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FINITE ELEMENT ANALYSIS FOR THIN-WALLED MEMBER SUBJECTED TO COMBINED LOADING

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Abstract. Thin-walled structures are widely used in various structural engineering applications due to their advantage of high bearing strength when compared to self-weight and used in a complex loading situation where subjected to combined loadings. When a thin-walled section is subjected to a combined load with restrained torsion, they are ineffective at resisting, resulting in a reduction in beam capacity due to torsion and additional warping stresses. A finite element calculation can be used to analyze a 3D bar of thin-walled structural sections. Different commercial software and studies commonly consider six degrees of freedom at each node of a member for a space frame without considering the effect of warping restraint at the member's ends. This paper presents a finite element calculation for thin-walled sections with restrained torsion using the 14x14 member stiffness matrix, which includes warping as an additional degree of freedom and is commonly used for open thin-walled sections. In this study, we considered two different methods for including the additional degree of freedom for the stiffness matrix, which are very close to each other for small values of characteristics number.

Keywords: Thin-walled structures, Finite element analysis, non-uniform warping, open section, stiffness matrix, restrained torsion

КОНЕЧНО-ЭЛЕМЕНТНЫЙ АНАЛИЗ ТОНКОСТЕННЫХ ЭЛЕМЕНТОВ ПОД ДЕЙСТВИЕМ КОМБИНИРОВАННЫХ НАГРУЗОК

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Аннотация: Тонкостенные конструкции широко используются в различных областях проектирования конструкций благодаря своему преимуществу высокой несущей прочности по сравнению с собственным весом и используются в сложной ситуации нагрузки, когда подвергаются комбинированным нагрузкам. Когда тонкостенные секции подвергаются комбинированной нагрузке со сдержанным торсионом, они неэффективны при сопротивлении, что приводит к снижению пропускной способности балки из-за торсионных и дополнительных деформационных напряжений. Расчет конечных элементов может быть использован для анализа 3D-стержня тонкостенных структурных секций. Различные коммерческие программы и исследования обычно рассматривают шесть степеней свободы в каждом узле члена для пространственной рамки без учета эффекта деформации сдерживания на концах элемента. В данной работе представлен расчет конечных элементов для тонкостенных секций с ограниченным кручением с использованием матрицы жесткости элементов 14х14, которая включает деформацию в качестве дополнительной степени свободы и обычно используется для открытых тонкостенных секций. В данном исследовании мы рассмотрели два различных метода включения дополнительной степени свободы для матрицы жесткости, которые очень близки друг к другу для малых значений числа характеристик.

Ключевые слова: тонкостенные структуры, конечно-элементный анализ, неравномерное деформация, открытое сечение, матрица жесткости, сдержанный кручение

1. INTRODUCTION

Steel members are now manufactured as thin-wall sections because of their high strength, highly flexible, ductility, quick construction, and effective space partitioning, and they are widely used in various engineering structures. Thin-walled beams are those that are primarily prone to bending. When a thin-walled section is subjected to a combined load, it is ineffective at resisting, resulting in a reduction in the beam's capacity. The behavior is poorly described by elementary formulations that reduce the mechanical components to stretching, bending, and uniform torsion (i.e., the simplest case of a uniform distribution of cross-sectional warping along the beam axis [1-2]. Warping effects occur primarily at the points of action of concentrated torsional moments (except at free end support of beam) and at sections with free-warping restrictions, and they are accounted for by an additional degree of freedom at each nodal point in the form of the first derivative of the angle of twist of the beam's cross-section [3-5].

The analysis for extension, bending and flexure is rather straight-forward, but the analysis for the coupled deformations of torsion, warping and distortion poses a major challenge[6]. Currently, most design specifications do not provide clear guidance for combined bending and torsion design and the need exists for a simple design equation. The variation of the displacement over a section of a member is expressed with a common function for stretching, torsion and bending[7-10]. I-shaped steel beams are widely used as structural elements because of their flexural efficiency about the strong axis. It considers the cross section as completely rigid in its own plane, and the effect of shearing deformations is neglected[11]. The solutions for thin-walled section with nonuniform torsion were developed as initial works and also there are studies considered to be as a design aids for simple cases[12-13]. This is limited for a slender beam and the shear deformation in middle surface is negligible but for short-deep beam and closed thin-walled beams, the shear deformation should be considered[4,14]. However, in many applications beams are eccentrically loaded and as a result experience torsional loads in combination with bending. The importance of restrained torsion of thin-walled section has grown significantly as the deformations and stresses caused by torsion affects the behavior of the structures with open as well as closed section[15-16]. Like all open sections, I-shaped steel beams are very inefficient at resisting torsion and the interaction effects due to torsion acting in combination with bending can significantly reduce the capacity of the beam.

