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THE EFFECT OF THE AXIAL AND SHEAR STIFFNESSES ON ELASTIC ROD'S STABILITY

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Abstract. This article is about the nonlinear problems of the theory of elastic Cosserat – Timoshenko's rods in the material (Lagrangian) description. The variational definition for the problem as finding the stationary point of the Lagrangian functional and differential formulation of static problems were given. The exact stability functional and stability equations of the plane problem for physically linear elastic rods taking into account the axial, shear and bending stiffnesses were received. The exact value of the critical load was obtained taking into account the axial, shear and bending deformations in the problem of the stability of a rod compressed by an axial force. In the present paper the stability of classical simplified rod's models such as the Timoshenko beam and the Euler–Bernoulli beam was investigated. Also, the stability of third simplified rod's model, based on beam's axial and bending stiffnesses, was explored. The stability functionals, the stability equations and critical loads formulatio ns for this three types of simplified models were derived as a particular case of the general theory. There were made the comparisons of described solutions which regards all the rod's stiffnesses and solutions, based on simplified models. The effect of the axial and shear stiffnesses on rod's stability was analyzed.

Keywords: stability of structures; variational formulation; the stability functional; the stability equations; the critical load

ВОЗДЕЙСТВИЕ ЖЕСТКОСТЕЙ НА РАСТЯЖЕНИЕ – СЖАТИЕ И СДВИГ НА УСТОЙЧИВОСТЬ УПРУГИХ СТЕРЖНЕЙ

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Аннотация. Данная статья посвящена нелинейным задачам теории упругих стержней Коссера – Тимошенко в материальном (Лагранжевом) описании. Приведено вариационное определение задачи в виде поиска точки стационарности функционала типа Лагранжа и дифференциальные постановки статических задач. Получены точные функционал устойчивости и уравнения устойчивости плоской задачи для физически линейных упругих стержней с учетом продольной, сдвиговой и изгибной жесткостей. Точное значение критической нагрузки получено с учетом продольных, сдвиговых и изгибающих деформаций в задаче устойчивости стержня, сжатого осевой силой. В настоящей работе исследована устойчивость классических упрощенных стержневых моделей, таких как балка Тимошенко и балка Бернулли - Эйлера. Также была исследована устойчивость третьей упрощенной модели стержня, основанной на учете продольной и изгибной жесткостей балки. Функционалы устойчивости, уравнения устойчивости и формулы критических сил для этих трех типов упрощенных моделей были выведены в качестве частного случая общей теории. Проведены сравнения описанных решений с учетом всех жесткостей стержня и решений, основанных на упрощенных моделях. Проанализировано влияние осевой и сдвиговой жесткости на устойчивость стержня.

Ключевые слова: устойчивость конструкций; вариационная формулировка; функционал устойчивости; уравнения устойчивости; критическая нагрузка

1. INTRODUCTION

The desire to reduce the material consumption of structures leads to the use of more flexible structural elements in modern construction. This increases the possibility of loss of stability of these elements. Traditional assessment methods of the rod's stability based on the classical Euler's formula, give only approximate values of critical loads for compliant elements. This is due to the fact that only the bending stiffness of the rods is taken into account in the Euler's formula. In this paper, we obtain exact solutions to the stability problems of rods that take into account, in addition to bending stiffness, also axial and shear stiffnesses.

The apparatus of the classical variational calculus is used to solve this problem. The traditional approach to the variational formulation of the problem of rod's nonlinear deformation is to use the variational equation in the form of the principle of possible displacements [1-18]. In this paper, it is shown that the variational problem can be formulated as a problem of finding the stationarity point of a Lagrange-type functional, using energetically conjugate vectors of forces and deformations [19]. In this case, it becomes possible to obtain exact stability equations as the Euler's equations for the second variation of the Lagrangian functional for the first time. From the exact stability functional and stability equations, it is possible as a consequence to obtain an approximate stability functional and stability equations, in which only bending stiffness (the Euler–Bernoulli beam) or only bending and shear stiffnesses (the Timoshenko beam) or only bending and axial stiffnesses (the Euler–Bernoulli beam taking into account the axial stiffness) are considered.

2. VARIATIONAL FORMULATIONS OF THE NONLINEAR STABILITY PROBLEMS OF ELASTIC RODS

Formulation of the geometrically nonlinear problem for the physically linear rod consists of three groups of differential equations: equilibrium equations, geometrical equations and physical equations.

