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FILTRATION PROBLEM WITH NONLINEAR FILTRATION AND CONCENTRATION FUNCTIONS

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Abstract: Ancient architectural buildings are of great value for all modern humanity. Over time, under the influence of vibrations, water and other man-made and natural factors, the foundations of such buildings are destroyed, the soil structure changes. Currently, one of the most popular methods of strengthening soils and strengthening foundations is the jet grouting technology. When the liquid grout passes through the porous rock, the suspended particles of the grout form a deposit. In this paper, we study a one-dimensional model of suspension deep bed filtration in a porous medium with different particle capture mechanisms. The considered filtration model consists of the balance equation for the masses of suspended and retained particles and the kinetic equation for deposit growth. In this case, the deposit growth rate is determined by the concentration function of suspended particles, which, in turn, depends on the properties of the suspension and the geometry of the porous medium. The solution to the problem is obtained for linear and non-linear concentration functions. An asymptotic solution of the problem is constructed for both types of functions near the concentration front of suspended and retained particles. It is shown that the asymptotic and numerical solutions are close over a long time interval.

Keywords: deep bed filtration, suspended and retained particles, suspension, porous medium, concentration function, asymptotic solution.

ЗАДАЧА ФИЛЬТРАЦИИ С НЕЛИНЕЙНЫМИ ФУНКЦИЯМИ ФИЛЬТРАЦИИ И КОНЦЕНТРАЦИИ

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Аннотация: Старинные архитектурные здания представляют собой огромную ценность для всего современного человечества. Со временем под воздействием вибраций, воды и других техногенных и природных факторов происходит разрушение фундаментов таких зданий, изменение структуры грунтов. В настоящее время одним из наиболее популярных методов усиления грунтов и укрепления фундаментов является технология струйной цементации. При прохождении жидкого раствора через пористую породу взвешенные частицы укрепителя образуют осадок. В работе исследуется одномерная модель долговременной глубинной фильтрации суспензии в пористой среде с различными механизмами захвата частиц. Рассматриваемая модель фильтрации состоит из уравнения баланса масс взвешенных и задержанных частиц и кинетического уравнение роста осадка. При этом скорость роста осадка определяется функцией концентрации осажденных частиц, которая в свою очередь зависит от свойств суспензии и геометрии пористой среды. Решение задачи получено для линейной и нелинейной функций концентрации. Построено асимптотическое решение задачи для обоих видов функций вблизи фронта концентрации взвешенных и осажденных частиц. Показано, что асимптотические и численные решения близки в большом временном интервале.

Ключевые слова: фильтрация, взвешенные и осажденные частицы, суспензия, пористая среда, функция концентрации, асимптотическое решение.

INTRODUCTION

Historical buildings are an integral part of the world cultural heritage. They reflect the historical trajectory of the country's development and are the product of the development of ancient history, culture, art and religion. Humanity needs history and culture, and it is through historical monuments that it comes into contact with both. Thanks to the protection and restoration of historical buildings, a person can complement the cultural heritage of his country, record history, and ensure his identity [1].

In most cases, the age of buildings with architectural value exceeds 100 years. Over a long service life, the foundations of monuments are under threat of destruction due to increased loads and vibrations, and the structure of soils changes over time. Water penetrating into the soil in various ways has a special effect (a sewer breakthrough, an increase in the groundwater level). To prevent the loss of historical values and cultural monuments, it is necessary to strengthen the soils under them, which will reduce the risk of deformations and destruction.

There are many examples of strengthening the soils of historical buildings in the world practice. For example, this is the Angel's Stradome in Krakow, the Branicki Palace and the historic Foxal 13 and 15 building in Warsaw, the Kutb Miner in Delhi (fig. 1), the Tsaritsyno State Museum-Reserve in Moscow (fig. 2), etc. [2-6].

Figure 1. The Qutb Miner

Figure 2. Tsaritsyno State Museum-Reserve

Since the 1970s, jet grouting technology has been used in the work on effective soil strengthening and foundation strengthening [7-9], which is currently one of the most popular in this field. Grout (fluid system), or grout with air (double fluid system), or grout with air and water (triple fluid system) is injected into the ground through nozzles of small diameter placed in a pipe for the grout lowered into the well. The tube rotates continuously at a constant speed and slowly rises to the surface of the earth. The jet propagates radially from the axis of the well, and after a while the injected solution solidifies, forming a body of cemented soil of a quasi-cylindrical shape [10].

Filtration of a suspension transporting through a porous medium, taking into account the retention of solid particles on a porous frame and a gradual decrease in the porosity and permeability of the medium, is of great interest in jet grouting. Suspension filtration is divided into two groups: cake filtration and deep bed filtration [11]. The first type of filtration consists in the retention of particles at the porous medium inlet, it is characteristic of suspensions with large particles and a high concentration, the second is associated with the deposit formation in the entire porous medium, if small suspended particles of a suspension with a low concentration can be freely transported through the pores [12]. In the case when the suspension has particles of different sizes, these two types of filtration are used simultaneously [13]. We will consider the deep bed filtration model.

