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THE LOOP RESULTANT METHOD FOR STATIC STRUCTURAL ANALYSIS

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Abstract: This work deals with the loop resultant method applied to linear static response of statically indeterminate rod-systems. The method uses the representation of this system in the form of a union of statically indeterminate loops. An algorithm for construction the flexibility matrix of the system is proposed. The unknowns of a system are the loop resultants. This method is based on the use of compatibility equations of deformations, the general solution of homogeneous equations of equilibrium is obtained by transposition of the compatibility matrix. The advantages of this method are the number and location of zero blocks and non-zero blocks of the system flexibility matrix depend only on the numbering of loops. Application of this method is considered for analysis of structural frame.

Keywords: force method; loop resultant method; flexibility matrix; compatibility equation of deformations; statically indeterminate system.

МЕТОД КОНТУРНЫХ УСИЛИЙ В СТАТИКЕ СТЕРЖНЕВЫХ СИСТЕМ

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Аннотация: В работе рассматривается метод расчета статически неопределимых стержневых систем, основанный на представлении системы в виде объединения статически неопределимых контуров. Предложен алгоритм построения матрицы податливости системы. Неизвестными в системе уравнений являются усилия в контурах. Основой метода являются уравнения совместности деформаций, общее решение однородных уравнений равновесия получается транспонированием матрицы совместности. Рассмотрен пример расчета статически неопределимой рамы. Достоинством метода является то, что структура матрицы разрешающей системы уравнений, то есть количество и расположение в ней нулевых и ненулевых блоков зависит только от нумерации контуров.

Ключевые слова: метод сил; метод контурных усилий; матрица податливости; уравнения совместности деформаций; статически неопределимая система.

1. INTRODUCTION

Most of the engineering structures in civil engineering are statically indeterminate systems [1,2], which are omnipresent nowadays. For determining the structural responses the force method and the displacement method [1-3] have already been applied successfully in hand-computation of static and dynamic analysis. However for hand calculations, solving a system of equations of more than 10 unknowns was a major challenge. Thus, matrix calculus was developed to write an algorithm (or a programme) in a programming language and it is used on a digital electronic computer for automation of the solution of the system of equations [4]. Through the matrix approaches and the finite element method [4,5] achieved reliable results in structural analysis. However, most of current programmes use the displacement method or the mixed method [5,6].

Besides that, the first matrix approaches have been applied to improvement and automation of the force method [7-12]. Many algorithms of the force method have been studied to apply to structural design, such as an efficient analysis for cyclically symmetric space truss structures using an orthogonal self-stress matrix [13], a new structural analysis and optimization algorithm for determining the minimum weight of structures under displacement and stress constraints [14], the genetic algorithm for nonlinear analysis and optimal design of structures [15].
Within the algorithm of the integrated force method [16,17] the compatibility conditions are used. The papers [18-20] are also noticed that the use of the compatibility conditions is important for the automatic proccess of the force method. An efficient algorithm of the force method, which was called the loop resultant method, was developed by [21], this method will be discussed in more detail in next sections.

2. FORMULATION OF THE FLEXIBILITY MATRIX

We formulate the different flexibility matrices for element-rod (ER) which are based on their strain energy as the following formula:

$$W = \frac{1}{2}\int_0^L \left[\frac{N^2(s)}{EA} + \frac{Q^2(s)}{kGA} + \frac{M^2(s)}{EI} \right] ds . \quad (2.1)$$

Let's find the integral of (3.1), we have

$$W = \frac{1}{2}\sigma^T \cdot \Lambda \cdot \sigma , \quad (2.2)$$

where L is the length of the element-rod, EA is the axial stiffness, kGA is the shear stiffness, EI is the bending stiffness.
The element-rod with rigid nodes, namely the type I, has the matrix (3x1) of internal forces $\sigma^T=[F_x \; F_y \; M]$ at any point A is shown as in Figure 2.1. The point A is any point, but the point A is the same point for all elements. Rigid cantilevers connect the element nodes with the point A. Thus, the bending moment M(s), the shear force Q as well as the axial force N of the element-rod are obtained, respectively:

$$M(s) = -(y(s) - y_A)F_x + \\ + (x(s) - x_A)F_y + M, \quad (2.3)$$

$$Q = F_x n_x + F_y n_y , \quad (2.4)$$

$$N = F_x t_x + F_y t_y . \quad (2.5)$$

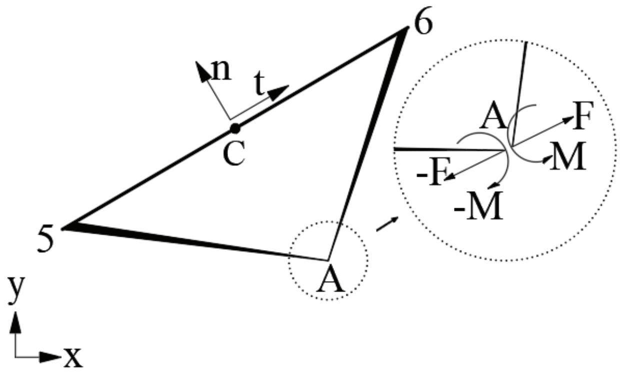

Figure 2.1. The nodal resultant of the element-rod (the type I)

Here C is the center point of the element-rod; t and n are the unit tangent and the unit normal vectors of the element-rod, respectively; F_x, F_y, n_x, n_y and t_x, t_y are the projections of three vectors F, n and t, respectively.
The type I (an element-rod with rigid ends), the type II (an element-rod hinged at one end and rigid at the other) and the type III (an element-rod with hinged ends) are shown in Figure 2.2.

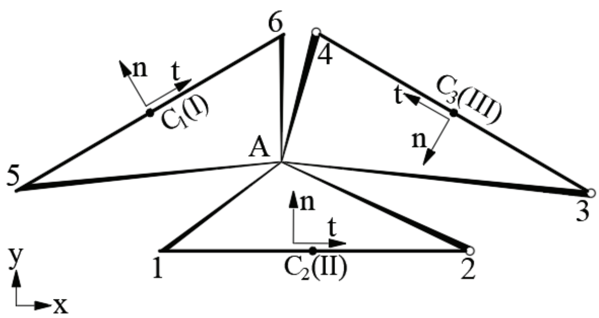

Figure 2.2. The different types of the elementrod

The flexibility matrices Λ of individual element-rods depend on their types. The flexibility matrix Λ (3x3) of the first type (its the shortest path from node 5 to node 6) can be expressed as

$$\Lambda_{(I)} = Q \cdot \tilde{\Lambda} \cdot Q^T + \tilde{\Lambda}_A, \quad (2.6)$$

where the diagonal flexibility matrix for the horizontal element-rod cooresponding to the natural coordinate system can be described as below

$$\tilde{\Lambda} = diag\left\{\frac{L}{EA}; \left(\frac{L}{kGA} + \frac{L^3}{12EI}\right); \frac{L}{EI}\right\}, \quad (2.7)$$

The orthogonal matrix (3x3) of a rotational element-rod is

$$Q = \begin{bmatrix} t_x & n_x & 0 \\ t_y & n_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (2.8)$$

and the adding matrix (3x3) of the vector C_1A is

$$\tilde{\Lambda}_A = \frac{L}{EI}\begin{bmatrix} y^2 & -x \cdot y & -y \\ -x \cdot y & x^2 & x \\ -y & x & 0 \end{bmatrix}, \quad (2.9)$$

here $x=x_{C1}-x_A$; $y=y_{C1}-y_A$ are the coordinates of a vector C_1A.

The second type of the element-rod hinged at one end, which is linked by a shortest path from node 1 to node 2, has the matrix (2x1) of internal forces $\sigma^T=[F_x\ F_y]$ and its the flexibility matrix (2x2) is expressed as:

$$\Lambda_{(II)} = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{bmatrix}, \quad (2.10)$$

where $\Lambda_{11} = L\left(\frac{t_x^2}{EA} + \frac{t_y^2}{kGA} + L^2\frac{t_y^2}{3EI}\right)$;

$$\Lambda_{12} = \Lambda_{21} = Lt_x t_y\left(\frac{1}{EA} - \frac{1}{kGA} - \frac{L^2}{3EI}\right);$$

$$\Lambda_{22} = L\left(\frac{t_y^2}{EA} + \frac{t_x^2}{kGA} + L^2\frac{t_x^2}{3EI}\right).$$

An element-rod with hinged ends is a third type in which two nodes 3, 4 are connected by one shortest path, has the matrix (1x1) of internal forces $\sigma=t_xF_x+t_yF_y$, its the flexibility matrix is presented as

$$\Lambda_{(III)} = \frac{L}{EA}. \quad (2.11)$$

The following kinematical variable *e* are energy conjugated with internal forces σ: $e^T=[e_x\ e_y\ \varphi]$, where e_x e_y - are relative displacements of the ends of the cantilevers at the point A, φ - is relative angle of rotation of the ends of the cantilevers at the point A.