Equilibrium equations for the plane problem are:

$$\begin{cases} (N\cos\varphi - Q\sin\varphi)' + q_x = 0; \\ (N\sin\varphi + Q\cos\varphi)' + q_y = 0; \\ M' + x'(N\sin\varphi + Q\cos\varphi) + \\ +y'(Q\sin\varphi - N\cos\varphi) + m = 0, \end{cases} \tag{1}$$

where N is axial force; Q is shear force; M is bending moment; q_x, q_y and m are distributed power and moment loads respectively. Functions $x(s)$, $y(s)$, and $\varphi(s)$ are three degrees of freedom in the plane problems of the geometrically nonlinear deformation of the rod. In the reference unstressed configuration every point of the rod can be identified by the s coordinate, where $0 \le s \le L$, L is length of the unstrained rod. (...)' denote derivative with respect to s.

The components of axial, shear and bending deformations ε, γ, ψ are defined through the functions $x(s)$, $y(s)$, and $\varphi(s)$ by geometrical equation:

$$\begin{cases} \varepsilon = x'\cos\varphi + y'\sin\varphi - 1; \\ \gamma = -x'\sin\varphi + y'\cos\varphi; \\ \psi = \varphi'. \end{cases} \tag{2}$$

Physical equations for the linear elastic material are:

$$N = k_1\varepsilon;\ Q = k_2\gamma;\ M = k_3\psi, \tag{3}$$

where k_1 = EA is axial stiffness; k_2 = GAk is shear stiffness; k_3 = EI is bending stiffness; E is Young's modulus; A is cross-section area of the rod; G is shear modulus; k is cross-section form coefficient; I is moment of inertia.

The Lagrange functional can be written in the following way:

$$L(x, y, \varphi) = \int_0^L \left[\frac{1}{2}\left(k_1\varepsilon^2 + k_2\gamma^2 + k_3\psi^2 - \right.\right. \\ \left.\left. -q_x(x - s) - q_y y - m\varphi\right)\right] ds - F_1(x(L) - \\ -L) - F_2 y(L) - M_1\varphi(L), \tag{4}$$

where: F_1 is "dead" load parallel to the X axis; F_2 is "dead" load, parallel to the Y axis; M_1 - external moment applied at the end of the rod at s = L.

In [20 - 23], it was proved that the differential formulation of the problem (1)-(3) is equivalent to the L→ STAT variational problem of the search of the stationary point of functional (4).

The stability functional of the plane problem for physically linear elastic rods taking into account the axial, shear and bending stiffnesses, resulting from the second variation the Lagrange functional, can be written in the following way:

$$\Phi_{\text{ст}}(u,\upsilon,\theta) = \frac{1}{2}\int_0^L \big[N_B\varepsilon_B + N\theta\big(2\gamma_B + \theta(\varepsilon+1)\big) + Q_B\gamma_B + Q(\theta\gamma - 2\varepsilon_B) + M_B\psi_B\big]\, ds, \qquad (5)$$

where the following notation is used:

$$\begin{gathered} N_{\text{в}} = k_1\varepsilon_{\text{в}}; \quad Q_{\text{в}} = k_2\gamma_{\text{в}}; \quad M_{\text{в}} = k_3\psi_{\text{в}}; \\ \varepsilon_{\text{в}} = u'\cos\varphi - x'\theta\sin\varphi + \upsilon'\sin\varphi + y'\theta\cos\varphi; \\ \gamma_{\text{в}} = -u'\sin\varphi - x'\theta\cos\varphi + \upsilon'\cos\varphi - y'\theta\sin\varphi; \\ \psi_{\text{в}} = \theta'. \end{gathered} \qquad (6)$$

The quantities x, y, φ, ε, γ, ψ, N, Q, M denote the equilibrium state characteristics, satisfying the system of equations (1) - (3), as well as boundary conditions. These quantities are characteristics of the equilibrium state, whose stability is studied. The quantities with the subscript "B" are denoted variations; $u(s)$, $\upsilon(s)$, and $\theta(s)$ variations of coordinates x, y and angle of rotation φ, respectively.

The stability equations are the Euler's equations for the variational problem $\Phi_{\text{ст}}$→ STAT. Euler's equations resulting from the condition $\delta\Phi_{\text{ст}} = 0$ are the further equations:

$$\begin{cases} (N_{\text{в}}\cos\varphi - Q_{\text{в}}\sin\varphi)' - \big(\theta(N\sin\varphi + Q\cos\varphi)\big)' = 0; \\ (N_{\text{в}}\sin\varphi + Q_{\text{в}}\cos\varphi)' + \big(\theta(N\cos\varphi - Q\sin\varphi)\big)' = 0; \\ M_{\text{в}}' + u'(N\sin\varphi + Q\cos\varphi) + \upsilon'(Q\sin\varphi - N\cos\varphi) + x'(N_{\text{в}}\sin\varphi + Q_{\text{в}}\cos\varphi + \theta(N\cos\varphi - Q\sin\varphi)) + y'(Q_{\text{в}}\sin\varphi - N_{\text{в}}\cos\varphi + \theta(N\sin\varphi + Q\cos\varphi)) = 0. \end{cases} \qquad (7)$$

System (7) is a system of equations for the functions u, υ, and θ. Functions x, y and φ, as well as N, Q, M are fixed and are solutions to problem (1) - (3).

