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ON THE QHASI CLASS AND ITS EXTENSION TO SOME GAUSSIAN SHEETS

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Abstract. Introduced in 2018 the generalized bifractional Brownian motion is considered as an element of the quasi-helix with approximately stationary increment class of real centered Gaussian processes conditioning by parameters. This paper proves that the generalized bifractional Brownian motion is an element of the above mentioned class with no condition on parameters. The quasi-helix with approximately stationary increment class of real centered Gaussian processes is extended to two-dimensional processes as the fractional Brownian sheet, the sub-fractional Brownian sheet, and the bifractional Brownian sheet. This generalized presentation of the class of stochastic processes is used to augment the training samples for generative adversarial networks in computer vision problem.

Keywords: centered Gaussian process, generalized bifractional Brownian motion, Gaussian sheet, generative adversarial network, computer vision

О КВАЗИ КЛАССЕ И ЕГО РАСШИРЕНИИ К НЕКОТОРЫМ ГАУССОВЫМ ЛИСТАМ

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Аннотация. Введенное в 2018 году обобщенное бифрактальное броуновское движение рассматривается как элемент квазиспирали класса действительных центрированных гауссовских процессов, обладающих приблизительно стационарными приращениями, обусловленных параметрами. В данной работе доказывается, что обобщенное бифракционное броуновское движение является элементом указанного выше класса без каких-либо условий на параметры. Свойство квазиспирали этого класса центрированных гауссовских процессов распространяется на двумерные процессы, такие как фрактальный броуновский лист, субфрактальный броуновский лист и бифрактальный броуновский лист. Это обобщенное представление класса процессов используется для расширения обучающих выборок для генеративно-состязательных сетей в задаче компьютерного зрения.

Ключевые слава: центрированный гауссовский процесс, обобщенное бифракционное броуновское движение, гауссовский листгенеративно-состязательная сеть, компьютерное зрение

1. INTRODUCTION

Trends in the development of information technology are changing the classical idea of how to solve many problems that arise in civil engineering. Accelerated analysis of large information flows of multivariate solutions from the concept of the project to the decommissioning moment for a certain construction object requires the use of artificial intelligence methods. It is clear that machine learning, deep learning, and reinforcement learning are becoming the leading information technologies. For example, the development of neural networks makes it possible to more accurately solve the problem of finding and classifying defects or pathologies hidden from the human eye on the surface of a structure, even at an early stage of the destruction process. In pursuit of the goal of increasing the reliability of the solutions obtained, the solution methodology itself is modified [1]. Popular in computer vision, convolutional neural networks very often use the so-called pseudo-samples for training, which result from generating data using various random noises [2]. This approach to training neural networks has led to the creation of generative adversarial networks (GANs) [3]. The main idea of the GAN is to compete with two neural networks in a zero-sum game, i.e. one network generates information and the other tries to please it. This competitive process must change over time to avoid overfitting the guessing network. When writing a scenario for generating pseudo data, it is necessary to use some universal multidimensional (even if two-dimensional) stochastic process or a class of processes that allows you to display reality as closely as possible - stationary or non-stationary dynamics of the phenomenon under study. The modern development of the theory of stochastic processes makes it possible to introduce a certain class of processes that can be successfully used to create a GAN, and as a result, to increase the reliability of solving computer vision problems.

In [4], a new class of centered Gaussian processes was introduced. More precisely, a centered Gaussian process $\{X(t), t \in I \subset \mathsf{R}\}$ belongs to the quasi-helix with approximately stationary increments (QHASI) class if it fulfills the five following assumptions:

- **A1**: $X(0) = 0$ with probability 1;
- **A2**: there exists $\lambda > 0$ such that X is self-similar with index λ;
- **A3**: there exist

$$0 < C_1 \le C_2 < +\infty;$$

such that $\forall\, (s,t) \in I^2$

$$C_1\,|\,t-s\,|^{2\lambda} \le \mathsf{E}(X(t)-X(s))^2 \le C_2\,|\,t-s\,|^{2\lambda};$$

- **A4**: there exists

$$C_3 \in [C_1, C_2]$$

such that $\forall\, (s,t) \in I^2$, $t \ge s$, $st \ne 0$, when $t-s \to 0$, $\mathsf{E}(X(t)-X(s))^2 \sim C_3\,(t-s)^{2\lambda}$,

- A5: there exists

$$C_4 \in [C_1, C_2]$$

such that $\forall\, t \in I$, $\mathsf{E}\,X(t)^2 = C_4\,|\,t\,|^{2\lambda}$.

