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NORMAL VIBRATIONS OF SAGGING CONDUCTORS OF OVERHEAD POWER LINES

Airat A. Valiullin [1], ***Alexander N. Danilin*** [2], ***Valery A. Feldshteyn*** [3]

[1] Moscow Institute of Physics and Technology (National Research University), Moscow, RUSSIA
[2] Institute of Applied Mechanics of the Russian Academy of Sciences, Moscow, RUSSIA
[3] JSC "Central Research Institute for Machine Building", Moscow, RUSSIA

Abstract. The phenomenon of self-excitation of thermomechanical vibrations of current-carrying conductors, experimentally discovered by academician A.F. Ioffe, is of practical interest as a possible explanation of the phenomenon of galloping conductors of overhead power transmission lines (OHL) – low-frequency vibrations with frequencies of ~ 1 Hz and with amplitudes of the order of the static conductor sagging. To build the theoretical foundations of this phenomenon, as a special class of self-oscillating systems, it is necessary, first of all, a model of conductor vibrations in the OHL span. With regard to the most studied type of conductor vibrations, high-frequency aeolian vibration, excited by sign-alternating aerodynamic forces from the Karman vortex street, the classical model of a straight string is reasonably applied. However, to study low-frequency vibrations of the galloping type, it is necessary to take into account the effect of sagging of the conductor, the associated elastic tension and, in some cases, the nonlinear nature of the vibrations. The article presents two models for calculating the natural vibrations of sagging conductors (cables) within the framework of the technical theory of flexible threads, assuming the constancy of the tension force along the length. The first model describes linear oscillations of an elastic conductor in the sagging plane. For a system of equations with respect to the displacement components given in natural coordinates, an exact solution of the Sturm-Liouville problem with estimates of the frequency ranges arising is obtained. The second model describes nonlinear vibrations of an elastic conductor in the sagging plane and pendulum vibrations accompanied by an exit from it. The solution of the problem is based on the principle of possible displacements using the Ritz method. The structure of the frequency spectrum and the natural forms of transverse vibrations are studied. The developed models are intended for further investigation of thermomechanical vibrations of conductor and flexible cable systems.

Keywords: sagging conductor, cable, flexible elastic thread, frequencies and modes of normal vibrations, Ritz method, spectrum structure

СОБСТВЕННЫЕ КОЛЕБАНИЯ ПРОВИСАЮЩИХ ПРОВОДОВ ВОЗДУШНЫХ ЛИНИЙ ЭЛЕКТРОПЕРЕДАЧИ

А.А. Валиуллин [1], ***А.Н. Данилин*** [2], ***В.А. Фельдштейн*** [3]

[1] Московский физико-технический институт (национальный исследовательский университет), Москва, РОССИЯ
[2] ФГБУН Институт прикладной механики Российской академии наук, Москва, РОССИЯ
[3] АО «Центральный научно-исследовательский институт машиностроения», Москва, РОССИЯ

Аннотация. Явление самовозбуждения термомеханических колебаний токонесущих проводников, экспериментально обнаруженное академиком А.Ф. Иоффе, представляет практический интерес в качестве возможного объяснения феномена пляски проводов воздушных линий электропередачи (ВЛЭ) – низкочастотных колебаний с частотами ~1 Гц и с амплитудами порядка стрелы статического провисания провода. Для построения теоретических основ этого явления, как особого класса автоколебательных систем, необходима, прежде всего, модель колебаний провода в пролете ВЛЭ. Применительно к наиболее изученному виду колебаний проводов, высокочастотной эоловой вибрации, возбуждаемой знакопеременными аэродинамическими силами со стороны вихревой дорожки Кармана, обоснованно применяется классическая модель прямолинейной струны. Однако для исследования низкочастотных колебаний типа пляски необходимо учитывать эффект провисания провода, связанное с этим упругое растяжение и, в ряде случаев, нелинейный характер колебаний. В статье представлены две модели для расчёта собственных колебаний провисающих проводов (тросов) в рамках технической теории гибких нитей, предполагающей постоянство силы натяжения по длине. Первая модель описывает линейные колебания упругого тяжелого провода в плоскости провисания. Для системы уравнений относительно компонент перемещения, заданных в естественных координатах, получено точное решение задачи Штурма-Лиувилля с оценками возникающих частотных диапазонов. Вторая модель описывает нелинейные колебания упругого провода, совершающего колебания в плоскости провисания и маятниковые колебания, сопровождающиеся выходом из нее. Решение задачи строится на основе принципа возможных перемещений с использованием метода Ритца. Изучена структура спектра частот и форм собственных поперечных колебаний провода. Разработанные модели предназначены для дальнейшего исследования термомеханических колебаний проводов и гибких тросовых систем.

Ключевые слова: провисающий провод, трос, гибкая упругая нить, частоты и формы собственных колебаний, метод Ритца, структура спектра

INTRODUCTION

Works [1-5] are devoted to the construction of a theory explaining the self-excitation of thermomechanical self-oscillations of a conductor that heats up when included in an electrical circuit. In [5], there are indications of the repetition of the experiment of A.F. Ioffe. A practical interest is the question of whether the self-excitation of thermomechanical vibrations is related to the phenomenon of the conductor galloping – low-frequency vibrations with frequencies of ~ 1 Hz and with amplitudes of the order of the static sagging [6,7].