Equations (7) are the exact equations of the problem of the equilibrium state of the rod for the case of the plane problem. We would like to stress that the derived system of the stability equations is exact. No simplifying assumptions were made about the displacement and rotation angles quantity, and the character of the equilibrium state of the rod. The resulting functional (5) and equations (7) are written in general terms and are applicable for any type of load and boundary conditions.

In [20 - 23], stability equations (7) were also obtained by the second way like the equations in variations of the equilibrium equations (1).

The classic Euler problem (hinged rod under the axial potential dead load shown in Figure 1) is considered as an example. The equilibrium configuration is rectilinear.

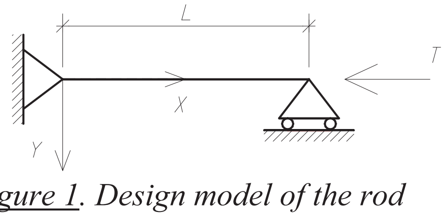

Figure 1. Design model of the rod

Boundary values for this example are:

$$\begin{gathered} s = 0:\ x(0) = 0;\ y(0) = 0;\ M(0) = 0; \\ s = \text{L}:\ (N\cos\varphi - Q\sin\varphi)|_{s=L} + T;\ y(\text{L}) = 0;\ M(\text{L}) = 0. \end{gathered} \qquad (8)$$

The exact solution of the nonlinear problem (1) - (3) and (8) is described by the formulas:

$$y = 0; \quad \varphi = 0; \quad \varepsilon = \frac{N}{k_1} = -\frac{T}{k_1};$$
$$\gamma = 0; \quad \psi = 0;$$
$$N = -T; \quad Q = 0; \quad M = 0. \tag{9}$$
$$x' = \varepsilon + 1 = \frac{N}{k_1} + 1 = 1 - \frac{T}{k_1}.$$

Substitute equations (9) into the stability functional (5) using expressions (6). The functional components containing u describes the axial deformations, not associated with the load T, and can be omitted when studying stability. Finally, the stability functional for this example can be written as:

$$\Phi_{\text{ст}}(v,\theta) = \frac{1}{2}\int_0^L \left[k_2\left(v' - \theta\left(1 - \frac{T}{k_1}\right)\right)^2 + \right.$$
$$\left. + k_3\theta'^2 + T\theta\left(\left(1 - \frac{T}{k_1}\right)\theta - 2v'\right)\right] ds. \tag{10}$$

The stability equations, which follow from the condition $\delta\Phi_{\text{ст}} = 0$, have the form:

$$\begin{cases} k_2\left(v' - \left(1 - \frac{T}{k_1}\right)\theta\right)' - T\theta' = 0; \\ k_3\theta'' + k_2\left(v' - \left(1 - \frac{T}{k_1}\right)\theta\right)\cdot \\ \quad \cdot\left(1 + T\left(\frac{1}{k_2} - \frac{1}{k_1}\right)\right) = 0. \end{cases} \tag{11}$$

A detailed solution to the system of equations (11) was considered in [20 - 23].

The critical (minimal) force value is calculated from the quadratic equation [20 - 23]:

$$T^2\left(\frac{1}{k_2} - \frac{1}{k_1}\right) + T - T_E = 0, \tag{12}$$

where $T_E = \frac{\pi^2 k_3}{L^2}$ – Euler's force for the hinged rod [24].

It is easy to show that the only positive value of the critical load, following from equation (12) is:

$$T_{cr} = \frac{\sqrt{1 + 4T_E\left(\frac{1}{k_2} - \frac{1}{k_1}\right)} - 1}{2\left(\frac{1}{k_2} - \frac{1}{k_1}\right)}, \tag{13}$$

Solution (13) is the exact solution of the problem of the hinged rod when axial, shear and bending stiffnesses are taken into account.