Let us make some comments about the assumptions. Assumptions (**A1**) and (**A5**) are done for sake of convenience. Then, assumption (**A2**) means that the process X is an attractive one. Finally, assumption (**A3**) means that the process X is a λ-quasi-helix in the sense of [5], whereas assumption (A4) means that the increments of X are approximately stationary for small increments, this notion having been introduced in [6]. The underlying idea of the QHASI class is to replace the stationary increments property by assumptions (**A3**) and (**A4**).

The QHASI class contains some famous Gaussian processes such that the fractional Brownian motion (fBM), the bifractional Brownian motion (bBM) and the sub-fractional Brownian motion (sfBM). The values of the associated constants $\left(\lambda, C_1, C_2, C_3, C_4\right)$ can be found in [4] for each of these processes. We refer on one hand to [6] for further information on the bBm and on the other hand to [7] for further information on the sfBm. Note also that the following processes are also elements of the QHASI class:

- the sub bifractional Brownian motion (sbBm) (see [8])
- the generalized fBM (gfBm) (see [9], [10])

In [10], the generalized bifractional Brownian motion (gbBm) $Y := Y_{\alpha,\beta,H,K}$, was introduced. It is defined as follows:

$$Y(t) := Y_{\alpha,\beta,H,K}(t) = \alpha\, B_{H,K}(t) + \beta\, B_{H,K}(-t), \quad t \ge 0,\ \alpha > 0,\ \beta > 0,$$

where $\{B_{H,K}(t), t \in \mathsf{R}\}$ is a bBm with indices $0 < H < 1$ and $0 < K \le 1$.

Set $\alpha(K)=\frac{1}{2^{(2-K)/2}}$, $0<K\le 1$. We insist on the fact that the process Y was already introduced for specific values of α, β, and K. More precisely, the sfBm corresponds to $Y_{\alpha(1),\alpha(1),H,1}$, the sbBm to $Y_{\alpha(K),\alpha(K),H,K}$ and the gfBm to $Y_{\alpha,\beta,H,1}$.
In [10], it was proved that the gbBm was an element of the QHASI class under some conditions on H and K. More precisely, the following result was established.

Theorem 1. *Assume that* $2HK\le 1$ *. Then the gbBm is an element of the QHASI class, with*

- $\lambda = HK$,
- $C_1=(\alpha+\beta)^2-2^{2-K}\ \alpha\ \beta$,
- $C_2=2^{1-K}\ ((\alpha+\beta)^2-2^{2HK}\alpha\beta)$,
- $C_3=2^{1-K}\left(\alpha^2+\beta^2\right)$,
- $C_4=\alpha^2+2(1-2^{2HK-K})\alpha\beta+\beta^2$.

■

The first aim of this paper is to show that the gbBm is an element of the QHASI class for any

$$(\alpha,\beta,H,K)\in]0,+\infty[\times]0,+\infty[\times]0,1[\times]0,1].$$

Our first result is stated in the following theorem.

Theorem 2. *Assume that* $2HK>1$. *Then the gbBm is an element of the QHASI class, with*

- $\lambda = HK$,
- $C_1=2\ (1-2^{2HK-1-K})\ (\alpha^2+\beta^2)$,
- $C_2=2^{1-K}\ (\alpha^2+\beta^2)$,
- $C_3=2^{1-K}\left(\alpha^2+\beta^2\right)$,
- $C_4=\alpha^2+2(1-2^{2HK-K})\alpha\beta+\beta^2$.

■

Let us make some comments on the above theorems. As it was already observed in [4], [8] and [12], the hyperbola $2HK=1$ plays a key role. It has also an influence on the values of the constants C_1 and C_2. Let focus our attention on two specific cases. First, when $\alpha=\beta=\alpha(K)=\frac{1}{2^{(2-K)/2}}$, theorem 2 generalizes proposition 1.1 in [8]. Next, when $K=1$ and $2H>1$, the values of the constant C_2 given in the above theorem and in [9] are similar, but the value of C_1 given in Theorem 2 is less precise than the value of C_1 given in [9]. It can be explained by the fact that, when $K=1$, direct computations are available.
The second aim of this paper is to answer to the following question: can we extend the QHASI class to two-dimensional processes? To this purpose, we introduce the following notation.

Let

$$\left\{X_1\left(s\right),s\ge 0\right\}$$

and

$$\left\{X_2\left(t\right),t\ge 0\right\}$$

be two elements of the QHASI class. For any $i\in\{1,2\}$, we denote by $\left(\lambda_i,C_{i1},C_{i2},C_{i3},C_{i4}\right)$ the associated constants. Set

$$\sigma_1^2\left(s_1,s_2\right)=\mathsf{E}\left(X_1(s_1)-X_1(s_2)\right)^2$$

and

$$\sigma_2^2\left(t_1,t_2\right)=\mathsf{E}\left(X_2(t_1)-X_2(t_2)\right)^2.$$

Set $u=\left(s,t\right)$ and $u_{ij}=(s_i,t_j)$, $1\le i,j\le 2$. We consider some Gaussian sheets $\left\{X\left(u\right),u\in\mathsf{R}^+\times\mathsf{R}^+\right\}$ such that

$$\mathsf{E}\left(X\left(u_{ij}\right)X\left(u_{i'j'}\right)\right)=\mathsf{E}\left(X_1\left(s_i\right)X_1\left(s_{i'}\right)\right)\times\mathsf{E}\left(X_2\left(t_j\right)X_2\left(t_{j'}\right)\right).$$