There are various mechanisms for capturing particles in a porous medium: size exclusion, electric forces (London-Van der Waals, double electric layer, etc.), gravitational segregation, multi particle bridging [14]. In this paper, size exclusion mechanism is considered, when large particles are captured by small pores, clog them, and pass unhindered through large pores [15-17], and a mixed mechanism that includes size exclusion mechanism and multi particle bridging mechanism [fig. 3]. In the case of multi particle bridging, suspension particles form arched bridges at the pores, thereby block them.

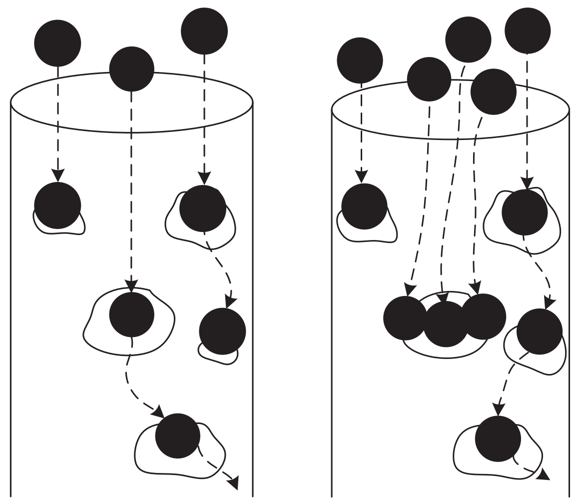


Figure 3. Particle capture mechanisms in the porous medium a) size exclusion b) size exclusion together with multi particle bridging

In the classical model of deep bed filtration, the system of partial differential equations consists of mass balance equation end kinetic equation of deposition. Exact solutions have been found for some classes of one-dimensional problems [18-20], for others it is possible to obtain only an asymptotic solution [21-23]. For a wide class of models, a solution is obtained by numerical methods [24-26].

MATHEMATICAL MODEL

Consider the classical model of deep bed filtration in the domain $\Omega = \{0 \le x \le 1,\ t \ge 0\}$

$$\frac{\partial C(x,t)}{\partial t} + \frac{\partial C(x,t)}{\partial x} + \frac{\partial S(x,t)}{\partial t} = 0, \quad (1)$$

$$\frac{\partial S(x,t)}{\partial t} = \Lambda\left(S(x,t)\right) F\left(C(x,t)\right) \quad (2)$$

with a boundary condition

$$x = 0: \quad C(0,t) = 1 \quad (3)$$

and initial conditions

$$t = 0: \quad C(x,0) = 0; \quad S(x,0) = 0. \quad (4)$$

Here $C(x,t)$ is the concentration of suspended particles, $S(x,t)$ is the concentration of retained particles, $\Lambda\left(S(x,t)\right)$ is the filtration function, $F\left(C(x,t)\right)$ is the concentration function.

The line $t = x$ determines the concentration front of suspended and retained particles. In the domain $\Omega^{(1)} = \{0 < x < 1,\ 0 < t < x\}$, located below this line, concentrations $C(x,t)$ and $S(x,t)$ are equal to zero. In the domain $\Omega^{(2)} = \{0 < x < 1,\ t > x\}$ concentrations $C(x,t)$ and $S(x,t)$ are positive. The solution $C(x,t)$ has a break along the front, the solution $S(x,t)$ is continuous everywhere and is zero at the concentration front.

In this paper, we will consider a cubic function of the general form as a filtration function

$$\Lambda\left(S\right) = \lambda_0 + \lambda_1 S + \lambda_2 S^2 + \lambda_3 S^3,$$

which is decreasing to zero, i.e. there is a value $\tilde{S}$ such that $\lambda_0 + \lambda_1 \tilde{S} + \lambda_2 \tilde{S}^2 + \lambda_3 \tilde{S}^3 \equiv 0$. The specific type of filtration function used in our calculations has been determined experimentally [27].

The system (1)-(2) takes the form

$$\frac{\partial C}{\partial t}+\frac{\partial C}{\partial x}+\left(\lambda_0+\lambda_1 S+\lambda_2 S^2+\lambda_3 S^3\right)F(C)=0\,,\quad(5)$$

$$\frac{\partial S}{\partial t}=\left(\lambda_0+\lambda_1 S+\lambda_2 S^2+\lambda_3 S^3\right)F(C)\qquad(6)$$

with boundary and initial conditions (3)-(4).
Turning in equation (5) to the characteristic variables $\tau=t-x$, $x=x$, we obtain the equation

$$\frac{\partial C}{\partial x}+\left(\lambda_0+\lambda_1 S+\lambda_2 S^2+\lambda_3 S^3\right)F(C)=0\,.$$