Many design methods have been developed to deal with combined bending and torsion, but none have been universally adopted by design standards. In the past decades, many relevant researches have been conducted and different commercial software commonly consider six degrees of freedom at each node of a member for a space frame without considering the effect of warping restraint at the ends of the member[9][17-18]. A finite element model is investigated based on a mixed variational formulation and numerical method of designing thin-walled bar systems using various theories and formulated matrices to provide an explicit way to calculate internal forces and stresses in thin-walled bar systems [19-22]. The bending and torsion behavior of cold-formed steel bars was studied experimentally based on the strengths of unbraced cold-formed steel channel beams loaded eccentrically [23-24]. Modern software packages for structural analysis use finite element types which consider up to six degrees of freedom at the structural nodes, which corresponds to the linear and angular displacements in these nodes as for the rigid bodies[25]. Moreover, various studies commonly consider with two degrees of freedom at each node of a member without considering the effect of warping restraint at node [26-27]. The warping part of the first derivative of the twist angle has been considered as the additional degree of freedom in each node at the element ends which can be regarded as part of the twist angle curvature caused by the warping moment [17][27][30]. Numerous studies developed the 14x14 member stiffness matrix including warping as an additional degree of freedom and commonly with open thin-walled section [18][25][28-29].

In this paper, a 3D frame element stiffness matrix will be presented which is more convenient for advanced structural analysis of 3D beam structures. The structures are analyzed or designed by using only the effect of Saint Venant torsion resistance thus the analysis may ignore the torsion part in the members and the design may be underestimated. To overcome this inaccuracy, several researchers tried to develop stiffness matrix with seven degrees of freedom at each node of a member for a space frame. This additional stiffness matrix considers the warping degree of freedom at the ends of the member with thin-walled section. This study deals with the Space frame finite element method regarding the first order theory based on the assumption is that the resulting deformations are small, and that the equilibrium may be formulated for the undeformed structure as an approximation. This is done by considering beam element and equation which are necessary for the computing deformations will be derived thus to calculate the displacements and internal forces and moments for frame structures.

2. METHOD

2.1. Geometry and concept of 3D thin-walled Frame

Considering Prismatic thin-walled beams of straight and of constant cross-section with y_1-axis is defined parallel to the longitudinal direction of the beam, while the y_2-axis and y_3-axis describe the transversal plane of the cross-section as shown in figure 2. The member is connected to local coordinate system and the corresponding displacement field adopted for the axial direction is v_1, while v_2 and v_3 are used for the cross-section's plane. Similarly, β_1, β_2 and β_3 are angles of rotation about the axis y_1, y_2 and y_3 and ψ is the sectional warping or twist of the section along y_1. Consider a point P with a member coordinate (y_1, y_2, y_3) in the member coordinate system. The basic assumption in the classical beam theory is that a cross-section orthogonal to the x-axis at the coordinate x remains plane and keeps its shape during deformation.

Due to the assumptions of the classical beam theory the cross-section orthogonal to the y_1-axis at the coordinate y_1 remains plane and keeps its shape during deformation and the general theory of elasticity for three-dimensional solids reduced to a special theory for space frames. The displacement of the point consists of a translation equal to that of the centroid C of section y_1 and a rotation displacement due to the rotation of the section as a rigid body about an axis through the centroid and a warping displacement normal to the section.

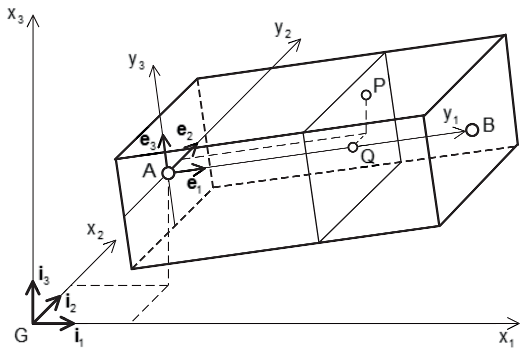

Figure 1. The orientation of coordinate systems for 3D beam section

Let S be a plane section normal to the axis of a member, which contains a point P and intersects the axis in point Q as shown in figure 1. The hypothesis for frame behavior [1] states that the shape of section S in its plane does not change under load, and that the displacement of point P is due to:

- The displacement of point Q
- A small rotation of section S about an axis passing through point Q
- A warping displacement in the y_1 direction, which is the product of a twist with a warping function

$$\begin{bmatrix} v_{1P} \\ v_{2P} \\ v_{3P} \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} + \begin{bmatrix} 0 & y_3 & -y_2 \\ -y_3 & 0 & 0 \\ y_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} + \omega \begin{bmatrix} \psi \\ 0 \\ 0 \end{bmatrix} \tag{1}$$

Where: $v_{kP}(y_1, y_2, y_3)$ the displacement coordinate of point P in the member space, $v_k(y_1)$ displacement coordinate of centroid C of section y_1, $\beta_k(y_1)$, coordinate of the rotation vector of the section, $\omega(y_2, y_3)$: warping function of center of rotation C $\psi(y_1)$ twisting of the section

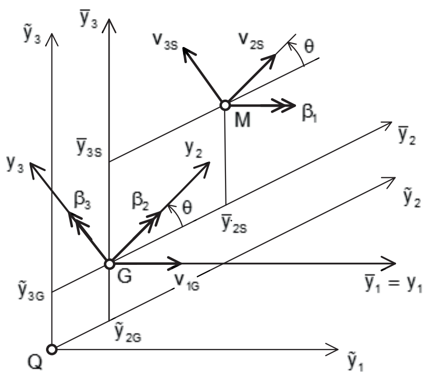

Figure 2. Beam kinematics, local and global reference systems for mass matrix

The strain coordinates are determined with the linear strain-displacement relations of the linear theory of elasticity. Because the frame hypothesis states that the shape of a section in its plane does not change, the strains are neglected.