3. THE STABILITY OF SIMPLIFIED ROD'S MODELS

3.1. The Timoshenko beam

The Timoshenko beam theory is based on taking into account the effect of shear deformation on the stress and strain state of the rod. The classic Euler problem (hinged rod under the axial potential dead load shown in Figure 1) is considered as an example. When analyzing the stability of the Timoshenko beam, the assumption is made that the change in the geometric dimensions of the rod under subcritical deformations is considered negligible. For instance, the length of the rod is unchanged in the process of loading. Thus, the rod is stressed but not deformed. So:

$$N = k_1\varepsilon = -T; \ \varepsilon = 0; \ \rightarrow \ k_1 \rightarrow \infty \ \rightarrow$$
$$\rightarrow (k_1)^{-1} = 0.$$

The stability functional for the Timoshenko beam follows from the stability functional in equation (10), taking $(k_1)^{-1} = 0$:

$$\Phi_{\text{ст}}(v,\theta) = \frac{1}{2}\int_0^L [k_3\theta'^2 + k_2(v' - \theta)^2 +$$
$$+ T\theta(\theta - 2v')] ds. \tag{14}$$

The stability equations for the Timoshenko beam, arising from the stability functional in equation (14), can be written in the following way:

$$\begin{cases} k_2(v' - \theta)' - T\theta' = 0; \\ k_3\theta'' + k_2(v' - \theta)\left(1 + \frac{T}{k_2}\right) = 0. \end{cases} \tag{15}$$

The solution to the system of equations (15), which is an exact solution to the stability problem for the Timoshenko beam, taking into account the shear and bending stiffnesses, can be written in the following way:

$$T_{cr} = \frac{k_2}{2}\left(\sqrt{1 + \frac{4T_E}{k_2}} - 1\right). \quad (16)$$

To assess the effect of the axial stiffness of an initially rectilinear rod compressed by an axial dead load T, as shown in Figure 1, let us compare the values of the critical load calculated by the exact formula, according to equation (13), taking into account the axial, shear and bending stiffnesses, with values, calculated by equation (16) for the Timoshenko beam, taking into account only the shear and bending stiffnesses. Figure 2 shows the graphs for the strut, made of the I-beam, as an example.

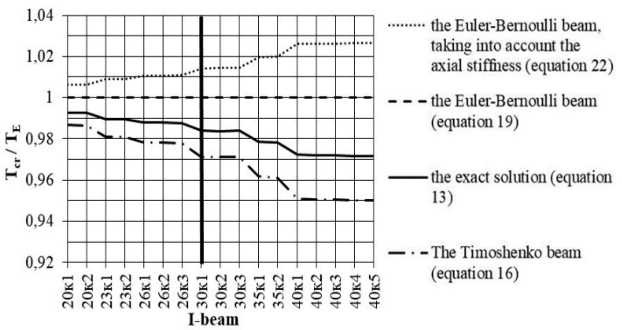

Figure 2. The effect of the axial and shear stiffnesses on values of the critical load

Based on the analysis of the results, we can conclude that the inclusion of the axial stiffness increases the critical load. Thus, we take into account the rod's internal "reserves" under the action of the "dead" axial load, adding the axial stiffness to the calculation of the rod's stability.

3.2. The Euler–Bernoulli beam

Let us consider the stability of the Euler-Bernoulli beam, which does not take into account the effect of shear deformation on the stress and strain state of the rod, as well as the hypothesis of non-deformability of the rod in the subcritical state is accepted. The classic Euler problem (hinged rod under the axial potential dead load shown in Figure 1) is considered as an example, as for the Timoshenko beam.

The stability functional for the Euler-Bernoulli beam follows from the stability functional in equation (10), taking $(k_1)^{-1} = 0; (k_2)^{-1} = 0$:

$$\Phi_{\text{ст}}(\upsilon, \theta) = \frac{1}{2}\int_0^L [k_3 \upsilon''^2 - T\upsilon'^2] ds. \quad (17)$$

Euler's equation, which follows from the condition $\delta\Phi_{\text{ст}} = 0$, can be written as:

$$k_3 \upsilon^{IV} + T\upsilon'' = 0. \quad (18)$$

The solution to the stability equation (18) is the classical common Euler formula for the critical load exclude the axial and shear stiffnesses.

$$T_E = \frac{\pi^2 k_3}{L^2}. \quad (19)$$

To assess the effect of the axial and shear stiffnesses of an initially rectilinear rod compressed by an axial dead load T, as shown in Figure 1, let us compare the values of the critical load calculated by the exact formula, according to equation (13), taking into account the axial, shear and bending stiffnesses, with values, calculated by equation (19) for the Euler-Bernoulli beam, taking into account only the bending stiffness. Figure 2 shows the graphs for the strut, made of the I-beam, as an example. For illustrative purposes, the figure 2 also shows the graph of the critical load values for the Timoshenko beam, calculated by equation (16), and the graph of the critical load values for the Euler-Bernoulli beam, taking into account axial stiffness, calculated by equation (22).