Let us find a solution $C(x,\tau)$ at $\tau=0$, which corresponds to the concentration front line $t=x$. Given that the solution $S(x,\tau)$ is zero at $\tau=0$, we get

$$\frac{\partial C}{\partial x}+\lambda_0 F(C)=0\,.$$

Consider concentration functions of the form $F\left(C\right)=C$ and $F\left(C\right)=(1-\alpha)C+\alpha C^3$, which correspond to the size-exclusion and mixed mechanisms of particle capture. For a linear function $F\left(C\right)=C$ we get

$$\frac{\partial C}{\partial x}=-\lambda_0 C\,.$$

Taking into account (3), we have

$$\int_{C_0(x)}^{1}\frac{dc}{C}=\lambda_0 x\,,$$

the solution $C_0(x)$ is determined with formula

$$C_0(x)=e^{-\lambda_0 x}\,.\qquad(7)$$

For a nonlinear function $F\left(C\right)=(1-\alpha)C+\alpha C^3$ we get

$$\int_{C_0(x)}^{1}\frac{dc}{(1-\alpha)C+\alpha C^3}=\lambda_0 x\,,$$

then

$$C_0(x)=\frac{\sqrt{1-\alpha}e^{-(1-\alpha)\lambda_0 x}}{\sqrt{1-\alpha e^{-2(1-\alpha)\lambda_0 x}}}\,.\qquad(8)$$

At the filter inlet $x=0$ taking into account the boundary condition (3), the solution $S_0(0,t)$ is determined by equation (6):

$$\frac{\partial S}{\partial t}=\left(\lambda_0+\lambda_1 S+\lambda_2 S^2+\lambda_3 S^3\right),$$

Then

$$\int_{0}^{S(0,t)}\frac{dS}{\lambda_0+\lambda_1 S+\lambda_2 S^2+\lambda_3 S^3}=t\,.\qquad(9)$$

ASYMPTOTIC SOLUTION NEAR THE CONCENTRATION FRONT

Linear concentration function

For linear concentration function $F\left(C\right)=C$ let us find asymptotic solutions near the concentration front $t=x$ in the form:

$$C(x,t)=C_0(x)+C_1(x)(t-x)+\\+\frac{1}{2}C_2(x)(t-x)^2+...,\qquad(10)$$

$$S(x,t)=S_1(x)(t-x)+\frac{1}{2}S_2(x)(t-x)^2+...\\+\frac{1}{6}S_3(x)(t-x)^3...,\qquad(11)$$

limiting in both solutions to the first three terms due to cumbersome calculations. The first term of the suspended particle asymptotic $C_0(x)$ is determined by the formula (7), $S_0(x)=0$ since the concentration of deposited particles is zero at the concentration front.

Differentiate the expansion (10) by x and t, the expansion (11) by t and substitute these expressions into the system of equations (5)-(6).

Grouping the terms at the same degrees of $(t-x)$, equate them to zero and get a system of equations:

$$(t-x)^0: \quad C_0'(x) = -\lambda_0 C_0(x),\ S_1(x) = \lambda_0 C_0(x);$$

$$(t-x)^1: \quad C_1'(x) = -\lambda_0 C_1(x) - \lambda_1 S_1(x) C_0(x),\ S_2(x) = \lambda_0 C_1(x) + \lambda_1 C_0(x) S_1(x);$$

$$(t-x)^2: \quad \frac{1}{2}C_2'(x) = -\frac{1}{2}\lambda_0 C_2(x) - \lambda_1 C_1(x) S_1(x) - \frac{1}{2}\lambda_1 C_0(x) S_2(x) - \lambda_2 C_0(x) S_1^2(x)$$

$$\frac{1}{2}S_3(x) = \frac{1}{2}\lambda_0 C_2(x) + \lambda_1 C_1(x) S_1(x) + \frac{1}{2}\lambda_1 C_0(x) S_2(x) + \lambda_2 C_0(x) S_1^2(x)$$

Solving this system of equations, we obtain:

$$C_0(x) = e^{-\lambda_0 x},\ S_1(x) = \lambda_0 e^{-\lambda_0 x};$$

$$C_1(x) = \lambda_1 e^{-\lambda_0 x}\left(e^{-\lambda_0 x} - 1\right),\ S_2(x) = \lambda_0 \lambda_1 e^{-\lambda_0 x}\left(2e^{-\lambda_0 x} - 1\right);$$

$$C_2 = e^{-\lambda_0 x}\left(\left(2\lambda_1^2 + \lambda_0\lambda_2\right)e^{-2\lambda_0 x} - 3\lambda_1^2 e^{-\lambda_0 x} + \left(\lambda_1^2 - 2\lambda_0\lambda_2\right)\right),$$

$$S_3(x) = \lambda_0 e^{-\lambda_0 x}\left(3(2\lambda_1^2 + \lambda_0\lambda_2)e^{-2\lambda_0 x} - 6\lambda_1^2 e^{-\lambda_0 x} + \lambda_1^2 - 2\lambda_0\lambda_2\right).$$

Note that the first solution of the obtained system coincides with solution (7).