$$\begin{aligned} \varepsilon_{11} &= v_{1P,1} = v_{1,1} + y_3\beta_{2,1} - y_2\beta_{3,1} + \omega\psi_{,1} \\ \varepsilon_{12} &= v_{1P,2} + v_{2P,1} = -\beta_3 + \omega_{,2}\psi + v_{2,1} - y_3\beta_{1,1} \\ \varepsilon_{13} &= v_{1P,3} + v_{3P,1} = \beta_2 + \omega_{,3}\psi + v_{3,1} + y_2\beta_{1,1} \end{aligned} \tag{2}$$

The expressions for the shear strains are rearranged so that the contributions of flexure, uniform torsion and torsion restraint are shown explicitly:

$$\varepsilon_{12} = \underbrace{(v_{2,1} - \beta_3)}_{\text{flexure}} - \underbrace{(y_3 + \omega_{,2})\beta_{1,1}}_{\text{uniform torsion}} + \underbrace{\omega_{,2}(\psi + \beta_{1,1})}_{\text{torsion restraint}} \tag{3}$$

$$\varepsilon_{13} = \underbrace{(v_{3,1} + \beta_2)}_{\text{flexure}} + \underbrace{(y_2 - \omega_{,3})\beta_{1,1}}_{\text{uniform torsion}} + \underbrace{\omega_{,3}(\psi + \beta_{1,1})}_{\text{torsion restraint}} \tag{4}$$

The constitutive hypothesis states that the strains due to the Poisson effect can be neglected in the analysis. For a linearly elastic material with modulus of elasticity E and shear modulus G the stress-strain may be calculated from equation (2) as follow:

$$\sigma_{11} = E\varepsilon_{11} = E(v_{1,1} + y_3\beta_{2,1} - y_2\beta_{3,1} + \omega\psi_{,1}) \tag{5}$$

$$\sigma_{12} = G\varepsilon_{12} = G(-\beta_3 + \omega_{,2}\psi + v_{2,1} - y_3\beta_{1,1}) \tag{6}$$

$$\sigma_{13} = G\varepsilon_{13} = G(\beta_2 + \omega_{,3}\psi + v_{3,1} + y_2\beta_{1,1}) \tag{7}$$

E modulus of elasticity

G *shear* modulus

The Virtual work of the inner forces δW_m done by the stresses σ_{11}, σ_{12} *and* σ_{13} of expressions (5)-(7), in the volume V of a member with length a, and area A due to virtual strains $\delta\varepsilon_{11}$, $\delta\varepsilon_{12}$ and $\delta\varepsilon_{13}$ is given by:

$$\delta W_m = \int_V \delta\varepsilon^T \sigma \, dv \tag{8}$$

Where: ε is state of strain vector (Voigt notation), σ state of stress vector (Voigt notation). The integrals of the products of the stress components with the geometric variables with the geometric quantities y_1, y_2, y_3 *and* ω over the area of the member are called stress resultants in the member and denoted as follows:

$$\begin{aligned} &\text{axial force in direction } y_1 && n_1 = \int_A \sigma_{11}\, dA \\ &\text{transverse force in direction } y_2 && n_2 = \int_A \sigma_{12}\, dA \\ &\text{transverse force in direction } y_3 && n_3 = \int_A \sigma_{13}\, dA \\ &\text{bending moment about axis } y_2 && m_2 = \int_A \sigma_{11} y_3\, dA \\ &\text{bending moment about axis } y_3 && m_3 = \int_A -\sigma_{11} y_2\, dA \\ &\text{bimoment due to warping} && m_\omega = \int_A \sigma_{11}\omega\, dA \\ &\text{primary torsion} && \\ &m_{Tp} = \int_A \left(\sigma_{13}(y_2 - \omega_{,3}) - \sigma_{12}(y_3 + \omega_{,2})\right) dA\ ) && \\ &\text{secondary torsion} && \\ &m_{Ts} = \int_A (\sigma_{13}\,\omega_{,3} + \sigma_{12}\,\omega_{,2})\, dA && \end{aligned} \tag{9}$$

The stress resultants acting on the positive face of a section are positive if they act in the positive direction of the axes of the member coordinate system. The stress resultants in the member of expression (9) are substituted into expression (8) and can be rewritten as follows:

$$\int_V \delta\boldsymbol{\varepsilon}^T \boldsymbol{\sigma}\, dv = \int_0^a \begin{pmatrix} n_1\,\delta v_1 + n_2(\delta v_{2,1} - \delta\beta_3) + \\ n_3(\delta v_{3,1} + \delta\beta_2) + \\ m_2\,\delta\beta_{2,1} + m_3\,\delta\beta_{3,1} + m_\omega\,\delta\psi_{,1} + \\ m_{Tp}\,\delta\beta_{1,1} + m_{Ts}(\delta\psi + \delta\beta_{1,1}) \end{pmatrix} dy_1 \quad (10)$$