Based on the analysis of the results, we can conclude that the inclusion of the axial and shear stiffnesses significantly reduces the critical load. Thus, the use of the classical Euler formula in equation (19) leads to the risk of loss of stability by the rod even before reaching the critical load calculated by the equation (13).

3.3. The Euler–Bernoulli beam taking into account the axial stiffness

Let us consider the stability of the Euler-Bernoulli beam, which takes into account the axial stiffness, but does not take into account the effect of shear deformation on the stress and strain state of the rod. However, the hypothesis of non-deformability of the rod in the subcritical state is not accepted in contrast to the classic the Euler-Bernoulli beam. The classic Euler problem (hinged rod under the axial potential dead load shown in Figure 1) is considered as an example, as for the Timoshenko beam.

The stability functional for this simplified rod's model follows from the stability functional in equation (10), taking $(k_2)^{-1} = 0$:

$$\Phi_{\text{ст}}(v,\theta) = \frac{1}{2}\int_0^L \left[\frac{k_3 v''^2}{\left(1-\frac{T}{k_1}\right)^2} - \frac{T v'^2}{\left(1-\frac{T}{k_1}\right)}\right] ds. \quad (20)$$

Euler's equation, which follows from the condition $\delta\Phi_{\text{ст}} = 0$, can be written as:

$$\frac{k_3 v^{IV}}{\left(1-\frac{T}{k_1}\right)^2} + \frac{T v''}{\left(1-\frac{T}{k_1}\right)} = 0. \quad (21)$$

The solution to the equation (21), which is an exact solution to the stability problem for the Euler-Bernoulli beam, taking into account the axial and bending stiffnesses, can be written in the following way:

$$T_{cr} = \frac{k_1}{2}\left(1 - \sqrt{1 - \frac{4T_E}{k_1}}\right). \quad (22)$$

To assess the effect of the shear stiffnesses of an initially rectilinear rod compressed by an axial dead load T, as shown in Figure 1, let us compare the values of the critical load calculated by the exact formula, according to equation (13), taking into account the axial, shear and bending stiffnesses, with values, calculated by equation (22) for the Euler-Bernoulli beam, taking into account the axial and bending stiffnesses. Figure 2 shows the graphs for the strut, made of the I-beam, as an example. For illustrative purposes, the figure 2 also shows the graph of the critical load values for the Timoshenko beam, calculated by equation (16), and the graph of the critical load values for the Euler-Bernoulli beam, calculated by equation (19).

Based on the analysis of the results, we can conclude that the value of the critical load obtained from equation (22) is greater than the value obtained from the exact formula (13) and the value obtained from Euler's formula (19). Thus, the use of equation (22) leads to the risk of loss of the rod's stability even before reaching the critical load calculated by the equation (22). Therefore, as shown in figure 2, it is unacceptable to take into account the axial stiffness without taking into account the shear stiffness, when analyzing the rod's stability.

4. CONCLUSIONS

1. The formulations of the problems are presented in the form of a system of differential equations and variational formulations in the form of the problem of finding the stationarity point functional of the Lagrange type.
2. For the plane problems, equations of equilibrium stability problems are obtained as the Euler equations for the second variation of the Lagrange functional
3. The exact universal solution in equation (13), taking into account axial, shear and bending stiffnesses, which gives the exact value of the critical load was obtained for the problem of the stability of a rod compressed by an axial force.
4. There were made the comparisons of exact solutions which regards all the rod's stiffnesses and solutions, based on three simplified models.
5. It was shown, that considering axial stiffness leads to increasing the values of the critical load. Thus, we take into account the rod's internal "reserves" under the action of the "dead" axial load, adding the axial stiffness to the calculation of the rod's stability.
6. It was shown, that inclusion of the axial and shear stiffnesses significantly reduces the critical load. Thus, the use of the classical Euler formula leads to the risk of loss of stability by the rod even before reaching the critical load calculated by the exact equation.

7. It was shown, that taking into account the axial stiffness, without taking into account the shear stiffness, significantly increases the critical load. Therefore, it is unacceptable to take into account the axial stiffness without taking into account the shear stiffness, when analyzing the rod's stability.
8. It was shown, that the obtained exact value of the critical compressive load, taking into account all rod's stiffnesses, has a lower value than the critical load value calculated by the classical Euler's formula. Since both formulas are equally simple for manual calculation, the resulting exact formula can be recommended for use in all cases in which Euler's formula was previously used.

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