3. THE BASIC LOOPS and COMPATIBILITY CONDITIONS

For the loop resultant method, Figure 3.1 shows two simple types of structural open-loop and closed-loop, which can be characterized by parameter: the degree of statically indeterminate loop, denoted by n^i_{st}.

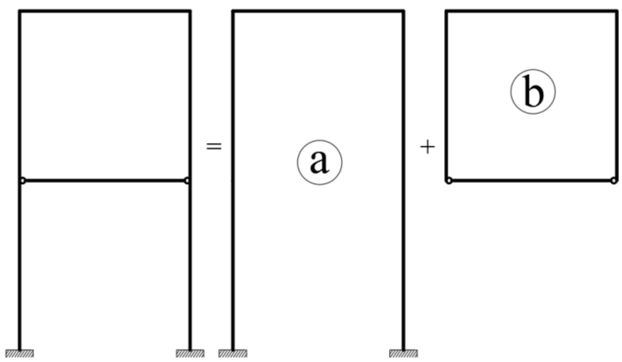

Figure 3.1. Two basic loops a (n^a_{st} =3), b (n^b_{st} =1)

In the following, we consider two principal rules to be applied for the process in building an algorithm.

Rule 1: The sum of the degree of each statically indeterminate loop $\sum n_{st}^i$ should be equal to the total degree of entire statically indeterminate structure n_{st}.

$$n_{st} = \sum_i n_{st}^i = n_{st}^a + n_{st}^b + ... + n_{st}^i . \qquad (3.1)$$

Rule 2: Selected loops can have one or several common element-rods and vice verca, but it is necessary to satisfy the condition of completeness of each loop, i.e. the individual element-rod must appear at least once in the basic loops.

The algorithm of the loop resultant method can be described in the following steps:

Step 1. Choose basic loops.

Step 2. Determine the flexibility matrix of element-rods Λ_i.

Step 3. Construct the flexibility block diagonal matrix

$$\Lambda = diag\{\Lambda_i\} . \qquad (3.2)$$

Step 4. Establish the compatibility matrix B.

Step 5. Complete the flexibility matrix of the framed structure

$$L = B \cdot \Lambda \cdot B^T . \qquad (3.3)$$

Step 6. Solve the system of equations

$$L \cdot X = -B \cdot e_o . \qquad (3.4)$$

Step 7. Compute the internal forces

$$\sigma = B^T \cdot X . \qquad (3.5)$$

We note that for the current algorithm, B^T is the transpose of a matrix B; X is the loop resultant matrix of the system; e_o is the initial deformations of the system.

To construct the compatibility matrix B, we consider the relationship of the deformation between different element-rods of the structural loop (Figure 3.1) using two transformation matrices H_1, H_2.

$$H_2 \cdot (e_1 + e_2 + e_3 - e_4) = 0 , \qquad (3.6)$$

where $H_2 = [t_{cx} \quad t_{cy} \quad (-y \cdot t_{cx} + x \cdot t_{cy})]$; t_c is the unit vector which passes through the two hinges; t_{cx}, t_{cy} are the projections of the vector t_c; $x = x_A - x_C$, $y = y_A - y_C$. Thus the compatibility matrix B (1x12) is

$$B = [H_2 \quad H_2 \quad H_2 \quad -H_2] . \qquad (3.7)$$

When the loop has only one hinge, the compatibility matrix B (2x12) is

$$B = [H_1 \quad H_1 \quad H_1 \quad -H_1] . \qquad (3.8)$$

here $H_1 = \begin{bmatrix} 1 & 0 & -y \\ 0 & 1 & x \end{bmatrix}$; $x = x_A - x_H$, $y = y_A - y_H$; H denotes the hinged position.

When the loop has not hinge, the compatibility matrix B (3x12) is

$$B = [I \quad I \quad I \quad -I] . \qquad (3.9)$$

where I is identity (3x3) matrix.