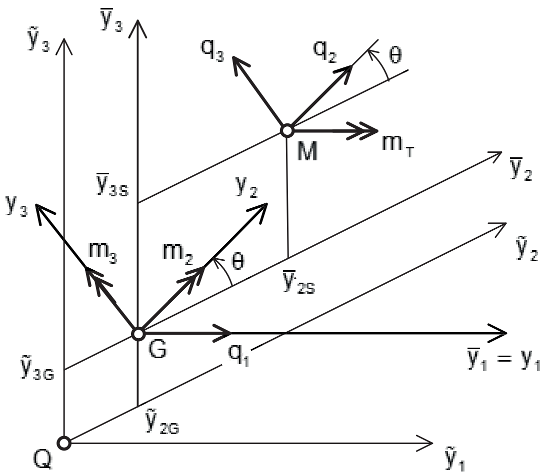

Figure 3. Local reference system and internal forces

The loads acting of the volume and the surface of the member in the theory of elasticity are replaced by line loads acting at axis y_1 and by nodal forces acting at the nodes of the member, as shown in figure 3. The nodal forces acting at the end node are equal to the stress resultants defined based on equation (8). The virtual work of the nodal forces due to variations δv_k of the displacement coordinates and $\delta\beta_k$ of the rotation coordinates is given by:

$$\begin{aligned} \delta W_n = {} & m_{TpB}\,\delta\beta_{1B} - m_{TpA}\,\delta\beta_{1A} + \\ & m_{TsB}(\delta\psi_B + \delta\beta_{1B,1}) - m_{TsA}(\delta\psi_A + \delta\beta_{1A,1}) + \\ & m_{2B}\,\delta\beta_{2B} - m_{2A}\,\delta\beta_{2A} + m_{3B}\,\delta\beta_{3B} - \\ & m_{3A}\,\delta\beta_{3A} + m_{\omega B}\,\delta\psi_{B,1} - m_{\omega A}\,\delta\psi_{A,1} + \\ & \sum_{k=1}^{3}(n_{kB}\,\delta v_{kB} - n_{kA}\,\delta v_{kA}) \end{aligned} \quad (11)$$

Where δW_n is virtual work of the nodal forces

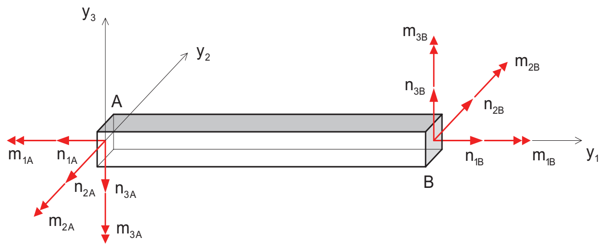

Figure 4. Positive directions of the member and nodal force coordinates

The virtual work of the inner forces in the volume of a member is expressed in terms of the strains and the virtual strains:

$$\delta W_m = \int_V (E\,\varepsilon_{11}\,\delta\varepsilon_{11} + G\,\varepsilon_{12}\,\delta\varepsilon_{12} + G\,\varepsilon_{13}\,\delta\varepsilon_{13})\, dV \quad (12)$$

Expressions (5) to (7) for the strains and the Prandtl stress function for $\beta_{2,1}$ *and* $\beta_{3,1}$ are substituted:

$$\begin{aligned} \delta W_m &= E\int_0^a\int_A h_1 h_2\, dA\, dy_1 + G\int_0^a\int_A h_3\, dA\, dy_1 \\ h_1 &= \delta v_{1,1} - y_2\delta v_{2,11} - y_3\delta v_{3,11} + \omega\,\delta\psi_{1,1} \\ h_2 &= v_{1,1} - y_2 v_{2,11} - y_3 v_{3,11} + \omega\,\psi_{1,1} \\ h_3 &= \left((y_2 - \omega_{,3})^2 + (y_3 + \omega_{,2})^2\right)\delta\beta_{1,1}\,\beta_{1,1} \end{aligned} \quad (13)$$

The integrals of functions of the coordinates and the warping function in (13) are called the shape parameters of the section or matrix section properties. To define the shape functions, we used a variable F for designations. They are defined and denoted as follows:

$$\delta W_m = E\int_0^a \delta k^T F\, k\, dy_1 + G\int_0^a J_T\,\delta\beta_{1,1}\,\beta_{1,1}\, dy_1$$

$$F = \begin{array}{|c|c|c|c|} \hline F_1 & F_2 & F_3 & F_\omega \\ \hline F_2 & F_{22} & F_{23} & F_{2\omega} \\ \hline F_3 & F_{32} & F_{33} & F_{3\omega} \\ \hline F_\omega & F_{2\omega} & F_{3\omega} & F_{\omega\omega} \\ \hline \end{array} \quad (14)$$