Substituting the obtained solutions in (10)-(11), we obtain asymptotic expansions:

$$C(x,t) = e^{-\lambda_0 x} + \lambda_1 e^{-\lambda_0 x}\left(e^{-\lambda_0 x} - 1\right)(t-x) +$$

$$+\frac{1}{2}e^{-\lambda_0 x}\left(\left(2\lambda_1^2 + \lambda_0\lambda_2\right)e^{-2\lambda_0 x} - 3\lambda_1^2 e^{-\lambda_0 x} + \left(\lambda_1^2 - 2\lambda_0\lambda_2\right)\right)(t-x)^2 + \ldots,$$

$$S(x,t) = \lambda_0 e^{-\lambda_0 x}(t-x) + \frac{1}{2}\lambda_0\lambda_1 e^{-\lambda_0 x}\left(2e^{-\lambda_0 x} - 1\right)(t-x)^2 +$$

$$+\frac{1}{6}\lambda_0 e^{-\lambda_0 x}\left(3(2\lambda_1^2 + \lambda_0\lambda_2)e^{-2\lambda_0 x} - 6\lambda_1^2 e^{-\lambda_0 x} + \lambda_1^2 - 2\lambda_0\lambda_2\right)(t-x)^3 + \ldots$$

Nonlinear concentration function

Consider the nonlinear concentration function $F(C) = (1-\alpha)C + \alpha C^3$. In order to avoid cumbersome calculations, we will limit ourselves in expansions (10)-(11) to the first two terms:

$$C(x,t) = C_0(x) + C_1(x)(t-x) + \ldots, \qquad (12)$$

$$S(x,t) = S_1(x)(t-x) + \frac{1}{2}S_2(x)(t-x)^2 + \ldots (13)$$

Here the main term of the asymptotic is given by formula (8).

Similarly to the case with a linear function, we obtain a system of equations:

$(t-x)^0$: $\quad C_0'(x) = -(1-\alpha)\lambda_0 C_0(x) - \alpha\lambda_0 C_0^3(x)$, $S_1(x) = (1-\alpha)\lambda_0 C_0(x) + \alpha\lambda_0 C_0^3(x)$;

$(t-x)^1$: $\quad C_1'(x) = -(1-\alpha)\lambda_0 C_1(x) - (1-\alpha)\lambda_1 C_0(x) S_1(x) - 3\alpha\lambda_0 C_0^2(x) C_1(x) - \alpha\lambda_1 C_0^3(x) S_1(x)$,

$S_2(x) = (1-\alpha)\lambda_0 C_1(x) + (1-\alpha)\lambda_1 C_0(x) S_1(x) + 3\alpha\lambda_0 C_0^2(x) C_1(x) + \alpha\lambda_1 C_0^3(x) S_1(x)$.

Solving the recurrent system, we obtain:

$$C_0 = \frac{\sqrt{1-\alpha}\,e^{-(1-\alpha)\lambda_0 x}}{\sqrt{1-\alpha e^{-2(1-\alpha)\lambda_0 x}}},\; S_1(x) = \frac{\sqrt{(1-\alpha)^3}\,\lambda_0 e^{-(1-\alpha)\lambda_0 x}}{\sqrt{1-\alpha e^{-2(1-\alpha)\lambda_0 x}}} + \frac{\alpha\sqrt{(1-\alpha)^3}\,\lambda_0 e^{-3(1-\alpha)\lambda_0 x}}{\sqrt{(1-\alpha e^{-2(1-\alpha)\lambda_0 x})^3}};$$

$$C_1(x) = \lambda_1 C_0(x)\big(C_0(x)-1\big)\big(1-\alpha+\alpha C_0^2(x)\big),$$

$$S_2(x) = 4\alpha^2\lambda_0\lambda_1 C_0^6(x) - 3\alpha^2\lambda_0\lambda_1 C_0^5(x) + 6\alpha\lambda_0\lambda_1 C_0^4(x)(1-\alpha) - 4\alpha\lambda_0\lambda_1 C_0^3(x)(1-\alpha) +$$
$$+2\lambda_0\lambda_1 C_0^2(x)(1-\alpha)^2 - \lambda_0\lambda_1 C_0(x)(1-\alpha)^2.$$

Substituting the obtained expressions into decompositions (12), (13), we obtain asymptotic solutions

$$C(x,t) = \frac{\sqrt{1-\alpha}\,e^{-(1-\alpha)\lambda_0 x}}{\sqrt{1-\alpha e^{-2(1-\alpha)\lambda_0 x}}} + \lambda_1 C_0(x)\big(C_0(x)-1\big)\big(1-\alpha+\alpha C_0^2(x)\big)(t-x),$$

$$S(x,t) = \left(\frac{\sqrt{(1-\alpha)^3}\,\lambda_0 e^{-(1-\alpha)\lambda_0 x}}{\sqrt{1-\alpha e^{-2(1-\alpha)\lambda_0 x}}} + \frac{\alpha\sqrt{(1-\alpha)^3}\,\lambda_0 e^{-3(1-\alpha)\lambda_0 x}}{\sqrt{(1-\alpha e^{-2(1-\alpha)\lambda_0 x})^3}}\right)(t-x) +$$
$$+\frac{1}{2}\Big(4\alpha^2\lambda_0\lambda_1 C_0^6(x) - 3\alpha^2\lambda_0\lambda_1 C_0^5(x) + 6\alpha\lambda_0\lambda_1 C_0^4(x)(1-\alpha) - 4\alpha\lambda_0\lambda_1 C_0^3(x)(1-\alpha) +$$
$$+2\lambda_0\lambda_1 C_0^2(x)(1-\alpha)^2 - \lambda_0\lambda_1 C_0(x)(1-\alpha)^2\Big)(t-x)^2.$$

