{u}_2}{\partial \varphi} + \frac{u_1}{r}\right)^2 + \left(\frac{1}{r}\frac{\partial u_1}{\partial \varphi} - \frac{u_2}{r}\right)^2\right], \tag{8}$$

$$\varepsilon_{12} = \frac{\partial u_2}{\partial r} - \frac{u_2}{r} + \frac{1}{r}\frac{\partial u_1}{\partial \varphi} + \frac{\partial u_1}{\partial r}\left(\frac{1}{r}\frac{\partial u_1}{\partial \varphi} - \frac{u_2}{r}\right) + $$
$$+ \frac{\partial u_2}{\partial r}\left(\frac{1}{r}\frac{\partial u_2}{\partial \varphi} + \frac{u_1}{r}\right).$$

The resulting deformations (8) contain expressions for the case of small deformations:

$$\varepsilon_{11} = \frac{\partial u_1}{\partial r}, \quad \varepsilon_{22} = \frac{1}{r}\frac{\partial u_2}{\partial \varphi} + \frac{u_1}{r},$$

$$\varepsilon_{12} = \frac{\partial u_2}{\partial r} - \frac{u_2}{r} + \frac{1}{r}\frac{\partial u_1}{\partial \varphi}$$

2.4. Relations between nonlinear strains and generalized stresses

We assume, according to [1, 2], the same form of recording the dependences between invariants of tensors and deviators of strains and stresses in geometrically linear media and the form of recording the dependences for invariants of tensors and deviators of nonlinear strains and generalized stresses in nonlinear media. The overall form of the nonlinear strain tensor:

$$T_\varepsilon = \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{pmatrix}, \quad (9)$$

where ε_{ij} are nonlinear strains in the curvilinear orthogonal coordinate system in the form (3) or nonlinear strains in the cylindrical (4) or polar (8) coordinate systems. The generalized stress tensor corresponding to the tensor T_ε is recorded as follows:

$$T_{\sigma^*} = \begin{pmatrix} \sigma^*_{11} & \sigma^*_{12} & \sigma^*_{13} \\ \sigma^*_{21} & \sigma^*_{22} & \sigma^*_{23} \\ \sigma^*_{31} & \sigma^*_{32} & \sigma^*_{33} \end{pmatrix} \quad (10)$$

We assume [7, 4, 16] the measure of deviation of isotropic material from the similarity principle for stress and strain deviators (deviator similarity phase) equal to zero: $\omega^* = 0$.

For a geometrically linear continuous medium, the relations for the stress and strain invariants are recorded as follows:

$$\sigma = 3K(\varepsilon, \Gamma)\cdot\varepsilon, \; T = 3G(\varepsilon, \Gamma)\cdot\Gamma$$

For a geometrically nonlinear continuous medium, the relations for the stress and strain invariants are recorded as follows:

$$\sigma^* = 3K^*(\varepsilon, \Gamma^*)\cdot\varepsilon, \; T^* = 3G^*(\varepsilon, \Gamma^*)\cdot\Gamma^*,$$

where ε is the first invariant of the nonlinear strain tensor (9), σ^* is the first invariant of the generalized stress tensor T_{σ^*} (10), T^* is the intensity of generalized shear stresses proportional to the second invariant of the generalized stress deviator, Γ^* is the intensity of nonlinear shear strains proportional to the second invariant of the strain deviator. The error in applying the similarity of the forms for recording the relations between stresses and strains depends on the adopted continuum model, the type of the stress-strain state, and the level of deformation development in the continuum.

Invariants K^*, G^*, ω^* characterize the mechanical properties of the isotropic material. The value K^* is referred to as the generalized volume expansion modulus; it characterizes the measure of resistance of the isotropic material to volume changes. G^* is the generalized shear modulus.

The physical relations for the plane deformation state of the body [12] are recorded in the following form:

$$\sigma^*_{ij} = 2G^*\tilde{\gamma}_{ij} + \frac{1}{3}\sigma^*\delta_{ij}, \quad (11)$$

$$\tilde{\gamma}_{ij} = \gamma_{ij} - \frac{1}{3}\varepsilon\delta_{ij},$$

where $\varepsilon = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = \varepsilon_{11} + \varepsilon_{22}$ is the first invariant of the nonlinear strain tensor in the form (9).

Let us record the expressions for the components of the strain tensor deviator $\tilde{\gamma}_{ij}, i, j = 1,2,3$:

$$\tilde{\gamma}_{11} = \varepsilon_{11} - \frac{1}{3}\varepsilon,\ \tilde{\gamma}_{22} = \varepsilon_{22} - \frac{1}{3}\varepsilon,$$
$$\tilde{\gamma}_{33} = \varepsilon_{33} - \frac{1}{3}\varepsilon = -\frac{1}{3}(\varepsilon_{11} + \varepsilon_{22}),$$
$$\tilde{\gamma}_{13} = \tilde{\gamma}_{31} = \frac{1}{2}\varepsilon_{13} = \frac{1}{2}\varepsilon_{31} = 0, \tag{12}$$
$$\tilde{\gamma}_{23} = \tilde{\gamma}_{32} = \frac{1}{2}\varepsilon_{23} = \frac{1}{2}\varepsilon_{32} = 0,$$
$$\tilde{\gamma}_{12} = \tilde{\gamma}_{12} = \frac{1}{2}\varepsilon_{12} = \frac{1}{2}\varepsilon_{21}.$$