In the cited works, such an assumption was made, but it has not yet received reasonable confirmation: there is no transfer of the effect, modeled theoretically and observed in a laboratory model, to the full-scale OHL conductors.

The purpose of this work is to study the normal frequencies and modes of a conductor in the OHL span, necessary for the mathematical model to determine the conditions for self-excitation of the galloping of full-scale conductors, based on the thermo-mechanical model.

1. A MODEL OF COUPLED LONGITUDINAL-TRANSVERSE VIBRATIONS OF A CONDUCTOR IN THE SAGGING PLANE

The natural oscillations of an OHL conductor in the plane of its sagging in a homogeneous field of gravity are considered. The conductor is considered as a flexible elastic heavy thread. The coordinate system and the selected natural basis are shown in Figure 1.

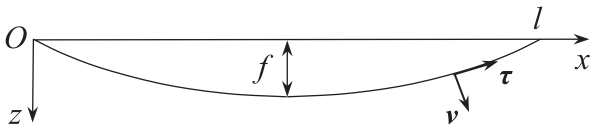

Figure 1. Orientation of the natural basis in the coordinate system Oxz

Conductor parameters: l – the distance between the suspension points (span length); m – linear mass; B – tensile stiffness; T_0 – static tension; k_0, f – the curvature of the static curve and its sag. By $\boldsymbol{p} = m\boldsymbol{g}$ denote the vertical linear load, where $\boldsymbol{g}$ is the vector of gravity acceleration. With small sag $(f << l)$, which is typical for most spans of OHL, tension $T_0 = B\varepsilon$ and curvature $k_0 = 8f/l^2$ can be considered constant along the length and related by the ratio:

$$T_0 = mg/k_0 = mgl^2/8f\ .$$

Conductor oscillation equation

$$(T\tau)' + \mathrm{p} - m(\ddot{u}\tau + \ddot{w}\nu) = 0$$

represent in projections onto the associated basis, using the Frenet formulas for a flat curve

$$\begin{gathered} T' - m\ddot{u} = 0, \\ -kT + mg - m\ddot{w} = 0. \end{gathered} \tag{1}$$

During vibrations, the conductor has an additional elongation deformation $\tilde{\varepsilon} = u' + kw$, increments of curvature $k = k_0 - w''$ and tension $T = T_0 + B\tilde{\varepsilon}$. Substituting these values into (1), limiting ourselves to a linear approximation and excluding time by substitution $u \to ue^{i\omega t}$, $w \to we^{i\omega t}$, we obtain

$$Bu'' + Bk_0 w' = -m\omega^2 u,$$
$$T_0 w'' - Bk_0 u' - Bk_0^2 w = -m\omega^2 w$$
$$x = 0, l: \; u = w = 0.$$

Here and below, the dashes denote the derivative with respect to the arc coordinate s, which, due to the flatness of the sag curve, is identified with the x coordinate on the horizontal projection.

Let us pass to dimensionless variables, choosing as the scales of length and frequency, respectively l and $\sqrt{T_0/m}/l$:

$$u'' + \alpha w' = \varepsilon\omega^2 u,$$
$$w'' - \frac{\alpha}{\varepsilon}\left(u' + \alpha w\right) = -\omega^2 w, \qquad (2)$$

where indicated: $\varepsilon = T_0/B$, $\alpha = k_0 l = 8f/l$. The parameter ε represents the deformation of the conductor elongation in the state of equilibrium and can be considered small. Note that equations (1) are similar to the equations of vibrations of an elongated cylindrical shell (panel with curvature k_0) with a vanishingly small bending stiffness [8]. The parameter α defines the connection between the longitudinal and transverse displacements of the wire section. At small values of this parameter, which are characteristic of a strongly stretched wire, system (2) breaks up into two independent equations: longitudinal vibrations of the rod and transverse vibrations of the string.

Assuming in (2) $u = Ue^{i\lambda x}$, $w = We^{i\lambda x}$, let's move on to the system

$$\left(\varepsilon\omega^2 - \lambda^2\right)U + i\alpha\lambda W = 0,$$
$$-\frac{i\alpha\lambda}{\varepsilon}U + \left(\omega^2 - \lambda^2 - \frac{\alpha^2}{\varepsilon}\right)W = 0. \qquad (4)$$

Let's write an equation for determining wavenumbers λ with respect to $z = \lambda^2$:

$$z^2 - z\omega^2\left(1+\varepsilon\right) + \varepsilon\omega^4 - \alpha^2\omega^2 = 0. \qquad (5)$$

At $\omega > \alpha/\sqrt{\varepsilon}$ (conditionally large frequencies) the roots $z_{1,2} > 0$ and all wavenumbers are real. At $\omega < \alpha/\sqrt{\varepsilon}$ (relatively low frequencies) $z_1 > 0$, $z_2 < 0$; in this case, one pair of wavenumbers is real, the other is imaginary. The frequency $\omega_{cr} = \alpha/\sqrt{\varepsilon}$, that delimits the low- and high-frequency regions is further called critical. From (4) follow the relationship between the displacement components (distribution coefficients) for each λ_k $(k = 1,...,4)$:

$$U_k = W_k \frac{i\alpha\lambda_k}{\lambda_k^2 - \varepsilon\omega^2} = W_k i\eta_k. \qquad (6)$$