We can easily derive the variance of the process X. We have

$$\mathsf{E}\left(X\left(u\right)^2\right)=\mathsf{E}\left(X_1\left(s\right)^2\right)\times\mathsf{E}\left(X_2\left(t\right)^2\right)=C_{X4}s^{2\lambda_1}t^{2\lambda_2},$$

where $C_{X4} = C_{14} \times C_{24}$.

Note that when the process X_1 is a fBm with Hurst index $0 < H_1 < 1$ and the process X_2 is a fBm with Hurst index $0 < H_2 < 1$, the process X is a fractional Brownian sheet (fBS) with indexes H_1 and H_2. There is a huge literature on the fBs. We refer to [13] for further information on this process.
The rest of the paper is organized as follows. In section 2, we prove Theorem 2, whereas the properties of the two-dimensional process X are studied in section 3. In section 4, we focus our attention on specific sheets and illustrations for the computer vision problem related to the surface anomalies detection. Section 5 concludes the main results of this research.

2. PROOF OF THEOREM 2

Recall first that $1 < 2HK < 2$, and therefore $H > 1/2$ and $K > 1/2$. Note that the values of λ, C_2, C_3 and C_4 were already given in [CEN18]. The proof of the theorem will be divided into four steps.

Step 1. Let us determine the value of the constant C_1. Combining proposition 10 with lemma 12 presented in [10], we have for $t \geq s \geq 0$

$$\begin{aligned}\sigma^2(s,t) &:= \sigma^2_{\alpha,\beta,H,K}(s,t) \\ &= \mathsf{E}\left(\left(Y_{\alpha,\beta,H,K}(t) - Y_{\alpha,\beta,H,K}(s)\right)^2\right) \\ &= 2^{1-K}\left(\alpha^2+\beta^2\right)(t-s)^{2HK} \\ &\quad -(\alpha+\beta)^2 F_{H,K}(s,t) \\ &\quad -2^{1-K+2HK}\alpha\beta\left|F_{1/2,2HK}(s,t)\right|\end{aligned}$$

where

$$F_{H,K}(s,t) = 2\left(\frac{t^{2H}+s^{2H}}{2}\right)^K - t^{2HK} - s^{2HK} \geq 0,$$

$$F_{1/2,2HK}(s,t) = 2\left(\frac{t+s}{2}\right)^K - t^{2HK} - s^{2HK} \leq 0.$$

Let us establish a suitable upper bound of

$$(\alpha+\beta)^2\, F_{H,K}(s,t) + 2^{1-K+2HK}\alpha\beta\left|F_{1/2,2HK}(s,t)\right|.$$

Recall that

$$2\,\alpha\,\beta \leq \alpha^2+\beta^2 \leq (\alpha+\beta)^2 \leq 2\,(\alpha^2+\beta^2).$$

Thus, we have

$$\begin{aligned}&(\alpha+\beta)^2\, F_{H,K}(s,t) + 2^{1-K+2HK}\alpha\,\beta\left|F_{1/2,2HK}(s,t)\right| \\ &\leq 2\left(\alpha^2+\beta^2\right)\Big(F_{H,K}(s,t) \\ &\quad +2^{2HK-1-K}\left|F_{1/2,2HK}(s,t)\right|\Big)\end{aligned}$$

Note that $2HK-1-K = (2H-1)K-1 < 0$.
Next, combining inequality (2.4) in [8] with straight computations, we get

$$\begin{aligned}&F_{H,K}(s,t) + 2^{2HK-1-K}\left|F_{1/2,2HK}(s,t)\right| \\ &= \tfrac{1}{2^K}\Big(2\left(t^{2H}+s^{2H}\right)^K \\ &\quad -\left(2^K-2^{2HK-1}\right)\left(t^{2HK}+s^{2HK}\right) - (t+s)^{2HK}\Big) \\ &\leq \tfrac{1}{2^K}\left((t-s)^{2HK} - \left(2^K-2^{2HK-1}\right)(t-s)^{2HK}\right).\end{aligned}$$

Hence, we get

$$\begin{aligned}\sigma^2(s,t) &\geq 2^{1-K}\left(\alpha^2+\beta^2\right)(t-s)^{2HK} - 2\left(\alpha^2+\beta^2\right) \\ &\quad \tfrac{1}{2^K}\left((t-s)^{2HK} - \left(2^K-2^{2HK-1}\right)(t-s)^{2HK}\right) \\ &= 2\left(1-2^{2HK-1-K}\right)\left(\alpha^2+\beta^2\right)(t-s)^{2HK}.\end{aligned}$$

The constant C_1 is now determined.