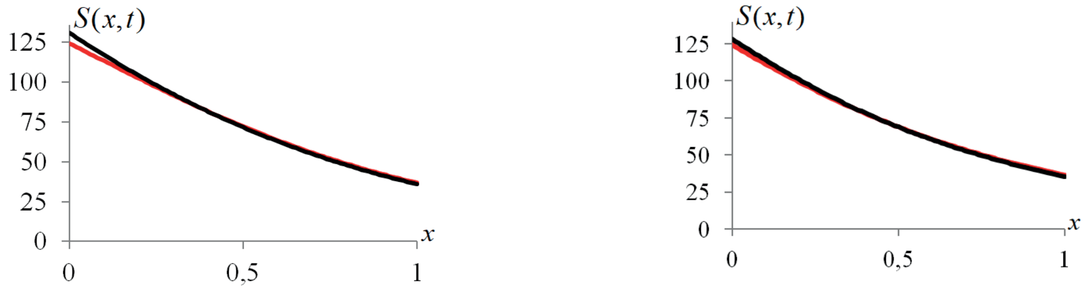
NUMERICAL SOLUTION OF THE PROBLEM

Let us solve the problem by the numerical method of finite differences. For the convergence of the method, the steps along the time axis t and the coordinate axis x are chosen $h_t = h_x = 0.01$, it ensures the fulfillment of the Courant condition [28]. The coefficients of the filtration function are obtained experimentally: $\lambda_0 = 1.551$, $\lambda_1 = -3.467\cdot 10^{-3}$, $\lambda_2 = -1.16\cdot 10^{-6}$, $\lambda_3 = -1.16\cdot 10^{-7}$ [29]. In this case, the solution at the filter inlet, determined by equation (9), is given implicitly

$$71.87\,\mathrm{arctg}(0.41+2S) - 59.12\ln(193.28-S) +$$
$$+29.56\ln(69177.9+203.28S+S^2) -$$
$$-71.87\,\mathrm{arctg}\,0.41 + 59.12\ln 193.28 -$$
$$-29.56\ln(69177.9) = t.$$

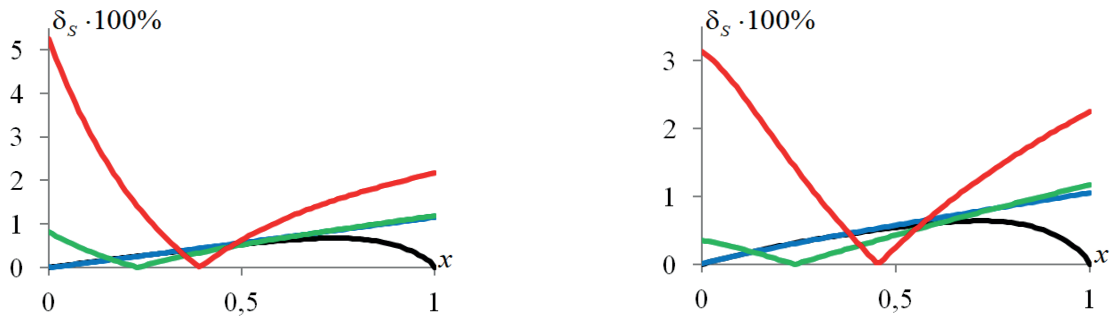
The graphs of suspended particle concentrations at a time $t = 100$ for linear (fig. 4a) and cubic (fig. 4b) concentration functions are shown in fig. 4: the red line corresponds to the numerical solution of the problem, the black line corresponds to the asymptotic solution. It can be seen from the figures that the asymptotics for both concentration functions gives a fairly good approximation, the greatest discrepancy is observed at the filter inlet. The relative error of the asymptotics in the case of a linear concentration function is about 5%, in the case of a nonlinear filtration function – 3%.

Fig. 5 shows the relative errors of the asymptotic concentration of suspended particles for different times t: $t=1$ (black line), $t=10$ (blue line), $t=50$ (green line), $t=100$ (red line).



a) $F(C)=C$ b) $F(C)=(1-\alpha)C+\alpha C^3$

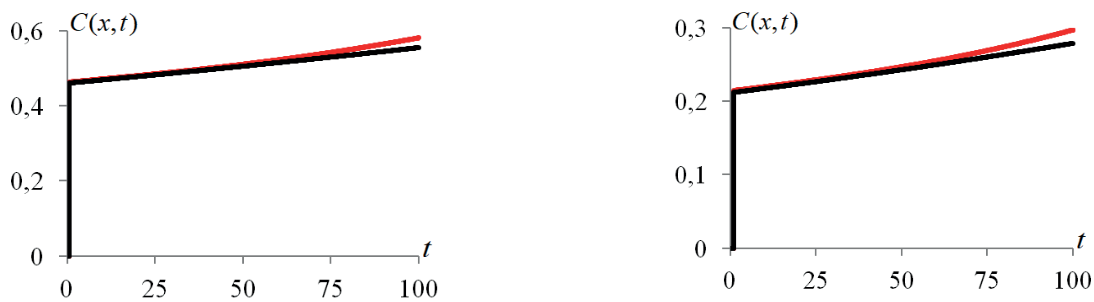
Figure 4. Numerical and asymptotic solutions of suspended particles concentrations at $t=100$