In the general case, the roots of equation (5) are:

$$z_{1,2} = \frac{\omega^2}{2}\left(1+\varepsilon\right) \pm$$
$$\pm\frac{\omega^2}{2}\left(1-\varepsilon\right)\sqrt{1 + \frac{4\alpha^2}{\omega^2\left(1-\varepsilon\right)^2}}. \qquad (7)$$

Let's first consider the high-frequency range: $\omega > \alpha/\sqrt{\varepsilon}$. Given the strong inequality, we assume that

$$\sqrt{1 + 4\alpha^2/\omega^2\left(1-\varepsilon\right)^2} \approx 1 + 2\alpha^2/\omega^2.$$

It follows that

$$z_1 \approx \omega^2 + \alpha^2, \; z_2 \approx \varepsilon\omega^2 - \alpha^2$$

and the wavenumbers and distribution coefficients are equal to:

$$\lambda_{1,2} = \pm\chi_1, \ \lambda_{3,4} = \pm\chi_2; \quad \eta_{1,2} = \pm\delta_1, \ \eta_{3,4} = \mp\delta_2. \tag{8}$$

Here the notations are used:

$$\chi_1 = \sqrt{\omega^2 + \alpha^2}, \quad \chi_2 = \sqrt{\varepsilon\omega^2 - \alpha^2}, \quad \delta_1 = \alpha/\chi_1, \quad \delta_2 = \chi_2/\alpha.$$

The general solution of system (2), taking into account correlation (6), has the form:

$$w = \sum_{k=1}^{4} A_k e^{i\lambda_k x}, \ u = \sum_{k=1}^{4} A_k \eta_k i e^{i\lambda_k x} \tag{9}$$

or in trigonometric form:

$$\begin{aligned} w &= B_1 \cos\chi_1 x + B_2 \sin\chi_1 x + \\ &+ B_3 \cos\chi_2 x + B_4 \sin\chi_2 x, \\ u &= -B_1\delta_1 \sin\chi_1 x + B_2\delta_1 \cos\chi_1 x + \\ &+ B_3\delta_2 \sin\chi_2 x - B_4\delta_2 \cos\chi_2 x. \end{aligned}$$

Subjecting the obtained solution to boundary conditions, we come to a homogeneous system of equations with respect to: B_k :

$$\begin{aligned} & B_1 + B_3 = 0, \\ & B_1 \cos\chi_1 + B_2 \sin\chi_1 + \\ & + B_3 \cos\chi_2 + B_4 \sin\chi_2 = 0, \\ & B_2\delta_1 - B_4\delta_2 = 0, \\ & -B_1\delta_1 \sin\chi_1 + B_2\delta_1 \cos\chi_1 + \\ & + B_3\delta_2 \sin\chi_2 - B_4\delta_2 \cos\chi_2 = 0. \end{aligned} \tag{10}$$

The condition for the existence of a nontrivial solution gives the frequency equation –

$$\begin{aligned} \Delta_1(\omega) &= 2\delta_1\delta_2\left(1 - \cos\chi_1 \cos\chi_2\right) + \\ &+ \left(\delta_1^2 + \delta_2^2\right)\sin\chi_1 \sin\chi_2 = 0. \end{aligned} \tag{11}$$

Assuming $B_2 = 1$ and defining the remaining integration constants from the first three equations of system (10), we represent the eigenfunctions (normal modes) in the form:

$$\begin{aligned} w &= \frac{1}{\delta_2}\left[\varphi_1(x) - \mu_1\psi_1(x)\right], \\ u &= \delta_1\psi_1(x) + \mu_1\theta_1(x). \end{aligned} \tag{12}$$

It is indicated here:

$$\begin{aligned} \varphi_1(x) &= \delta_2 \sin\chi_1 x + \delta_1 \sin\chi_2 x, \\ \psi_1(x) &= \cos\chi_1 x - \cos\chi_2 x, \\ \theta_1(x) &= \nu \sin\chi_1 x + \sin\chi_2 x, \\ \nu &= \delta_1/\delta_2, \ \mu_1 = \varphi_1(1)/\psi_1(1). \end{aligned}$$

Consider the low-frequency range, when $\omega < \alpha/\sqrt{\varepsilon}$. In this case, the wavenumbers and distribution coefficients are equal to

$$\lambda_{1,2} = \pm\chi_1, \ \lambda_{3,4} = \pm i\chi_3; \quad \eta_{1,2} = \pm\delta_1, \ \eta_{3,4} = \mp i\delta_2.$$

where now: $\chi_2 = \sqrt{\alpha^2 - \varepsilon\omega^2}$, $\delta_2 = \chi_2/\alpha$. The general solution (9) takes the form:

$$\begin{aligned} w &= B_1 \cos\chi_1 x + B_2 \sin\chi_1 x + \\ &+ B_3 \,\mathrm{ch}\chi_2 x + B_4 \,\mathrm{sh}\chi_2 x, \\ u &= -B_1\delta_1 \sin\chi_1 x + B_2\delta_1 \cos\chi_1 x - \\ &- B_3\delta_2 \,\mathrm{sh}\chi_2 x - B_4\delta_2 \,\mathrm{ch}\chi_2 x. \end{aligned}$$