Step 2. The aim of this step is to show that $C_1 \leq C_3 = C_2$. Since $1 < 2HK < 2$ and $0 < K \leq 1$, we have

$$2^{K} \leq 2 \leq 1 + 2^{2HK-1},$$

and therefore

$$2 - 2^{2HK-K} \leq 2^{1-K}.$$

The last inequality can be rewritten as follows

$$2\,(1 - 2^{2HK-K-1}) \leq 2^{1-K}.$$

Hence $C_1 \leq C_3 = C_2$.

Step 3. Let us show that $C_4 \leq C_2 = C_3$. To determine the sign of $C_3 - C_4$, it suffices to study the function $T_{H,K}$ defined by

$$T_{H,K}(x) = (2^{1-K} - 1)x^2 + 2(2^{2HK-K} - 1)x + 2^{1-K} - 1, \quad x \in \mathsf{R}.$$

We will distinguish the following two cases.

Case 1. $K = 1$ and $2H > 1$.
We have $T_{H,1}(x) = 2(2^{2H-1} - 1)x$. Keep in mind that $\alpha > 0$ and $\beta > 0$. Once $x > 0$, it follows that $T_{H,1}(x) > 0$. Thenceforward, $C_3 > C_4$.

Case 2. $K < 1$ and $1 < 2HK$.
The function $T_{H,K}$ has a unique minimum at the point

$$x_0 = -\frac{2^{2HK-K} - 1}{2^{1-K} - 1}.$$

Since

$$2HK - K > 0,$$

we obviously have $x_0 < 0$. Moreover, recall that, when $x \leq x_0$, $T_{H,K}$ is a non-increasing function, otherwise a non-decreasing one. Note that

$$T_{H,K}(0) = 2^{1-K} - 1 > 0.$$

Thus we have $T_{H,K}(x) > 0$ for any $x > 0$, and therefore $C_3 > C_4$.

Step 4. Let us show that $C_1 \leq C_4$. It suffices to verify

$$2\left(1 - 2^{2HK-1-K}\right)\left(\alpha^2 + \beta^2\right) \leq \alpha^2 + 2\left(1 - 2^{2HK-K}\right)\alpha\beta + \beta^2.$$

This inequality can be rewritten in the form

$$\left(1 - 2^{2HK-K}\right)\left(\alpha^2 + \beta^2\right) \leq 2\left(1 - 2^{2HK-K}\right)\alpha\beta,$$

which is equivalent to

$$\left(1 - 2^{2HK-K}\right)\left(\alpha - \beta\right)^2 \leq 0.$$

Since $2HK - K > 0$, $C_1 \leq C_4$.
This completes the proof of the theorem.

3. PROPERTIES OF THE PROCESS X

Let us state some basic properties of the process X

Proposition 3. *We have*

- $X(\cdot,\cdot)$ *is a Gaussian process,*
- $X(s,0) = X(0,t) = 0$,
- *for any* $s_0 > 0$, *the one-dimensional process* $\left\{s_0^{-\lambda_1} X\left(s_0, t\right), t \geq 0\right\}$ *is a* $\sqrt{C_{14}} \times X_2$ *process,*
- *for any* $t_0 > 0$, *the one-dimensional process* $\left\{t_0^{-\lambda_2} X\left(s, t_0\right), s \geq 0\right\}$ *is a* $\sqrt{C_{24}} \times X_1$ *process.*
-

Proof. The first two points are obvious. To prove the third point, it suffices to compute

$$\mathsf{E}\left(s_0^{-\lambda_1} X\left(s_0, t_1\right) s_0^{-\lambda_1} X\left(s_0, t_2\right)\right).$$

We have

$$\mathsf{E}\left(s_0^{-\lambda_1}X(s_0,t_1)s_0^{-\lambda_1}X(s_0,t_2)\right)$$
$$=s_0^{-2\lambda_1}\mathsf{E}\left(X_1(s_0)X_1(s_0)\right)\times\mathsf{E}\left(X_2(t_1)X_2(t_2)\right)$$
$$=s_0^{-2\lambda_1}C_{14}s_0^{2\lambda_1}\times\mathsf{E}\left(X_2(t_1)X_2(t_2)\right)$$
$$=C_{14}\times\mathsf{E}\left(X_2(t_1)X_2(t_2)\right).$$

We omit the proof of the last point. ■

Keep in mind that the flavor of the QHASI class consists in the quasi-helix property in the sense of Kahane [2] and its approximately stationary one. To extend these concepts to two-dimensional processes, let us recall that the increment Δ of X between the points $u_{11}=(s_1,t_1)$ and $u_{22}=(s_2,t_2)$ is defined as follows

$$\Delta=X(u_{11})+X(u_{22})-X(u_{12})-X(u_{21}),$$

where $u_{12}=(s_1,t_2)$ and $u_{21}=(s_2,t_1)$.
Set

$$\mathsf{E}\Delta^2=\sigma^2(u_{11},u_{22}).$$

We can establish the following essential proposition.