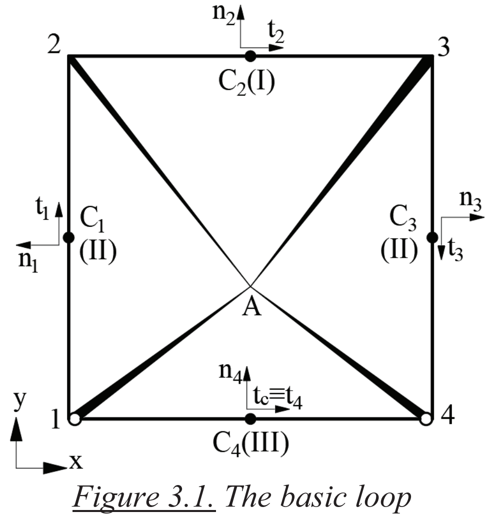

Figure 3.1. The basic loop

4. THE NUMERICAL EXAMPLE

The example is shown in Figure 4.1. The flexural rigidity EI is constant, the Young's modulus is $E=3\cdot10^7$ kN/m^2, the sizes of a rectangular cross-section are b=200 mm, h=200 mm, L=2 m. Determine the bending moments of structural frame under external loads P_1=8 kN, P_2=4 kN.

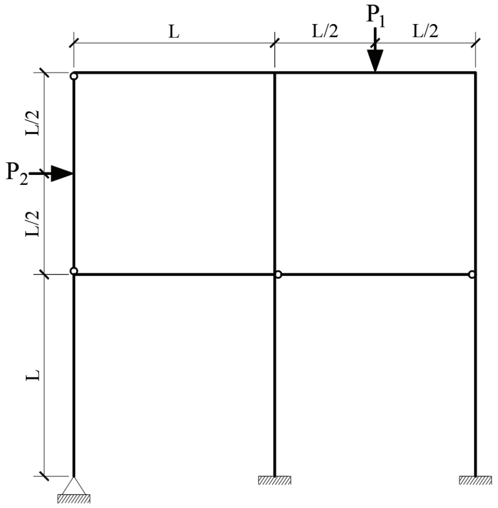

Figure 4.1. The framed structure under dead loads P_1 and P_2

First, the loop resultant method requires the component of information system which is shown in Figure 4.2, includes the point A, the type of the ER, the coordinate of the center point of the ER.

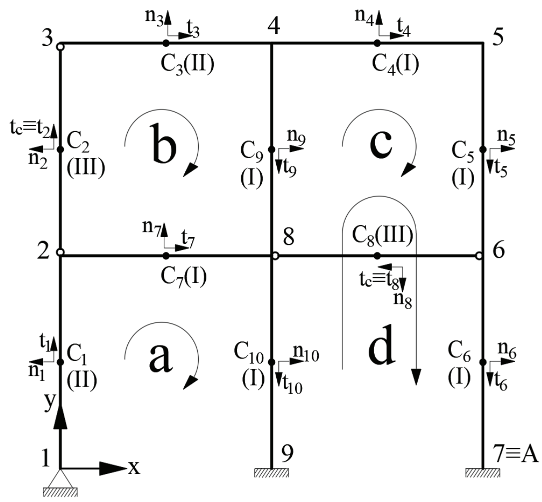

Figure 4.2. The four loops

In the loop resultant method basic loops may be chosen as shown in Figure. 4.3, which has four statically indeterminate loops: $n_{st}^{a}=2$, $n_{st}^{b}=1$, $n_{st}^{c}=1$, $n_{st}^{d}=3$.

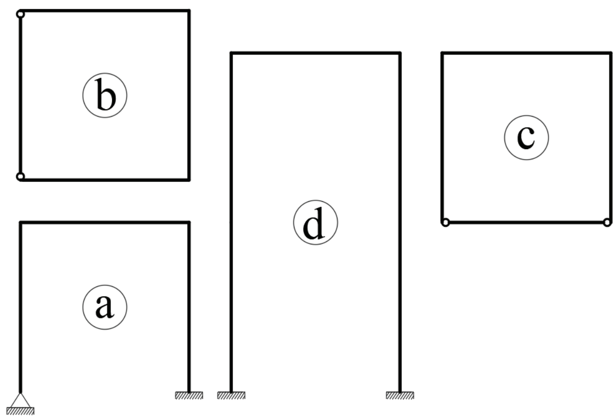

Figure 4.3. Selected basic loops.