Frequency equation is

$$\begin{aligned} \Delta_2(\omega) &= 2\delta_1\delta_2\left(1 - \cos\chi_1 \mathrm{ch}\chi_2\right) + \\ &+ \left(\delta_1^2 - \delta_2^2\right)\sin\chi_1 \mathrm{sh}\chi_2 = 0. \end{aligned} \tag{13}$$

Note that the boundary frequency $\omega_b = \alpha/\sqrt{\varepsilon}$ simultaneously satisfies both frequency equations (11) и (13) and, therefore, is a natural frequency. Native functions:

$$w = \frac{1}{\delta_2}\left[\varphi_2(x) - \mu_2\psi_2(x)\right], \quad u = -\delta_1\psi_2(x) + \mu_2\theta_2(x). \tag{14}$$

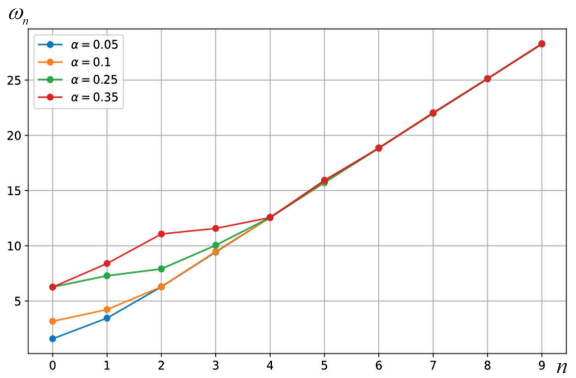

Figure 2. Natural frequency spectra for various α *and* $\varepsilon = 10^{-3}$

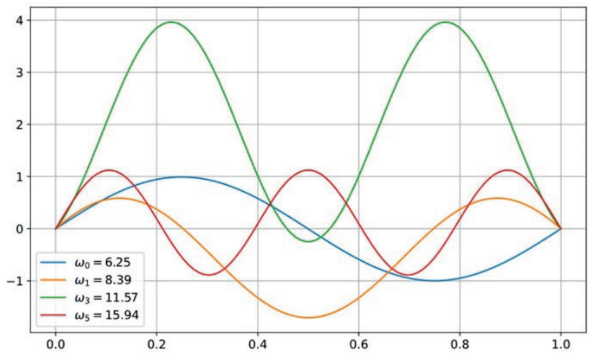

Figure 3. Characteristic forms of transverse vibrations of the lower (n=0, 1) *and upper* (n=3, 5) *harmonics; here* $\alpha = 0.35$, $\varepsilon = 10^{-3}$, *critical frequency* $\omega_{cr} = 11.07$

It is indicated here:

$$\varphi_2(x) = \delta_2 \sin\chi_1 x + \delta_1 \mathrm{sh}\chi_2 x,$$
$$\psi_2(x) = \mathrm{ch}\chi_2 x - \cos\chi_1 x,$$
$$\theta_2(x) = \mathrm{sh}\chi_2 x - \nu \sin\chi_1 x;$$
$$\mu_1 = \varphi_1(1)/\psi_1(1).$$

Note that for small sag $(\alpha \to 0)$, the high-frequency equation (11) transforms into $\sin\chi_1 \sin\chi_2 = 0$ and the spectrum splits into groups of quasi-transverse (string) and quasi-longitudinal frequencies: $\omega_n = \pi n$ and $\omega_n = \pi n/\sqrt{\varepsilon}$.

The low-frequency equation (13) takes the form $\sin\chi_1 \mathrm{sh}\chi_2 = 0$ and defines only transverse frequencies. For high harmonics, string asymptotic is manifested for all, not necessarily small values α. The spectrum features are characterized by Figure 2, which shows the frequencies of the modes corresponding to the harmonics with a number n calculated for different α and $\varepsilon = 10^{-3}$.

The structure of the spectrum, which is quite complex in the low-frequency region, becomes regular with the growth of the harmonic number. The forms of vibrations in the low-frequency region differ significantly from the forms of transverse vibrations of a string and a beam: the difference is that the amplitudes of adjacent half-waves (of different signs) vary greatly in amplitude, which is not typical for a string. This difference decreases with the growth of the harmonic number, as follows from the graphs in Figure 3, and for high harmonics, the shapes do not differ from the shapes of the string.

2. MODEL OF SPATIAL VIBRATIONS

Let's introduce the coordinate system $Oxyz$, directing the axis Ox through the conductor fixing points, as shown in Figure 2.