Proposition 4. *We have*

$$\sigma^2(u_{11},u_{22})=\sigma_1^2(s_1,s_2)\times\sigma_2^2(t_1,t_2).$$

Proof. As far as we know, the above proposition has not been written yet. Therefore we will prove it. Direct computations yield

$$\begin{aligned}&\sigma^2(u_{11},u_{22})=\\&\left(\mathsf{E}X_1^2(s_1)+\mathsf{E}X_1^2(s_2)\right)\times\left(\mathsf{E}X_2^2(t_1)+\mathsf{E}X_2^2(t_2)\right)\\&+2\times\mathsf{E}\left(X_1(s_1)X_1(s_2)\right)\times\mathsf{E}\left(X_2(t_1)X_2(t_2)\right)\\&-\left(\mathsf{E}X_1^2(s_1)+\mathsf{E}X_1^2(s_2)\right)\times\mathsf{E}\left(X_2(t_1)X_2(t_2)\right)\\&-\left(\mathsf{E}X_2^2(t_1)+\mathsf{E}X_2^2(t_2)\right)\times\mathsf{E}\left(X_1(s_1)X_1(s_2)\right).\end{aligned}$$

Since

$$\sigma_1^2(s_1,s_2)=\mathsf{E}X_1^2(s_1)-2\,\mathsf{E}\left(X_1(s_1)X_1(s_2)\right)+\mathsf{E}X_1^2(s_2)$$

and

$$\sigma_2^2(t_1,t_2)=\mathsf{E}X_2^2(t_1)-2\,\mathsf{E}\left(X_2(t_1)X_2(t_2)\right)+\mathsf{E}X_2^2(t_2),$$

we have

$$\begin{aligned}&\sigma^2(u_{11},u_{22})=\\&\left(\mathsf{E}X_1^2(s_1)+\mathsf{E}X_1^2(s_2)\right)\left(\mathsf{E}X_2^2(t_1)+\mathsf{E}X_2^2(t_2)\right)\\&-2\times\sigma_1^2(s_1,s_2)\mathsf{E}\left(X_2(t_1)X_2(t_2)\right)\\&-\left(\mathsf{E}X_2^2(t_1)+\mathsf{E}X_2^2(t_2)\right)\mathsf{E}\left(X_1(s_1)X_1(s_2)\right)\\&=-2\times\sigma_1^2(s_1,s_2)\times\mathsf{E}\left(X_2(t_1)X_2(t_2)\right)\\&+\left(\mathsf{E}X_2^2(t_1)+\mathsf{E}X_2^2(t_2)\right)\\&\times\left(\mathsf{E}X_1^2(s_1)+\mathsf{E}X_1^2(s_2)-2\mathsf{E}\left(X_1(s_1)X_1(s_2)\right)\right)\\&=-2\times\sigma_1^2(s_1,s_2)\times\mathsf{E}\left(X_2(t_1)X_2(t_2)\right)\\&+\left(\mathsf{E}X_2^2(t_1)+\mathsf{E}X_2^2(t_2)\right)\times\sigma_1^2(s_1,s_2)\\&=\sigma_1^2(s_1,s_2)\times\left(\mathsf{E}X_2^2(t_1)+\mathsf{E}X_2^2(t_2)\right.\\&\left.-2\times\mathsf{E}\left(X_2(t_1)X_2(t_2)\right)\right)\\&=\sigma_1^2(s_1,s_2)\times\sigma_2^2(t_1,t_2).\end{aligned}$$

The proof of the proposition is now complete. ■

Combining the above proposition with the fact that the processes X_1 and X_2 are elements of the QHASI class, we get the following results.

Proposition 5. *We have*

$$C_{X1}\left|s_1-s_2\right|^{2\lambda_1}\left|t_1-t_2\right|^{2\lambda_2}\le\sigma^2(u_{11},u_{22})\le C_{X2}\left|s_1-s_2\right|^{2\lambda_1}\left|t_1-t_2\right|^{2\lambda_2},$$

where $C_{X1}=C_{11}\times C_{21}$ *and* $C_{X2}=C_{12}\times C_{22}$.