Where the section constants are expresses as given below:

$$F_1 = \int_A 1\, dA$$
$$F_2 = \int_A y_2\, dA \qquad F_3 = \int_A y_3\, dA$$
$$F_\omega = \int_A \omega\, dA$$
$$F_{22} = \int_A y_2^2\, dA \qquad F_{23} = \int_A y_2 y_3\, dA$$
$$F_{33} = \int_A y_3^2\, dA \qquad F_{2\omega} = \int_A y_2\, \omega\, dA$$
$$F_{3\omega} = \int_A y_3\, \omega\, dA \qquad F_{\omega\omega} = \int_A \omega^2\, dA$$
$$J_T = \int_A \left((y_2 - \omega_{,3})^2 + (y_3 + \omega_{,2})^2 \right) dA$$

The virtual work of the nodal forces due to variations of the displacement coordinates and of the rotation coordinates is given by:

$$\begin{aligned} \delta W_n \;=\;& m_{TpB}\,\delta\beta_{1B} - m_{TpA}\,\delta\beta_{1A} + \\ & m_{TsB}(\delta\psi_B + \delta\beta_{1B,1}) - \\ & m_{TsA}(\delta\psi_A + \delta\beta_{1A,1}) + \\ & m_{2B}\,\delta\beta_{2B} - m_{2A}\,\delta\beta_{2A} + m_{3B}\,\delta\beta_{3B} - \\ & m_{3A}\,\delta\beta_{3A} + m_{\omega B}\,\delta\psi_{B,1} - m_{\omega A}\,\delta\psi_{A,1} + \\ & \sum_{k=1}^{3}(n_{kB}\,\delta v_{kB} - n_{kA}\,\delta v_{kA}) \end{aligned} \qquad (15)$$

δW_n virtual work of the nodal forces

The virtual work of member loads due to variations δv_k of the displacement coordinates and $\delta\beta_k$ of the rotation coordinates is given by:

$$\delta W_q = \int_0^a \left(t_\omega \psi + \sum_{i=1}^{3} (q_i\, \delta v_i + t_i\, \delta\beta_i) \right) dy_1 \qquad (16)$$

q_i distributed force load in the direction of axis i
t_i distributed moment load in the direction of axis i
t_ω distributed bimoment load

2.2. Governing equations for 3D thin-walled frames

The governing equations for a member and frame are derived by applying the principle of virtual work to the frame. The sum over the members of the virtual work δW_m of the inner forces in (14) equals the sum over the members of the virtual work δW_{md} of the member loads.

The differential governing equations for the generalized member displacements are satisfied for arbitrary virtual displacements and expressed as follows:

$$\begin{aligned} E A v_{1,1} + q_1 &= 0 \\ E J_3 v_{2,1111} - q_2 + m_{3,1} &= 0 \\ E J_2 v_{3,1111} - q_3 - m_{2,1} &= 0 \\ E J_\omega \beta_{1,1111} - G J_T \beta_{1,11} - m_1 - m_{\omega,1} &= 0 \end{aligned} \qquad (17)$$

Similarly for frames, The sum over the members of the virtual work δW_m of the inner forces in (14) equals the sum over the members of the virtual work δW_{md} of the member loads and the virtual work δW_n of the nodal loads:

$$\sum_{m=1}^{M} \delta W_m = \sum_{m=1}^{M} \delta W_{md} + \delta W_n \qquad (18)$$

3. RESULT AND DISCUSSION

3.1. Element *stiffness matrix for a combined load:*

Stiffness matrix as it is known, the relationship between the generalized force vector q_m and the generalized displacement vector v_m is established by the stiffness matrix K_m of the element.

$$q_m = K_m\, v_m \qquad (19)$$

The displacement variation over the length of a member is related to the nodal displacements by solving the differential equations the differential governing equations for the generalized member displacements such that the values of the displacement functions at the nodes equal the unknown nodal displacement values. For non-uniform torsion, a trigonometric interpolation of rotation β_1 is used as an initial parameter and finally compared with the approximation solution.

$$v_1 = \mathbf{h}_1^T \mathbf{v}_1 \qquad 0 \le z \le 1$$

$$\mathbf{h}_1 = \begin{bmatrix} 1-z \\ z \end{bmatrix} \qquad \mathbf{v}_1 = \begin{bmatrix} v_{1A} \\ v_{1B} \end{bmatrix}$$

$$v_2 = \mathbf{h}_2^T \mathbf{v}_2$$

$$\mathbf{h}_2 = \begin{bmatrix} 1-3z^2+2z^3 \\ a\,z(1-z)^2 \\ z^2(3-2z) \\ -a\,z^2(1-z) \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} v_{2A} \\ \beta_{3A} \\ v_{2B} \\ \beta_{3B} \end{bmatrix}$$

$$v_3 = \mathbf{h}_3^T \mathbf{v}_3$$

$$\mathbf{h}_3 = \begin{bmatrix} 1-3z^2+2z^3 \\ -a\,z(1-z)^2 \\ z^2(3-2z) \\ a\,z^2(1-z) \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} v_{3A} \\ \beta_{2A} \\ v_{3B} \\ \beta_{2B} \end{bmatrix}$$