The expressions for the generalized stresses given (12) are recorded as follows:

$$\sigma_{11}^{*} = 2G^{*}(\varepsilon,\Gamma)(\varepsilon_{11} - \frac{1}{3}\varepsilon) + \frac{1}{3}\sigma^{*},$$
$$\sigma_{22}^{*} = 2G^{*}(\varepsilon,\Gamma)(\varepsilon_{22} - \frac{1}{3}\varepsilon) + \frac{1}{3}\sigma^{*},$$
$$\sigma_{33}^{*} = 2G^{*}(\varepsilon,\Gamma)[-\frac{1}{3}(\varepsilon_{11} + \varepsilon_{22}) + \frac{1}{3}\sigma^{*}, \tag{13}$$
$$\sigma_{23}^{*} = 2G^{*}\tilde{\gamma}_{23} = 0,\ \sigma_{13}^{*} = 2G^{*}\tilde{\gamma}_{13} = 0,$$
$$\sigma_{12}^{*} = \sigma_{21}^{*} = 2G^{*}\tilde{\gamma}_{12} = G^{*}(\varepsilon,\Gamma)\varepsilon_{12},$$

where $\sigma^{*} = \sigma^{*}_{11} + \sigma^{*}_{22} + \sigma^{*}_{33}$ is the first invariant of generalized stresses.
Given the generalized modulus in the form:

$$K^{*}(\varepsilon,\Gamma) = \frac{1}{3}\frac{\sigma^{*}}{\varepsilon},$$

the physical relations for the generalized stresses in a geometrically nonlinear continuum are recorded as

$$\sigma_{11}^{*} = 2G^{*}(\varepsilon,\Gamma)(\varepsilon_{11} - \frac{1}{3}\varepsilon) + K^{*}(\varepsilon,\Gamma)\,\varepsilon,$$
$$\sigma_{22}^{*} = 2G^{*}(\varepsilon,\Gamma)(\varepsilon_{22} - \frac{1}{3}\varepsilon) + K^{*}(\varepsilon,\Gamma)\,\varepsilon,$$
$$\sigma_{33}^{*} = [K^{*}(\varepsilon,\Gamma) - \frac{2}{3}G^{*}(\varepsilon,\Gamma)]\,\varepsilon, \tag{14}$$
$$\sigma_{12}^{*} = \sigma_{21}^{*} = G^{*}(\varepsilon,\Gamma)\,\varepsilon_{12}.$$

Here, $\sigma^{*} = \sigma^{*}_{11} + \sigma^{*}_{22} + \sigma^{*}_{33}$ is the first invariant of generalized stresses tensor (10), $\varepsilon = \varepsilon_{11} + \varepsilon_{22}$ is the first invariant of the nonlinear strain tensor (9), the generalized mechanical properties $K^{*}(\varepsilon,\Gamma), G^{*}(\varepsilon,\Gamma)$ that depend on the first invariant of the nonlinear strain tensor ε and the nonlinear shear strain intensity:

$$\Gamma = \sqrt{\frac{2}{3}}\sqrt{(\varepsilon_{11} - \varepsilon_{22})^{2} + \varepsilon_{11}^{2} + \varepsilon_{22}^{2} + \frac{2}{3}\varepsilon_{12}^{2}}.$$

The expressions for finite deformations (8) and generalized stresses (14) in the case of the plane deformation state of a nonlinear continuum were obtained for the formulation of the boundary value problem for continuum mechanics in consideration of geometric and physical nonlinearity.

RESULTS

The expressions for finite deformations (4) in the cylindrical coordinate system are obtained, the expressions for finite deformations (8) for the case of plane deformation in the polar coordinate system are obtained, and the relations between finite deformations and generalized stresses (14) in the polar coordinate system are recorded.

DISCUSSION

The deformation expressions (5) for small deformations in the polar coordinate system are obtained by examining the displacements of the minor area element $r\,dr\,d\varphi$. The application of the classical approach for obtaining expressions of finite deformations is complicated by taking into account the rotation of the element, including the radial displacements of points *M* and N prior to deformation to points M^* and N^* post-deformation.

The known relations for finite deformations in Cartesian coordinate system are also difficult to translate into the polar coordinate system. When considering the displacement field, the displacement vector is projected to the axes of the local trihedron plotted at the point M of application of the displacement vector $\vec{u} = u_1\vec{k_1} + u_2\vec{k_2} + u_1\vec{k_3}$, $\vec{k_i}$ are the coordinate axis orthodes. The direction of vectors $\vec{k_i}$ changes when passing from point to point, so the projections on these axes change. The derivatives $\frac{\partial \vec{k}_i}{\partial \alpha_i}$ form a vector field. The differentiation rule valid for the Cartesian coordinate system is violated: the projection of the vector derivative on the coordinate α_i is equal to the derivative of its projection on this coordinate. Therefore, the orthogonal curvilinear coordinate system is used to derive expressions of finite deformations. The generalized stresses differ from the real ones by accounting for the change in the areas of the faces of the oblique parallelepiped during the deformation.

CONCLUSIONS

The expressions for nonlinear deformations (8) and generalized stresses (14) in the case of the plane deformation state of a nonlinear continuum were obtained for the formulation of the boundary value problem for continuum mechanics in consideration of geometric and physical nonlinearity.

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