Now, we consider the selected loops in Figure 4.3, which are reduced to the statically determinate loops in Figure 4.4, and we then calculate bending moments M_{Pi} (Figure 4.4) of the loops caused by external loads. When the basic loop subjected to external loads, it is necessary that the added internal forces allow it to reach balance, which can be mutually excluded in the whole system.

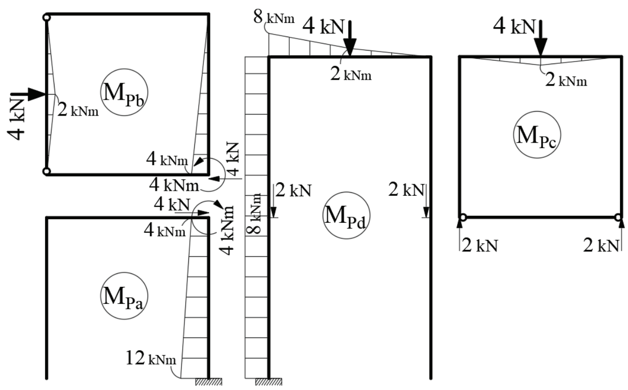

Figure 4.4. Bending moments M_{Pi} of loop fragments

Next, we compute bending moments on statically determinate loops caused by unit force in the direction of the desired deformation as in

Figuge 4.5 (their bending moment diagrams are shown in Figures 4.6, 4.7).

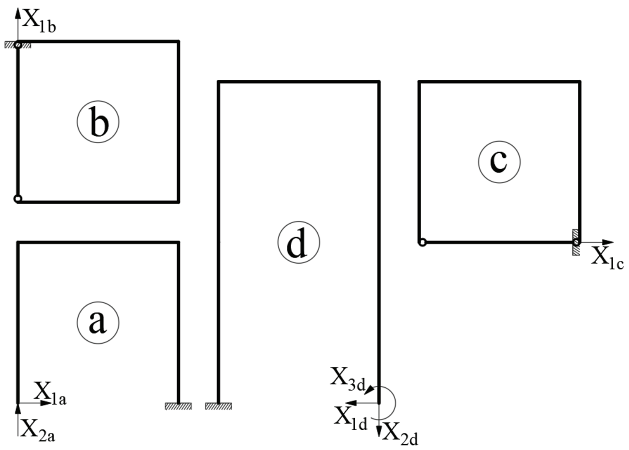

Figure 4.5. *The loops subjected to unit force*

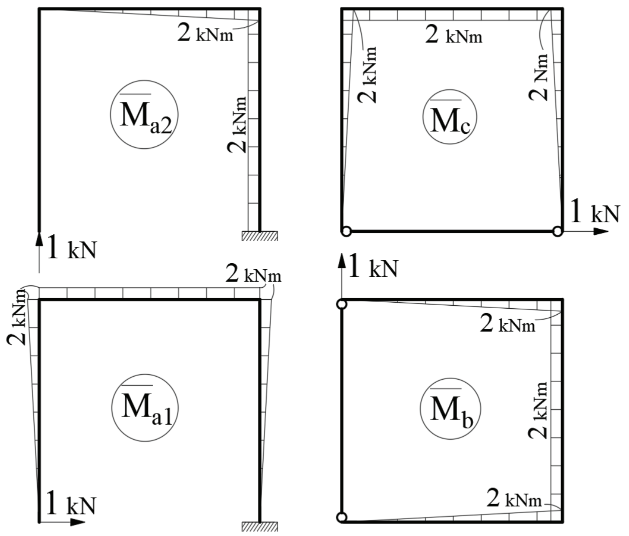

Figure 4.6. *The loops a, b, c in the unit state*

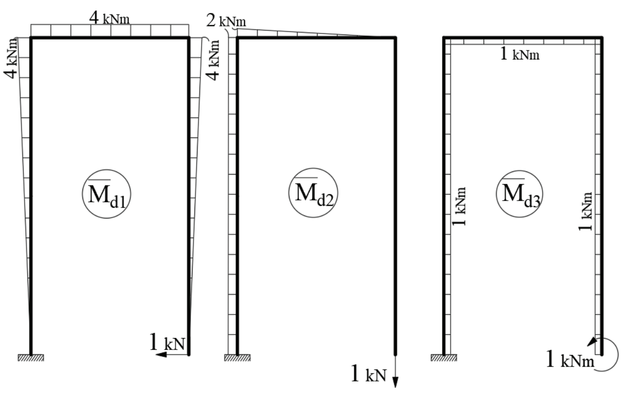

Figure 4.7. *The loop d in the unit state*

For calculating the initial deformations e_o, it is necessary to construct bending moment diagrams $\bar{M}_{ij}$ due to unit primary unknowns X_{ij}, j = 1, 2, 3, i = a, b, c, d and diagrams M $_{Pi}$ due to external load.