To consider the warping of the restrained member, additional degrees of freedoms are introduced at the nodes and added to member displacement vector. An interpolation function containing hyperbolic functions of y_1, which satisfies the governing differential equation (16) for torsion considered:

$$\beta_1(y_1) = g(y_1)^T b$$

$$g^T = \begin{bmatrix} g_1(y_1) & g_2(y_1) & g_3(y_1) & g_4(y_1) \end{bmatrix} \tag{20}$$

$$b^T = \begin{bmatrix} \beta_{1A} & \beta_{1,1A} & \beta_{1B} & \beta_{1,1B} \end{bmatrix}$$

$$\beta_1 = h_\omega^T C$$

$$h_\omega^T = \begin{bmatrix} \sinh\theta z \\ \cosh\theta z \\ z \\ 1 \end{bmatrix} \qquad C = \begin{bmatrix} C_1 & C_2 & C_3 & C_4 \end{bmatrix}$$

The derivatives in the integrand on the left-hand side of equation (20) are formed:

$$v_{1,1} = g_1^T v_1 \qquad v_{2,11} = g_2^T v_2 \qquad 0 \le z \le 1$$

$$g_1 = \frac{1}{a}\begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad g_2 = \frac{1}{a^2}\begin{bmatrix} 12z-6 \\ a(6z-4) \\ -(12z-6) \\ a(6z-2) \end{bmatrix}$$

$$v_{3,11} = \mathbf{g}_3^T \mathbf{v}_3$$

$$\mathbf{g}_3 = \frac{1}{a^2}\begin{bmatrix} 12z-6 \\ -a(6z-4) \\ -(12z-6) \\ -a(6z-2) \end{bmatrix}$$

The interpolation functions are substituted into the left-hand side of (18) and the integration over the length of the member is performed for axial and bending loads but separately considered for torsion as it developed based on the two different methods.

$$EA\int_0^a \delta v_1 v_{1,1}\,dy_1 = \delta v_1^T K_1 v_1$$

$$EJ_3\int_0^a \delta v_2 v_{2,11}\,dy_1 = \delta v_2^T K_2 v_2 \qquad EJ_2\int_0^a \delta v_3 v_{3,11}\,dy_1 = \delta v_3^T K_3 v_3$$

$$K_1 = \frac{EA}{a}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} k_1 & k_2 \\ k_2 & k_1 \end{bmatrix}$$

$$K_2 = \frac{EJ_2}{a^3}\begin{bmatrix} 12 & 6a & -12 & 6a \\ 6a & 4a^2 & -6a & 2a^2 \\ -12 & -6a & 12 & -6a \\ 6a & 2a^2 & -6a & 4a^2 \end{bmatrix} = \begin{bmatrix} k_3 & k_4 & k_6 & k_4 \\ k_4 & k_5 & k_7 & k_8 \\ k_6 & k_7 & k_3 & k_7 \\ k_4 & k_8 & k_7 & k_5 \end{bmatrix}$$

$$K_3 = \frac{EJ_3}{a^3}\begin{bmatrix} 12 & -6a & -12 & -6a \\ -6a & 4a^2 & 6a & 2a^2 \\ -12 & 6a & 12 & 6a \\ -6a & 2a^2 & 6a & 4a^2 \end{bmatrix} = \begin{bmatrix} k_9 & k_{10} & k_{12} & k_{10} \\ k_{10} & k_{11} & k_{13} & k_{14} \\ k_{12} & k_{13} & k_9 & k_{13} \\ k_{10} & k_{14} & k_{13} & k_{11} \end{bmatrix}$$

The contribution of torsion to the internal virtual work of the governing differential equation (16) is given as the following expressions:

$$\int_0^a (EC_\omega \delta\beta_{1,11}\beta_{1,11} + GJ_T \delta\beta_{1,1}\beta_{1,1})\,dA = \delta b^T (K_{\omega 1} + K_{\omega 2})\, b$$

$K_{\omega 1}$ warping stiffness matrix
$K_{\omega 2}$ stiffness matrix for torsion with out warping restraint

Stiffness matrices $K_{\omega 1}$ and $K_{\omega 2}$ are added to the member stiffness matrix K_m in the usual manner.

$$K_T = \frac{EC_\omega}{a^3}\begin{bmatrix} k_{T1} & k_{T2} & k_{T3} & k_{T4} \\ k_{T2} & k_{T6} & k_{T7} & k_{T8} \\ k_{T3} & k_{T7} & k_{T11} & k_{T12} \\ k_{T4} & k_{T8} & k_{T12} & k_{T16} \end{bmatrix}$$

$$K_{T1} = K_{T11} = S*\theta\sinh\theta,$$

$$K_{T6} = K_{T16} = S*(\cosh\theta - \frac{\sinh\theta}{\theta})*a^2$$

$$K_{T2} = K_{T4} = S*(\cosh\theta - 1)*a,$$

$$K_{T8} = S*\left(\frac{\sinh\theta}{\theta} - 1\right)*a^2$$

$$S = \left(\frac{\theta^2}{Q}\right), \quad Q = 2\left(1-\cosh\theta\right)+\theta\sinh\theta,$$

$$K_{T3} = -K_{T1}, \quad K_{T7} = K_{T12} = -K_{T2}$$

The above element stiffness matrix for torsion with restrain warping can be used by divided into two matrices. The parameters K_{T1}, K_{T2}, K_{T6} and K_{T8} can be replace by approximation as shown below:

$$K_{T1a} = 12 + \frac{6}{5}*\theta^2 \qquad K_{T2a} = 6 + \frac{1}{10}*\theta^2$$

$$K_{T6a} = 4 + \frac{2}{15}*\theta^2 \qquad K_{T8a} = 2 - \frac{1}{30}*\theta^2$$