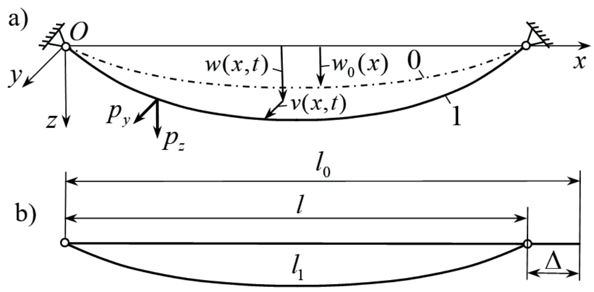

Figure 4. Parameters of static (0) and dynamic (1) states: a) displacements and components of the external load (a); lengths of the initial and current states (b)

Let $u(x,t)$, $v(x,t)$, $w(x,t)$ be the displacements of the points of the conductor axial line along the axes Ox, Oy, and Oz respectively. As before, the conductor is considered as a weighty elastic thread, fixed at the ends in a stretched state. Denote by p_y the given lateral linear load in the plane Oxy, by $p_z = mg$ the linear load of gravity forces in the plane Oxz.

The positive directions of the entered values are shown in the Figure 4 a). Figure 4 (b) shows: l – span length; l_0 – the conductor length in the span without elastic deformation at normal temperature; l_1 – the length of the stretched conductor; initial elongation –

$$\Delta = l_0(1+\alpha T) - l, \tag{15}$$

where T is the increment of temperature relative to its normal value, α is the coefficient of linear thermal expansion.

Neglecting the longitudinal inertial forces, we assume that the tensile force T and tensile stiffness B are constant along the conductor length. It follows that the deformation of the conductor is also constant along the length, i.e. $\varepsilon(x,t) \approx \varepsilon(t)$. Using this assumption, we determine the longitudinal deformation in a quadratic approximation. It's obvious that

$$\begin{aligned} dl_1^2 &= (dx+du)^2 + dv^2 + dw^2 = \\ &= \left[(1+u')^2 + v'^2 + w'^2\right] dx^2. \end{aligned}$$

Here and below, primes denote the derivative with respect to x. Neglecting the square of a small value du, we have

$$dl_1 \;;\; \sqrt{1+2u'+v'^2+w'^2}\,dx.$$

Expanding the last expression into a Taylor series and restricting ourselves to the first two terms, we obtain

$$dl_1 \approx \left[1+u'+\frac{1}{2}\left(v'^2+w'^2\right)\right]dx.$$

Integrating the last expression by x gives

$$l_1 - l = u_1 - u_0 + \frac{1}{2}\int_0^l \left(v'^2+w'^2\right)dx.$$

This makes it possible to determine the deformation of the conductor elongation in the form

$$\begin{aligned} \varepsilon &= \frac{l_1-l_0}{l_0} = \frac{l_1-l}{l_0} - \frac{\Delta}{l_0} = \\ &= \frac{1}{2l_0}\int_0^l \left(v'^2+w'^2\right)dx - \frac{\Delta}{l_0}. \end{aligned} \tag{16}$$

where Δ is determined by expression (15) and it is taken into account that at the fixing points $u_0 = u_1 = 0$.

2.1. Nonlinear vibration equations

We will obtain the conductor vibration equations based on the principle of possible displacements in generalized coordinates with nonlinear elastic forces [9-11]:

$$\delta U - \delta A_p - \delta A_{in} = 0. \tag{17}$$

where δU is the variation of the potential energy of the system; δA_p, δA_{in} are the variation of the work of external and inertial forces. It is assumed that the initial configuration is known from the solution of static equilibrium equations.

We will search for displacements using the Ritz method:

$$\begin{aligned} \overline{w}(x,t) &= \frac{w}{l} = \sum_i \left(\overline{q}_{0i} + \overline{q}_i(t)\right)\sin\frac{i\pi x}{l}, \\ \overline{v}(x,t) &= \frac{v}{l} = \sum_j \overline{r}_i(t)\sin\frac{j\pi x}{l}. \end{aligned} \tag{18}$$

where $\overline{q}_{0i}$ are the generalized coordinates of the static (initial) state; $\overline{q}_i$, $\overline{r}_i$ are generalized coordinates describing the dynamic process.

Let us determine the axial deformation by formula (16) in the form:

$$\varepsilon = \frac{l}{l_0}\left(\frac{\pi}{2}\right)^2 \left[\sum_i i^2 \left(\overline{q}_{0i} + \overline{q}_i\right)^2 + \right. \\ \left. + \sum_j j^2 \overline{r}_j^2 \right] - \frac{\Delta}{l_0}. \tag{19}$$

Then the potential energy of longitudinal deformation and its variation are respectively equal to

$$\Pi = \frac{l_0}{2} B\varepsilon^2; \\ \delta\Pi = \sum_i \frac{\partial \Pi}{\partial \overline{q}_i} \delta\overline{q}_i + \sum_j \frac{\partial \Pi}{\partial \overline{r}_j} \delta\overline{r}_j . \tag{20}$$

Here:

$$\frac{\partial \Pi}{\partial \overline{q}_i} = l_0 T \frac{\partial \varepsilon}{\partial \overline{q}_i} = lT \frac{(i\pi)^2}{2} \left(\overline{q}_{0i} + \overline{q}_i\right), \\ \frac{\partial \Pi}{\partial \overline{r}_j} = l_0 T \frac{\partial \varepsilon}{\partial \overline{r}_j} = lT \frac{(j\pi)^2}{2} \overline{r}_j; \quad T = B\varepsilon,$$

where the deformation ε is determined by the nonlinear expression (19).