The components of the matrix deformation Be_o are a set of displacements and rotations of basic loops, which are calculated by the formula.

$$\Delta_{ji,Pi} = \sum \int \frac{\bar{M}_{ji} \cdot M^*_{Pi}}{EI} dS, \qquad (4.1)$$

The matrix deformation Be_o may be constructed for all structure

$$(Be_o)^T = [\Delta_{1a,Pa} \quad \Delta_{2a,Pa} \quad \Delta_{1b,Pb} \quad \Delta_{1c,Pc} \quad \Delta_{1d,Pd} \; \Delta_{2d,Pd} \; \Delta_{3d,Pd}]. \qquad (4.2)$$

In our case we get

$$(Be_o)^T = (1/EI) \times [-29.33 \; 64 \; 40 \; -26.67 \; 104 \; 110.67 \; -56]. \qquad (4.3)$$

In structural frame, we consider only the bending moment, i.e. the axial flexibility (1/EA) and the shear flexibility (1/kGA) are equal to zero. The corresponding flexibility matrices (2.6), (2.10), (2.11) are

$$\Lambda_{(I)} = \frac{L}{EI} \begin{bmatrix} (\frac{L^2}{12} t_y^2 + \frac{y^2}{EI}) & -(\frac{L^2}{12} t_x t_y + xy) & -y \\ & (\frac{L^2}{12} t_x^2 + x^2) & x \\ symm. & & 1 \end{bmatrix},$$

$$\Lambda_{(II)} = \begin{bmatrix} \frac{L^3}{3EI} t_y^2 & -\frac{L^3}{3EI} t_x t_y \\ symm. & \frac{L^3}{3EI} t_x^2 \end{bmatrix}, \; \Lambda_{(III)} = 0 \quad (4.4).$$

So the flexibility matrix Λ_i for each element-rod of the frame can be calculated by using the formulas (4.4).

$\Lambda_1 = \begin{bmatrix} 2.7 & 0.0 \\ 0.0 & 0.0 \end{bmatrix}$, $\Lambda_2=0$,

$\Lambda_3 = \begin{bmatrix} 0.0 & 0.0 \\ 0.0 & 2.7 \end{bmatrix}$, $\Lambda_4 = \begin{bmatrix} 32 & 8.0 & -8.0 \\ 8.0 & 2.7 & -2.0 \\ -8.0 & -2.0 & 2.0 \end{bmatrix}$,

$\Lambda_5 = \begin{bmatrix} 18.7 & 0 & -6.0 \\ 0.0 & 0 & 0.0 \\ -6.0 & 0 & 2.0 \end{bmatrix}$, $\Lambda_6 = \begin{bmatrix} 2.7 & 0 & -2.0 \\ 0.0 & 0 & 0.0 \\ -2.0 & 0 & 2.0 \end{bmatrix}$,

$\Lambda_7 = \begin{bmatrix} 8.00 & 12.0 & -4.0 \\ 12.0 & 18.7 & -6.0 \\ -4.0 & -6.0 & 2.0 \end{bmatrix}$, $\Lambda_8=0$,

$\Lambda_9 = \begin{bmatrix} 18.7 & 12.0 & -6.0 \\ 12.0 & 8.0 & -4.0 \\ -6.0 & -4.0 & 2.0 \end{bmatrix}$, $\Lambda_{10} = \begin{bmatrix} 2.7 & 4.0 & -2.0 \\ 4.0 & 8.0 & -4.0 \\ -2.0 & -4.0 & 2.0 \end{bmatrix}$.

After the flexibility matrices of element-rods are found, the following block diagonal matrix must be established.

$\Lambda=(1/EI)\times\text{diag}\{\Lambda_1; \Lambda_2; \Lambda_3; \Lambda_4; \Lambda_5; \Lambda_6; \Lambda_7; \Lambda_8; \Lambda_9; \Lambda_{10}\}$.