Considering the above series expressions, the alternative matrices can express as shown below:

$$K_{Ta} = \frac{EC_\omega}{a^3}\begin{bmatrix} 12 & -6a & -12 & 6a \\ 6a & 4a^2 & 6a & 2a^2 \\ -12 & -6a & 12 & 6a \\ 6a & 2a^2 & 6a & 4a^2 \end{bmatrix} + \frac{GJ}{30a}\begin{bmatrix} 36 & -3a & -36 & -3a \\ -3a & 4a^2 & 3a & -a^2 \\ -3a & 3a & 36 & 3a \\ -3a & -a^2 & 3a & 4a^2 \end{bmatrix}$$

Comparing both methods, we can conclude that both are similar for small value of θ and which is commonly considered for open thin-walled section as their value of θ is small as shown in figure 5.

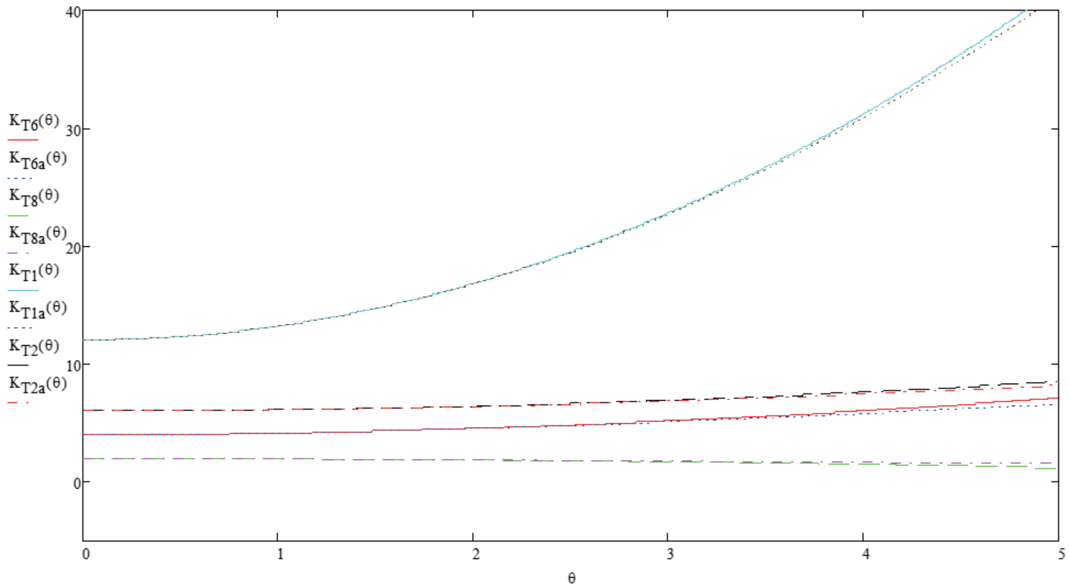

Figure 5. Evaluation of exact and approximate methods for various values of θ

If the member is free to warp, $C_w = 0$ and the torsional moment is carried by *St Venant's* torsion which is considered as uniform torsion.

Considering Expression 21, only the second part of the matrix or the uniform torsion stiffness matrix can be used as given below.

$$\mathbf{K}_T = \frac{G J_T}{a} \begin{array}{|c|c|}\hline 1 & -1 \\ \hline -1 & 1 \\ \hline \end{array} \quad \text{or}$$

$$\mathbf{K}_T = \frac{G J}{30a} \begin{array}{|c|c|c|c|}\hline 36 & -3a & -36 & -3a \\ \hline -3a & 4a^2 & 3a & -a^2 \\ \hline -3a & 3a & 36 & 3a \\ \hline -3a & -a^2 & 3a & 4a^2 \\ \hline \end{array}$$

Element load vector: Consider loads which vary linearly from start node A to end node B:

$$q_k = \mathbf{h}_1^T \mathbf{q}_k$$

$$m_k = \mathbf{h}_1^T \mathbf{m}_k \qquad \mathbf{q}_k = \begin{array}{|c|}\hline q_{kA} \\ \hline q_{kB} \\ \hline \end{array} \qquad \mathbf{m}_k = \begin{array}{|c|}\hline m_{kA} \\ \hline m_{kB} \\ \hline \end{array}$$

These loads and the interpolation functions are substituted into the right-hand side of equation (17). The integration over the length of the member is shown for q_1 *and* q_3.