a) $F(C)=C$ b) $F(C)=(1-\alpha)C+\alpha C^3$

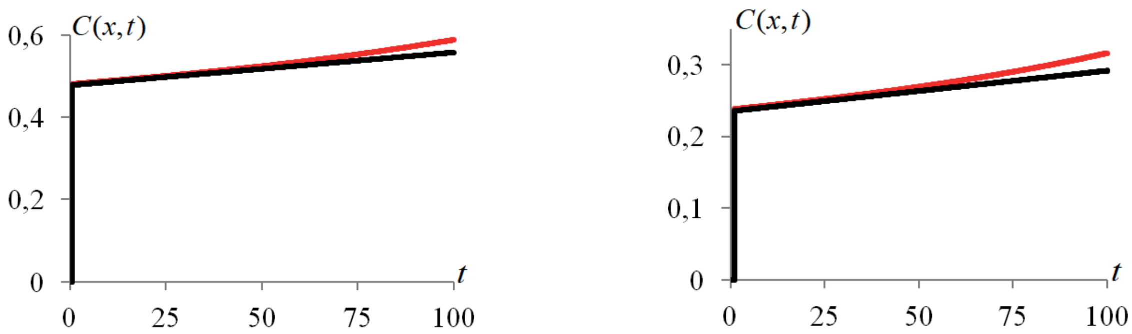
Figure 5. Relative errors of suspended particles concentration asymptotics for various t

In fig. 6 the graphs of retained particle concentrations at $x=0.5$ (fig. 6a) and at the filter outlet $x=1$ (fig. 6b) for the concentration function $F(C)=C$ are presented. The numerical solution is indicated by a red line, the asymptotic solution is indicated by a black line. Similar graphs for the case $F(C)=(1-\alpha)C+\alpha C^3$ are shown in fig. 7. The relative errors of the asymptotic concentration of retained particles for various filtration functions are shown in fig. 8. The case $x=0.5$ corresponds to the red line and the case $x=1$ corresponds to the black line.



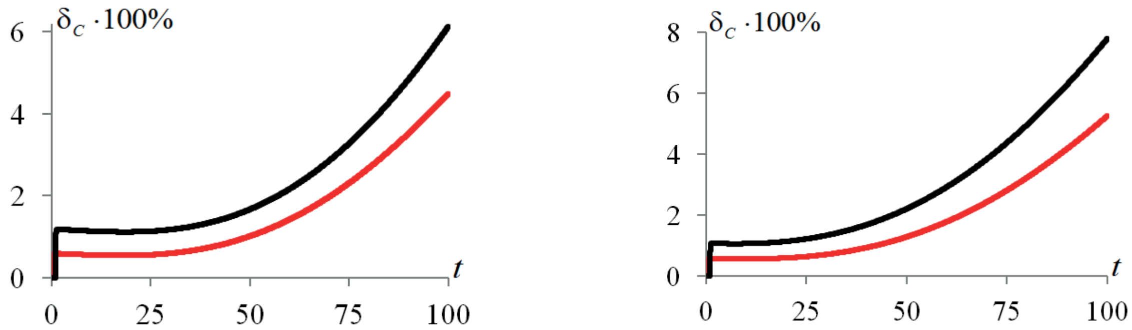
a) $x=0.5$ b) $x=1$

Figure 6. Numerical and asymptotic solutions of retained particle concentrations for $F(C)=C$



a) $x = 0.5$ b) $x = 1$

Figure 7. Numerical and asymptotic solutions of retained particle concentrations for $F(C) = (1-\alpha)C + \alpha C^3$



a) $F(C) = C$ b) $F(C) = (1-\alpha)C + \alpha C^3$

Figure 8. Relative errors of suspended particles concentration asymptotics at $x = 0.5$ *and* $x = 1$

The graphs show that the deviation of the asymptotic from the numerical solution increases with increasing time, approximately up to the time moment $t = 50$, the relative error of the asymptotic for both concentration functions is less than 2%, at the time $t = 100$ at the filter outlet, the relative error of the asymptotic in the case of a linear concentration function is slightly more than 6%, with a nonlinear function – 8%.

CONCLUSION

The mathematical model of deep filtration with a dimensional particle capture mechanism and mixed mechanism that includes size exclusion mechanism and multi particle bridging mechanism is considered in paper. With the first particle capture mechanism, solid particles freely pass through pores whose diameter is larger than the particle size and clog pores with a smaller diameter. With mixed mechanism, some solid particles freely pass through the pore necks, some get stuck in them, and others can block the pores, forming arch bridges. Several particles are connected, attached to the edges of the pore, thereby blocking it and preventing the suspended particles from entering the pore. In the considered model, the linear concentration function $F(C) = C$ describes size exclusion mechanism, the nonlinear function $F(C) = (1-\alpha)C + \alpha C^3$ describes mixed mechanism. In the filtration model, a polynomial of the third degree was used as a filtration function $\Lambda(S)$.