We now write down the variations of the work of inertial and external forces:

$$\delta A_{in} = -\frac{l^3 m^*}{2} \left(\sum_i \ddot{\bar{q}}_i \delta\bar{q}_i + \sum_j \ddot{\bar{r}}_j \delta\bar{r}_j \right), \\ \delta A_p = l\left(\sum_i (Q_{0i} + Q_i)\delta\bar{q}_i + \sum_j R_j \delta\bar{r}_j\right), \tag{21}$$

using the notations

$$m^* = m\left(1 + \frac{\Delta}{l}\right); \quad Q_{0i} = lm^* g \frac{1 - \cos i\pi}{i\pi}, \\ Q_i = \int_0^l p_y \sin\frac{i\pi x}{l} dx, \quad R_j = \int_0^l p_z \sin\frac{j\pi x}{l} dx.$$

The equations of spatial oscillations of the conductor follow from the variational principle of possible displacements (17) taking into account expressions (20), (21). As a result, we have

$$m^* l^2 \ddot{\bar{q}}_i + T(i\pi)^2(\bar{q}_{0i} + \bar{q}_i) = 2(Q_{0i} + Q_i), \\ m^* l^2 \ddot{\bar{r}}_j + T(j\pi)^2 \bar{r}_j = 2R_j; \\ i, j = 1,2,3,\ldots. \tag{22}$$

Let's omit the terms in the first equation, the sum of which turns to zero due to static conditions. To do this, we will write the longitudinal deformation in the form (19) as the sum of the static and dynamic components:

$$\varepsilon(t) = \varepsilon_0 + \varepsilon_d(t), \tag{23}$$

where

$$\varepsilon_0 = \frac{l}{l_0}\left(\frac{\pi}{2}\right)^2 \sum_i i^2 \overline{q}_{0i}^2 - \frac{\Delta}{l_0}, \\ \varepsilon_d = \frac{l}{l_0}\left(\frac{\pi}{2}\right)^2 \left[\sum_i i^2 \left(2\overline{q}_{0i}\overline{q}_i + \overline{q}_i^2\right) + \sum_j j^2 \overline{r}_j^2 \right].$$

Substituting expression (23) into the first equation of system (22), we obtain nonlinear equations of spatial vibrations of the wire in the form:

$$m^* l^2 \ddot{\bar{q}}_i + B(i\pi)^2(\varepsilon_d \bar{q}_{0i} + \varepsilon \bar{q}_i) = 2Q_i, \\ m^* l^2 \ddot{\bar{r}}_j + (j\pi)^2 \varepsilon \bar{r}_j = 2R_j; \\ i, j = 1,2,3,\ldots,$$

where ε is determined by formula (23). Passing to the quantities

$$\tau = t\sqrt{\frac{B}{m^* l^2}}; \quad \overline{Q}_i = 2\frac{Q_i}{B}, \quad \overline{R}_j = 2\frac{R_j}{B},$$

we obtain the final form of the equations in dimensionless form:

$$\frac{d^2\overline{q}_i}{d\tau^2} = \overline{Q}_i - (i\pi)^2 \left(\varepsilon_d \overline{q}_{0i} + \varepsilon \overline{q}_i\right),$$
$$\frac{d^2\overline{r}_j}{d\tau^2} = \overline{R}_j - (j\pi)^2 \varepsilon \overline{r}_j; \qquad (24)$$
$$i, j = 1,2,3,....$$

2.2. Solution of the static problem

In this case, instead of expressions (18), (19) we have

$$w_0(x) = \sum_k q_{0k} \sin\frac{k\pi x}{l};$$
$$\varepsilon_0 = \frac{l}{l_0}\left(\frac{\pi}{2}\right)^2 \sum_k \left(k\overline{q}_{0k}\right)^2 - \frac{\Delta}{l_0}.$$

The equilibrium equation follows from (22):

$$l\frac{(k\pi)^2}{2} T_0 k^2 \overline{q}_{0k} = -l^2 m^* g \frac{1}{k\pi}\left(1-\cos k\pi\right),$$

whence it follows that $\overline{q}_{0k} = 0$ at $k = 2,4,6,...$. We rewrite the last equation in the form:

$$\frac{k\pi}{2}\overline{q}_{0k} = -\frac{2}{(k\pi)^2}\frac{l\,m^* g}{N_0} \quad (k = 1,3,5,...)$$

and substitute in the expression for deformation (24). As a result, we get

$$\varepsilon_0 = \frac{l}{l_0}\left(\frac{2l\,m^* g}{\pi^2 EF\varepsilon_0}\right)^2 \sum_{k=1,3,...}\frac{1}{k^4} - \frac{\Delta}{l_0},$$

whence it follows that the deformation ε_0 is determined from the solution of the cubic equation $\varepsilon_0^3 + b\varepsilon_0^2 + d = 0$, where

$$b = \frac{\Delta}{l_0} \geq 0, \quad d = -\frac{l}{l_0}\left(\frac{2l\,m^* g}{\pi^2 B}\right)^2 \sum_{k=1,3,...}\frac{1}{k^4} < 0,$$

the solution of which is found by the Cardano formula.