Proposition 6. *When* $s_2 - s_1 \to 0$, $s_2 \ge s_1 > 0$, *and* $t_2 - t_1 \to 0$, $t_2 \ge t_1 > 0$, *we have*

$$\sigma^2(u_{11}, u_{22}) \sim C_{X3}(s_2 - s_1)^{2\lambda_1}(t_2 - t_1)^{2\lambda_2},$$

where $C_{X3} = C_{13} \times C_{23}$.

It is obvious that $C_{X1} \le C_{X3} \le C_{X2}$ and $C_{X1} \le C_{X4} \le C_{X2}$. Roughly speaking, we can say that the process X is a quasi-helix in the sense of Kahane [2] and has approximately stationary increments. We can associate to X the six constants $(\lambda_1, \lambda_2, C_{X1}, C_{X2}, C_{X3}, C_{X4})$. Thus, we answer to the question stated in the introduction. Indeed we are able, on one hand to extend the definition of the QHASI class to two dimensional processes, and on the other hand to create new Gaussian sheets.

4. SOME SPECIFIC SHEETS

4.1. The fractional Brownian sheet

Let X_1 be a fBm with Hurst index $0 < H_1 < 1$ and X_2 be a fBm with Hurst index $0 < H_2 < 1$. As already mentioned, the process X constructed as described earlier is the fBs. Note that its six associated constants are $(H_1, H_2, 1, 1, 1, 1)$. It implies that calculi are quite convenient for the fBs. This partially explains its popularity.

4.2. The subfractional Brownian sheet

Let X_1 be a sfBm with Hurst index $0 < H_1 < 1$ and X_2 be a sfBm with Hurst index $0 < H_2 < 1$. We can construct the process X. To determine the six associated constants, we have to consider the four following cases:

- when $H_1 < \frac{1}{2}$ and $H_2 < \frac{1}{2}$, the constants are
$$\left(H_1, H_2, 1, \left(2 - 2^{2H_1 - 1}\right)\left(2 - 2^{2H_2 - 1}\right),\right.$$
$$\left.1, \left(2 - 2^{2H_1 - 1}\right)\left(2 - 2^{2H_2 - 1}\right)\right);$$
- when $H_1 < \frac{1}{2}$ and $H_2 \ge \frac{1}{2}$, the constants are
$$\left(H_1, H_2, 2 - 2^{2H_2 - 1}, 2 - 2^{2H_1 - 1},\right.$$
$$\left.1, \left(2 - 2^{2H_1 - 1}\right)\left(2 - 2^{2H_2 - 1}\right)\right);$$
- when $H_1 \ge \frac{1}{2}$ and $H_2 < \frac{1}{2}$, the constants are
$$\left(H_1, H_2, 2 - 2^{2H_1 - 1}, 2 - 2^{2H_2 - 1},\right.$$
$$\left.1, \left(2 - 2^{2H_1 - 1}\right)\left(2 - 2^{2H_2 - 1}\right)\right);$$
- when $H_1 \ge \frac{1}{2}$ and $H_2 \ge \frac{1}{2}$, the constants are
$$\left(H_1, H_2, \left(2 - 2^{2H_1 - 1}\right)\left(2 - 2^{2H_2 - 1}\right),\right.$$
$$\left.1, 1, \left(2 - 2^{2H_1 - 1}\right)\left(2 - 2^{2H_2 - 1}\right)\right).$$

4.3. The bifractional Brownian sheet

Let X_1 be a bBm with Hurst indices $0 < H_1 < 1$ and $0 < K_1 \le 1$ as well as X_2 be a bBm with Hurst indices $0 < H_2 < 1$ and $0 < K_1 \le 1$. We can construct the process X. Its six associated constants are

$$\left(H_1 K_1, H_2 K_2, 2^{-K_1 - K_2}, 2^{2 - K_1 - K_2}, 2^{2 - K_1 - K_2}, 1\right).$$

4.4. Other possible sheets

Following the same ideas, we can construct the sub-bifractional Brownian sheet, the generalized fractional Brownian sheet and the generalized bifractional sheet. There is no difficulty to give the six associated constants. We can also mix the different elements of the QHASI class in order to create new sheets. For example, let X_1 be a fBm with Hurst index $0 < H_1 < 1$ and X_2 be an element of the QHASI class with the associated constants $(\lambda, C_1, C_2, C_3, C_4)$. We can construct the process X using six associated constants

$$\left(H_1, \lambda, C_1, C_2, C_3, C_4\right)$$

In some sense, the influence of the fBm vanishes. This is not really surprising since the fBm has stationary increments.