Using the formula (3.7), (3.8), (3.9), we construct the compatibility matrix B (7x24) for the whole system:

$$B = \begin{bmatrix} H_1 & 0 & 0 & 0 & 0 & 0 & H_1 & 0 & 0 & H_1 \\ 0 & H_2 & H_2 & 0 & 0 & 0 & H_2 & 0 & H_2 & 0 \\ 0 & 0 & 0 & H_2 & H_2 & 0 & 0 & H_2 & H_2 & 0 \\ 0 & 0 & 0 & I_{3\times3} & I_{3\times3} & I_{3\times3} & 0 & 0 & I_{3\times3} & I_{3\times3} \end{bmatrix}, \quad (4.9)$$

where I_{nxn} is the unit matrix of size nxn.
Using the formula (3.3) for the flexibility matrix of structural frame, we have

$$L = \frac{1}{EI}\begin{bmatrix} 13.33 & -8.00 & 4.00 & 0.00 & -2.67 & -4.00 & 2.00 \\ -8.00 & 10.67 & -2.67 & 0.00 & 4.00 & 8.00 & -4.00 \\ 4.00 & -2.67 & 13.33 & -4.00 & 12.0 & 8.00 & -4.00 \\ 0.00 & 0.00 & -4.00 & 13.33 & -29.3 & -8.00 & 8.00 \\ -2.67 & 4.00 & 12.00 & -29.3 & 74.7 & 24.0 & -24.0 \\ -4.00 & 8.00 & 8.00 & -8.00 & 24.0 & 18.67 & -10.0 \\ 2.00 & -4.00 & -4.00 & 8.00 & -24.0 & -10.0 & 10.00 \end{bmatrix}. \quad (4.10)$$

The solution of equations (3.4) leads to following results.
The loop resultants are

$$X = [-793.776 \quad -2687.37 \quad -677.853 \quad -283.478 \quad 925.688 \quad -4937.73 \quad 1923.37]. \quad (4.11)$$

According to the formula (3.5) we get the stress resultant as

$$\begin{aligned} \sigma = [&-793.776 \quad -2687.37 \quad -677.853 \quad 0.0000 \quad -677.853 \quad 1209.17 \\ &-4937.73 \quad 2490.33 \quad 1209.17 \quad -4937.73 \quad 2490.33 \quad 925.688 \\ &-4937.73 \quad 1923.37 \quad -793.776 \quad -2009.51 \quad 8038.05 \quad 283.478 \\ &-1209.17 \quad 4259.88 \quad -5201.74 \quad -1719.46 \quad 2250.37 \quad -12672.8]. \end{aligned} \quad (4.12)$$

The final bending moment which are shown in Table 4.1, may be calculated by the formula $M_{final} = M_{Pn} + M$, where $M_p = \Sigma M_{Pi}$ (Figure 4.4), M is calculated by formula (2.3).

Table 4.1. The bending moments of the structural frame.

№ ER	Bending moments M_P (N·m)	Bending moments of the resultant system M (N·m)	Bending moments M_{final} (N·m)
1s	0.000000	0.0000000	0.0000000
1e	0.000000	-1587.550	-1587.550
2s,2e	0.000000	0.0000000	0.0000000
2m	-2000.00	0.0000000	-2000.000
3s	0.000000	0.0000000	0.0000000
3e	0.000000	-1355.700	-1355.700
4s	-8000.00	+7529.120	-470.8800
4m	0.000000	+2591.390	+2591.390
4e	0.000000	-2346.330	-2346.330
5s	0.000000	-2346.330	-2346.330
5e	0.000000	+71.99000	+71.99000
6s	0.000000	+71.99000	+71.99000
6e	0.000000	+1923.370	+1923.370
7s	0.000000	+1587.550	+1587.550
7e	0.000000	-2431.470	-2431.470
8s,8e	0.000000	0.0000000	0.0000000
9s	+8000.00	-8884.830	-884.8300
9e	+12000.0	-11303.16	+696.8400
10s	+12000.0	-13734.63	-1734.630
10e	+20000.0	-17173.56	+2826.44

We note that the numbers 1…10 describe the ordinal ERs; the value of the bending moment at the start, middle and end of the ER are denoted by s, m and e, respectively.

5. CONCLUSIONS

It is well know that the classical force method is not easily fully automated. In the present paper the technique, the so-called loop resultant method, is used to automate the classical force method by using an algorithm, which may be constructed the computationally easy way to static analysis for statically indeterminate structures.

The extension of this algorithm that could be applied to dynamic analysis is still under investigation.

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