As an example, a wire fixed at the ends with the characteristics given in Table 1 is considered. Table 2 shows the results of calculations for $n = 1,3,...,9$.

2.3. Natural vibrations

The linearization of equations (24) leads to a system of linear equations

$$\frac{d^2\overline{q}_i}{d\tau^2} + (i\pi)^2\left(\varepsilon_d \overline{q}_{0i} + \varepsilon_0 \overline{q}_i\right) = 0,$$
$$\frac{d^2\overline{r}_j}{d\tau^2} + (j\pi)^2 \varepsilon_0 \overline{r}_j = 0; \quad i, j = 1,2,3,..., \qquad (25)$$

where ε_0, $\overline{q}_{0i}$ are determined from the solution of a static problem; $\varepsilon_d = \frac{l}{l_0}\frac{\pi^2}{2}\sum_i i^2 \overline{q}_{0i}\overline{q}_i$.

Equations (25) will be written in matrix form by introducing column vectors

$$\overline{\boldsymbol{q}} = \left(\overline{q}_1 \;...\; \overline{q}_n\right)^T, \; \overline{\boldsymbol{r}} = \left(\overline{r}_1 \;...\; \overline{r}_m\right)^T$$

and a diagonal matrix K with elements $\kappa_{ii} = i^2$. For simplicity, we will assume that $n = m$. Then instead of equations (25) we have two unrelated matrix equations

$$\frac{d^2\overline{\boldsymbol{q}}}{d\tau^2} + \mathrm{M}^q \overline{\boldsymbol{q}} = 0, \quad \frac{d^2\overline{\boldsymbol{r}}}{d\tau^2} + \mathrm{M}^r \overline{\boldsymbol{r}} = 0, \quad (26)$$

where

$$\mathrm{M}^q = \pi^2 \mathrm{K}\left(\frac{l}{l_0}\frac{\pi^2}{2}\overline{\boldsymbol{q}}\,\overline{\boldsymbol{q}}^T \mathrm{K} + \varepsilon_0 \mathrm{E}\right), \; \mathrm{M}^r = \pi^2 \varepsilon_0 \mathrm{K};$$

E – unit matrix.

The solution of equations (26) is represented as

$$\bar{q} = \mathrm{A}\,\widetilde{\sin\omega}\,\tau, \bar{r} = \mathrm{C}\,\widetilde{\sin\omega}\,\tau;$$
$$\widetilde{\omega} = \omega\sqrt{\frac{m^* l^2}{B}}. \qquad (27)$$

Substituting expressions (27) into equations (25) leads to two unrelated systems of algebraic equations

$$(M^q - \tilde{\omega}^2 E)\mathrm{A} = 0, \; (M^r - \tilde{\omega}^2 E)\mathrm{C} = 0, \quad (28)$$

Since the matrix M^r is diagonal, the eigenvalues of the second equation are the values $\tilde{\omega}_j^2 = (j\pi)^2 \varepsilon_0$. Then the frequency spectrum of the natural vibrations of the conductor in the horizontal direction is a sequence $\tilde{\omega}_j = j\pi\sqrt{\varepsilon_0}$, $j = 1, 2, ..., n$.

From the condition of non-triviality of the solution of the first equation of system (28), a frequency equation follows for determining the frequency spectrum of natural vibrations in the vertical direction:

$$det\|M^q - \tilde{\omega}^2 E\| = 0.$$

The vibration modes $\boldsymbol{A}$ are determined from the solution of the first equation (28) with the normalization condition $\sqrt{\boldsymbol{A}^T \boldsymbol{A}} = 1$. The vibration modes $\boldsymbol{C}$ are trivial as a sequence $(1\,0 ... 0)$, $(0\,1 ... 0)$, ..., $(0\,0 ... 1)$.

As an example, consider a conductor with characteristics from Table 1 for $n = 5$. The solution of the static problem is given in Table 2. The calculation results are shown in Table 3.

The first frequency of horizontal oscillations can be estimated using the equation of oscillations of a physical pendulum $J\, d^2\varphi / dt^2 = M$, where J is the moment of inertia of the sagging conductor about the axis Ox, M is the total moment of the gravitational load.