$$\int_0^a q_1 \delta v_1 \, dy_1 = \int_0^a \delta \mathbf{v}_1^T \mathbf{h}_1 \mathbf{h}_1^T \mathbf{q}_1 \, dy_1 = \delta \mathbf{v}_1^T \mathbf{B}_1 \mathbf{q}_1$$

$$\int_0^a q_3 \delta v_3 \, dy_1 = \int_0^a \delta \mathbf{v}_3^T \mathbf{h}_3 \mathbf{h}_3^T \mathbf{q}_3 \, dy_1 = \delta \mathbf{v}_3^T \mathbf{B}_3 \mathbf{q}_3$$

$$\mathrm{B}_1 \quad \frac{a}{6} \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 1 & 2 \\ \hline \end{array} = \begin{array}{|c|c|}\hline b_1 & b_2 \\ \hline b_2 & b_1 \\ \hline \end{array}$$

$$\mathrm{B}_3 = \frac{a}{60} \begin{array}{|c|c|}\hline 21 & 9 \\ \hline -3a & -2a \\ \hline 9 & 21 \\ \hline 2a & 3a \\ \hline \end{array} = \begin{array}{|c|c|}\hline b_3 & b_4 \\ \hline b_5 & b_6 \\ \hline b_4 & b_3 \\ \hline b_7 & b_8 \\ \hline \end{array}$$

The results are compared with different studies in both methods to include the aadditional degrees of freedom and are introduced at the nodes and added to member displacement vector[29][30][31]. The member variables are collected in member displacement vector V_m and member load vector q_m and the matrices are arranged correspondingly in member stiffness matrix k_m.

$$\mathbf{K}_m = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline k_1 & & & & & & & k_2 & & & & & & \\ \hline & k_3 & & & & k_4 & & & k_6 & & & & k_4 & \\ \hline & & k_9 & & k_{10} & & & & & k_{12} & & k_{10} & & \\ \hline & & & k_{T1} & & & k_{T2} & & & & k_{T3} & & & k_{T4} \\ \hline & & k_{10} & & k_{11} & & & & & k_{13} & & k_{14} & & \\ \hline & k_4 & & & & k_5 & & & k_7 & & & & k_8 & \\ \hline & & & k_{T5} & & & k_{T6} & & & & k_{T7} & & & k_{T8} \\ \hline k_2 & & & & & & & k_1 & & & & & & \\ \hline & k_6 & & & & k_7 & & & k_3 & & & & k_7 & \\ \hline & & k_{12} & & k_{13} & & & & & k_9 & & k_{13} & & \\ \hline & & & k_{T9} & & & k_{T10} & & & & k_{T11} & & & k_{T12} \\ \hline & & k_{10} & & k_{14} & & & & & k_{13} & & k_{11} & & \\ \hline & k_4 & & & & k_5 & & k_7 & & & & & k_8 & \\ \hline & & & k_{T13} & & & k_{T14} & & & & k_{T15} & & & k_{T16} \\ \hline \end{array}$$

$$\mathbf{q}_m = \begin{array}{|c|}\hline b_1 q_{1A} + b_2 q_{3B} \\ \hline b_1 q_{2A} + b_2 q_{2B} \\ \hline b_7 q_{2A} + b_8 q_{2B} \\ \hline b_3 q_{3A} + b_4 q_{3B} \\ \hline b_5 q_{3A} + b_6 q_{3B} \\ \hline b_1 m_{1A} + b_2 m_{1B} \\ \hline b_5 m_{1A} + b_6 m_{1B} \\ \hline b_1 q_{1A} + b_2 q_{1B} \\ \hline b_7 q_{2A} + b_8 q_{2B} \\ \hline b_7 q_{2A} + b_8 q_{2B} \\ \hline b_3 q_{3A} + b_4 q_{3B} \\ \hline b_5 q_{3A} + b_6 q_{3B} \\ \hline b_1 m_{1A} + b_2 m_{1B} \\ \hline b_5 m_{1A} + b_6 m_{1B} \\ \hline \end{array} \qquad \mathbf{v}_e = \begin{array}{|c|}\hline v_{1A} \\ \hline v_{2A} \\ \hline v_{3A} \\ \hline \beta_{1A} \\ \hline \beta_{2A} \\ \hline \beta_{3A} \\ \hline \psi_A \\ \hline v_{1B} \\ \hline v_{2B} \\ \hline v_{3B} \\ \hline \beta_{1B} \\ \hline \beta_{2B} \\ \hline \beta_{3B} \\ \hline \psi_B \\ \hline \end{array}$$

4. CONCLUSION

The frequently used finite element method for thin-walled sections only considers six degrees of freedom (DOFs) in each node of a beam, but it has been demonstrated that including warping of the section as an additional DOF in structural analysis can result in a safe and optimal design.

According to this study the following conclusions are drawn. The simple geometric properties of the section are used to generate the stiffness matrix for thin-walled beam sections with retrained torsion. By considering an additional degree of freedom at each node, the trigonometric and approximation solutions of an interpolation function are used to express the stiffness matrix for non-uniform torsion. The stiffness matrix for 3D thin-walled sections subjected to combined loading is presented, making advanced structural analysis bar elements more convenient. This stiffness matrix is more applicable for open thin-walled sections because the value of characteristics number for open section is very small comparing to the closed thin-walled sections. To include the additional degree of freedom both trigonometric and approximate methods are considered and for characteristics number (θ) =1 and 2 the errors range between 6.7 % to – 9.7 % which is considered reasonable and both methods are acceptable for open thin-walled sections. The length of the member is limited based on the section type and with maximum value of characteristics number (θ) less than 2.

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