4.5. Illustrations

Now we give several illustrations of an image generation, using the fractional Brownian sheet (see Fig. 1) and the subfractional Brownian sheet (see Fig.3, Fig. 5, Fig. 7, and Fig. 9). As it is possible to notice these images are similar with the pictures which one can obtain by thermal camera, say for some heated surface. Since our goal is only to augment quantity of training samples, we just suppose that minimal values of the generated process correspond to "black" pixels and maximal values corresponds to "white" pixels. Setting "red" color as a normal temperature for the heated surface, it is possible to see "overheated" areas. To make the corrupted areas more visible we apply color-based segmentation using k-means clustering (see Fig. 2, Fig. 4, Fig. 6, Fig. 8, and Fig. 10).

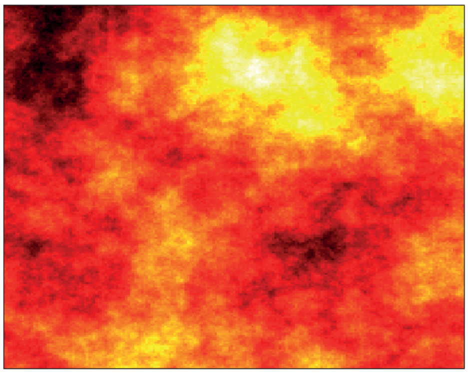

Figure 1. Test 1 - the fractional Brownian sheet with parameters $(0.75, 0.75, 1, 1, 1, 1)$

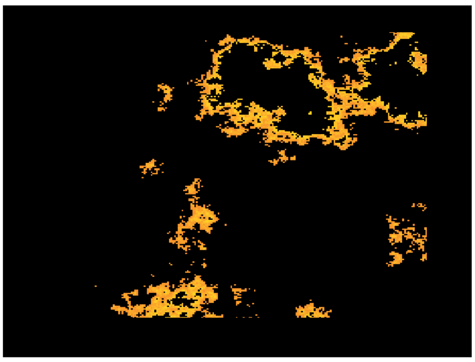

Figure 2. Segmented areas for Test 1

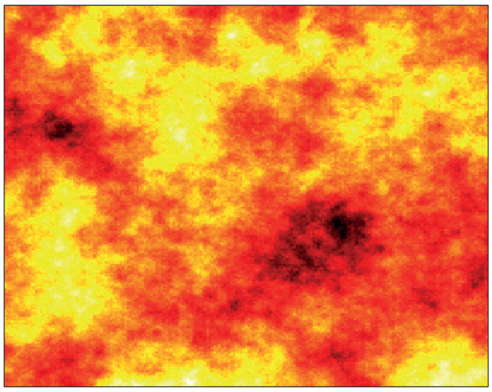

Figure 3. Test 2 – the subfractional Brownian sheet with parameters $(0.75, 0.75, 1, 0.34, 1, 0.34)$

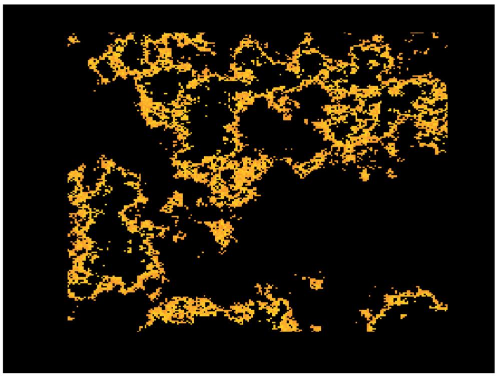

Figure 4. Segmented areas for Test 2

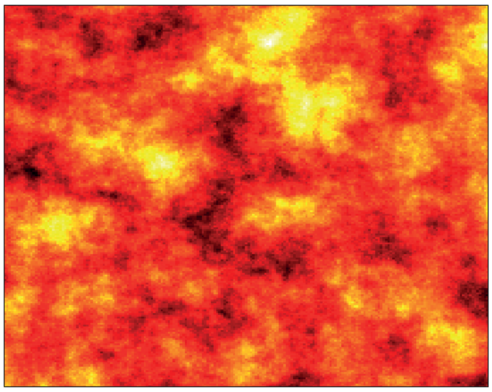

Figure 5. Test 3 – the subfractional Brownian sheet with parameters $(0.75, 0.25, 1, 0.76, 1, 0.76)$