Table 1. Conductor parameters

Tensile stiffness	$B = 7.3 \cdot 10^6$ N
Linear mass	$m = 0.23$ kg/m
Conductor length	$l_0 = 21$ m
Span length	$l = 20$ m
Gravity acceleration	$g = 9.81$ m/s^2

Table 2. Results of solving a static problem

n	1	3	5	7	9
D	$-6.59 \cdot 10^{-18}$	$-6.67 \cdot 10^{-18}$	$-6.68 \cdot 10^{-18}$	$-6.68 \cdot 10^{-18}$	$-6.68 \cdot 10^{-18}$
ε_0	$5.88 \cdot 10^{-6}$	$5.92 \cdot 10^{-6}$	$5.92 \cdot 10^{-6}$	$5.92 \cdot 10^{-6}$	$5.92 \cdot 10^{-6}$
q_{01}	0.14236	0.14149	0.14138	0.14135	0.14134
q_{03}	-	0.00524	0.00523	0.00523	0.00523
q_{05}	-	-	0.00113	0.00113	0.00113
q_{07}	-	-	-	0.00041	0.00041
q_{09}	-	-	-	-	0.00019

Table 3. Frequencies and waveforms for $n = 5$

Vertical vibrations					Horizontal vibrations				
Natural frequencies, Hz									
0.669	0.957	1.338	1.646	45.202	0.335	0.669	1.004	1.338	1.673
Vibration modes									
0	0.932	0	0.930	0.932	1	0	0	0	0
1	0	0	0	0	0	1	0	0	0
0	0.311	0	0.310	0.310	0	0	1	0	0
0	0	1	0	0	0	0	0	1	0
0	0.189	0	0.195	0.187	0	0	0	0	1

Using the expansion

$$w_0(x) = \sum_k q_{0k} \sin\left(k\pi x/l\right),$$

we get

$$J = \frac{m^* l^3 \tilde{J}}{2}, M = -\frac{2m^* l^2 g \tilde{S} \phi}{\pi},$$

where $\tilde{J} = \sum_{i=1,3,\dots} \bar{q}_{0i}^2$, $\tilde{S} = \sum_{i=1,3,\dots} \frac{\bar{q}_{0i}}{i}$.

For the angle of rotation of the pendulum (sagging wire) in the form $\varphi = A \sin \omega t$ from the condition of non-triviality of the solution of the equation of vibrations, we obtain a formula for calculating the circular frequency of oscillations $\omega = \sqrt{\frac{4g\tilde{S}}{\pi l \tilde{J}}}$ or in Hertz $f = \frac{1}{\pi} \cdot \sqrt{\frac{g\tilde{S}}{\pi l \tilde{J}}}$.

The calculation for the above example gives the value $f = 0.333$ Гц, which is completely consistent with the first oscillation frequency in the horizontal direction $f = 0.335$ Гц.

CONCLUSION

When constructing the theory of self-excitation of conductor vibrations, classified in operational OHL practice as a galloping, it is necessary to proceed from the model of a flexible heavy thread that performs spatial vibrations. Galloping modes are observed in the frequency range of the order of 1 Hz, which in the typical OHL spans correspond to the first 1-3 harmonics [6, 7]. Model experiments have shown [5] that vibrations in the vertical plane, which excite parametric vibrations with exit from the sag plane, are essential for such processes. The model of self-excitation should be based on the data of the modal analysis of the system as its basic characteristics.

The methods developed and described in the article for calculating the natural frequencies and vibration modes of the OHL conductors reflect the features of the conductors that determine their tendency to self-excitation of vibrations. It is shown that in the frequency domain of interest, transverse stretching vibrations and pendulum vibrations are essential; longitudinal elastic waves do not play a significant role.

The developed methods of modal analysis of conductor vibrations will be used in the construction of a model of self-excitation of vibrations of OHL conductors of both thermomechanical and aerodynamic nature.

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Valiullin Airat A., student of the Moscow Institute of Physics and Technology (National Research University) (MIPT); 9, Institutsky Lane, Dolgoprudny, Moscow region, 141701; +7 (495) 408-45-54; valiullin.aa@phystech.edu

Валиуллин Айрат Анасович, студент Московского физико-технического института (национального исследовательского университета) (МФТИ); 141701, Московская область, г. Долгопрудный, Институтский пер., д. 9; +7 (495) 408-45-54; valiullin.aa@phystech.edu

Danilin Alexander N., Doctor of Physical and Mathematical Sciences, Chief Researcher of the Institute of Applied Mechanics of the Russian Academy of Sciences (IAM RAS), Deputy Director of IAM RAS for Scientific Work, Professor of the Moscow Aviation Institute; 125040, Moscow, Leningradsky Prospekt, 7, p. 1; +7 (495) 946-18-06; andanilin@yandex.ru

Данилин Александр Николаевич, доктор физико-математических наук, главный научный сотрудник Института прикладной механики РАН (ИПРИМ РАН), заместитель директора ИПРИМ РАН по научной работе, профессор Московского авиационного института; 125040, г. Москва, Ленинградский проспект, д. 7, стр. 1; +7 (495) 946-18-06; andanilin@yandex.ru

Feldshteyn Valery A., Doctor of Technical Sciences, Chief Research Associate of JSC "Central Research Institute for Machine Building" (JSC "TsNIIMash"), Professor at MIPT; Russia, 141070, Moscow region, Korolev, Pionerskaya str., 4; +7 (495) 513-59-51; dinpro@mail.ru

Фельдштейн Валерий Адольфович, доктор технических наук, главный научный сотрудник АО «Центральный научно-исследовательский институт машиностроения» (АО "ЦНИИМаш"), профессор МФТИ; Россия, 141070, Московская область, г. Королёв, ул. Пионерская, д. 4; +7 (495) 513-59-51; dinpro@mail.ru