The constructed asymptotics of the concentrations of suspended and retained particles near the concentration front give good approximations to numerical solutions even in sufficiently large time intervals. The asymptotic solution of the suspended particle concentration of the second order, constructed for a nonlinear concentration function, is closer to the numerical solution, the relative error at the time is about 3%, while the error of the third-order asymptotic in the case of a linear concentration function is slightly more than 5%. For retained particles, the second-order asymptotic in the case of a linear concentration function better approximates the numerical solution than the first-order asymptotic when using a nonlinear concentration function, for example, at the filter outlet at a time $t = 100$, the relative error of the second-order asymptotic is 2% less than the error of the first-order asymptotic.

It should be noted that by limiting the asymptotic of suspended particle concentration to two terms with a nonlinear concentration function, an approximate solution for the linear filtration function is actually constructed. Thus, we can determine when a solution with a nonlinear filtering function begins to deviate significantly from the asymptotic of the linear function.

In field studies of reservoir water filtration in porous rock, measurement errors are at least 10 % [30]. Therefore, the proposed asymptotic formulas adequately describe the filtration process up to time $t = 100$.

REFERENCES

1. **Li B., Han J.** Conservation and regeneration of historical buildings // IOP Conference Series: Earth and Environmental Science, 2021, vol. 787, 012179.
2. **Makowski A., Polanska B.** Application of jet grouting technology for the renovation of historic buildings // IOP Conference Series: Journal of Physics, 2020, vol. 1425, 012201.
3. **Redina T.** Reconstuction of the historical building foksal 13 and 15 in Warsaw // Acta Polytechnica CTU Proceedings, 2019, vol. 23, pp. 44-48.
4. **Sengupta R.** Grouting for strengthening the Qutb Minar, Delhi // Mortars, Cements and Grouts used in the Conservation of Historic Buildings, 1981, pp. 147-164.
5. **Wanik K.** Use of jet grouting technique to realize substructures of historic buildings: The example of an apartment building in Warsaw // Geotechnical Engineering for the Preservation of Monuments and Historic Sites, 2013, pp. 769-777.
6. **Chernyakov A.V.** Application of jet grouting during foundation reinforcement and reconstruction of historic buildings on land occupied by the state museum and reservation «Tsarutsyno» // Soil Mechanics and Foundation Engineering, 2011, vol. 48(5), pp. 769-777.
7. **Hofmann H.** Unterfangungen mit einpressung von zementsuspension unter hohem Druck // Beton- und Stahlbetonbau, 1989, vol. 84(8), pp. 199-202.
8. **Morey J.J.** Les domaines d'application du jet-grouting // Revue Francaise de Geotechnique, 1992, vol. 61, 17-30.
9. **Drooff E.R., Furth A.J., Scarborough J. A.** Jet grouting to support historic buildings // Geotechnical Special Publication, 1995, vol. 50, pp. 42-55.
10. **Modoni G., Croce P., Mongiovi L.** Theoretical modelling of jet grouting // Geotechnique, 2006, vol. 56(5), pp. 335-347.
11. **Tien C.** Principles of filtration // Elsevier: Amsterdam, 2012.
12. **Tien C.** Introduction to cake filtration // Elsevier: Amsterdam, 2006.
13. **Sacramento R.N., Yang Y., You Z., Waldmann A., Martins A.L., Vaz A.S.L., Zitha P.L.J., Bedrikovetsky P.** Deep bed and cake filtration of two-size particle suspension in porous media // Journal of Petroleum Science and Engineering, 2015, vol. 126, pp. 201-210.
14. **Fallah H., Barzegar Fathi H., Mohammadi H.** The mathematical model for particle suspension flow through porous medium // Geomaterials, 2012, vol. 2, pp. 57-62.