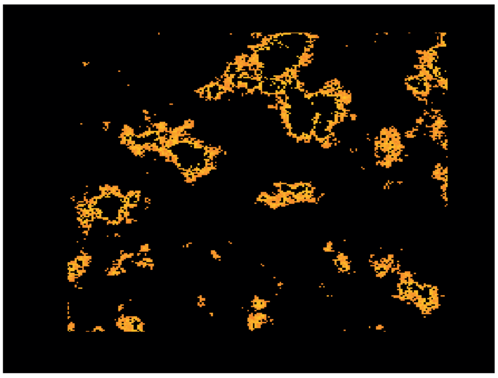

Figure 6. Segmented areas for Test 3

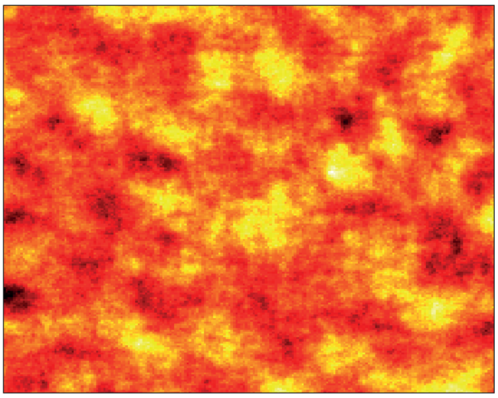

Figure 7. Test 4 – the subfractional Brownian sheet with parameters

$$(0.25, 0.75, 1, 0.76, 1, 0.76)$$

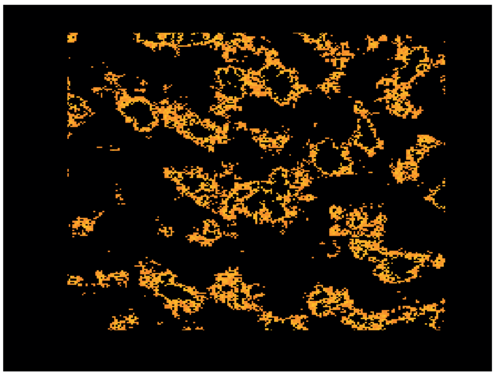

Figure 8. Segmented areas for Test 4

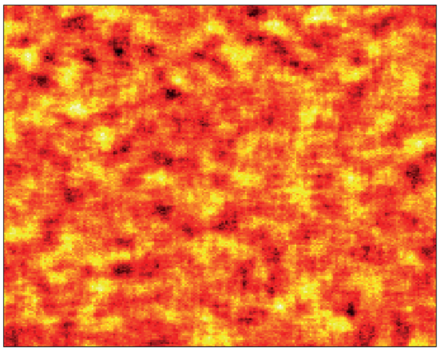

Figure 9. Test 5 – the subfractional Brownian sheet with parameters

$$(0.25, 0.25, 1, 1.67, 1, 1.67)$$

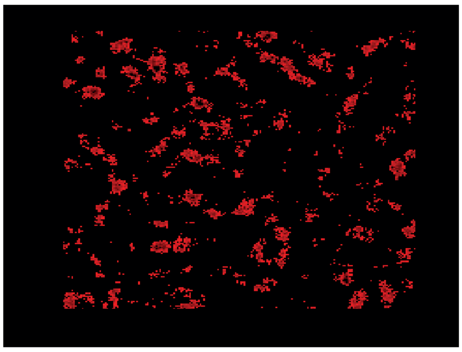

Figure 10. Segmented areas for Test 5

It is obvious that only by changing the parameters of the stochastic process we get different corruption processes for the surface. Moreover, any repetition of the generation even with the same parameters gives new image preserving the main tendency of the corruption process.

5. CONCLUDING REMARKS

We have completed previous results by proving that the gbBm is an element of the QHASI class with no condition on the parameters. When $2HK > 1$, the constant C_1 has been determined. Then we have proposed a construction of several Gaussian sheets based on the QHASI class. We have studied the main properties of these sheets such that the self-similarity one, the quasi-helix one and the approximately stationary one. The QHASI

class is therefore extended to two dimensional processes. The associated constants are determined. We have also focused our attention on new specific sheets, the well-known fractional Brownian one becoming a particular case.
We insist on the fact that a natural extension can be done for three dimensional processes. In this case, the increment Δ of X between the points

$$u_{111} = (x_1, y_1, z_1)$$

and

$$u_{222} = (x_2, y_2, z_2)$$

is defined as follows

$$\begin{aligned}\Delta = X\left(u_{222}\right) + X\left(u_{112}\right) + X\left(u_{121}\right) + X\left(u_{211}\right) \\ -X\left(u_{122}\right) - X\left(u_{212}\right) - X\left(u_{221}\right) - X\left(u_{111}\right),\end{aligned}$$

where

$$u_{ijk} = (x_i, y_j, z_k), 1 \le i, j, k \le 2$$

are points in R^3. We can also determine the seven associated constants: the first three ones deal with self-similarity whereas the last ones deal with the constants C_i, $1 \le i \le 4$. Following the same lines, we can build n dimensional processes. However, the increment Δ has no simple expression. This is why we omit this extension.
The numerical illustrations were shown for the Gaussian sheets. This generalized presentation of the class of stochastic processes was used to augment the training samples for generative adversarial networks in computer vision problem. The same approach can be used in R^3, which permits solve many applied problems devoted to default diagnostics by computer vision.

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