15. **Fallah H., Fallah A., Rahmani A., Afkhami M., Ahmadi A.** Size exclusion mechanism, suspension flow through porous medium // International Journal of Modern Nonlinear Theory and Application, 2012, vol. 1, pp. 113-117.
16. **Santos A., Bedrikovetsky P.** Size exclusion during particle suspension transport in porous media: stochastic and averaged equations // Computational and Applied Mathematics, 2004, vol. 23(2-3), pp. 259-284.
17. **You Z., Bedrikovetsky P., Kuzmina L.** Exact solution for long-term size exclusion suspension-colloidal transport in porous media // Hindawi Publishing Corporation Abstract and Applied Analysis, 2013, vol. 2013, 680693.
18. **Kuzmina L.I., Osipov Yu.V.** Exact solution for 1D deep bed filtration with particle capture by advection and dispersion // International Journal of Non-Linear Mechanics, 2021, vol. 137, 103830.
19. **Nazaikinskii V.E., Bedrikovetsky P.G., Kuzmina L.I., Osipov Y.V.** Exact solution for deep bed filtration with finite blocking time // SIAM Journal on Applied Mathematics, 2020, vol. 80 (5), pp. 2120-2143.
20. **Kuzmina L.I., Osipov Y.V. and Zheglova Y.G.** Analytical model for deep bed filtration with multiple mechanisms of particle capture // International Journal of Non-linear Mechanics, 2018, vol. 105, pp. 242-248.
21. **Kuzmina L.I., Osipov Y.V., Gorbunova T.N.** Asymptotics for filtration of polydisperse suspension with small impurities // Applied Mathematics and Mechanics (English Edition), 2021, vol. 42(1), pp. 109-126.
22. **Osipov Y.V.** Calculation of filtration of polydisperse suspension in a porous medium // MATEC Web of Conferences, 2017, vol. 117, 00131.
23. **Kuzmina L.I., Osipov Yu.V.** Deep bed filtration asymptotics at the filter inlet // Procedia Engineering, 2016, vol. 153, pp. 366-370.
24. **Safina G.L.** Calculation of retention profiles in porous medium // Lecture Notes in Civil Engineering, 2021, vol. 170, pp. 21-28.
25. **Safina G.L.** Numerical solution of filtration in porous rock // E3S Web of Conferences, 2019, vol. 97, 05016.
26. **Osipov Yu., Safina G., Galaguz Yu.** Filtration model with multiple particle capture // Journal of Physics Conference Series, 2019, vol. 1425(1), 012110.
27. **Yang S., Russell T., Badalyan A., Schacht U., Woolley M., Bedrikovetsky P.** Characterisation of fines migration system using laboratory pressure measurements // Journal of Natural Gas Science and Engineering, 2019, vol. 65, pp. 108-124.
28. **Osipov Yu., Safina G., Galaguz Yu.** Calculation of the filtration problem by finite differences methods // MATEC Web of Conferences, 2018, vol. 251, 04021.
29. **You Z., Osipov Y., Bedrikovetsky P., Kuzmina L**. Asymptotic model for deep bed filtration // Chemical Engineering Journal, 2014, vol. 258, pp. 374-385.
30. **Dev Burman G.K., Das P.K.** Groundwater exploration in hard rock terrain: An experience from eastern India // The Hydrological Basis for Water Resources Management, 1990, vol. 197, pp. 19-30.

СПИСОК ЛИТЕРАТУРЫ

1. **Li B., Han J.** Conservation and regeneration of historical buildings // IOP Conference Series: Earth and Environmental Science, 2021, vol. 787, 012179.
2. **Makowski A., Polanska B.** Application of jet grouting technology for the renovation of historic buildings // IOP Conference Series: Journal of Physics, 2020, vol. 1425, 012201.

3. **Redina T.** Reconstuction of the historical building foksal 13 and 15 in Warsaw // Acta Polytechnica CTU Proceedings, 2019, vol. 23, pp. 44-48.
4. **Sengupta R.** Grouting for strengthening the Qutb Minar, Delhi // Mortars, Cements and Grouts used in the Conservation of Historic Buildings, 1981, pp. 147-164.
5. **Wanik K.** Use of jet grouting technique to realize substructures of historic buildings: The example of an apartment building in Warsaw // Geotechnical Engineering for the Preservation of Monuments and Historic Sites, 2013, pp. 769-777.
6. **Chernyakov A.V.** Application of jet grouting during foundation reinforcement and reconstruction of historic buildings on land occupied by the state museum and reservation «Tsarutsyno» // Soil Mechanics and Foundation Engineering, 2011, vol. 48(5), pp. 769-777.
7. **Hofmann H.** Unterfangungen mit einpressung von zementsuspension unter hohem Druck // Beton- und Stahlbetonbau, 1989, vol. 84(8), pp. 199-202.
8. **Morey J.J.** Les domaines d'application du jet-grouting // Revue Francaise de Geotechnique, 1992, vol. 61, 17-30.
9. **Drooff E.R., Furth A.J., Scarborough J. A.** Jet grouting to support historic buildings // Geotechnical Special Publication, 1995, vol. 50, pp. 42-55.
10. **Modoni G., Croce P., Mongiovi L.** Theoretical modelling of jet grouting // Geotechnique, 2006, vol. 56(5), pp. 335-347.
11. **Tien C.** Principles of filtration // Elsevier: Amsterdam, 2012.
12. **Tien C.** Introduction to cake filtration // Elsevier: Amsterdam, 2006.
13. **Sacramento R.N., Yang Y., You Z., Waldmann A., Martins A.L., Vaz A.S.L., Zitha P.L.J., Bedrikovetsky P.** Deep bed and cake filtration of two-size particle suspension in porous media // Journal of Petroleum Science and Engineering, 2015, vol. 126, pp. 201-210.
14. **Fallah H., Barzegar Fathi H., Mohammadi H.** The mathematical model for particle suspension flow through porous medium // Geomaterials, 2012, vol. 2, pp. 57-62.
15. **Fallah H., Fallah A., Rahmani A., Afkhami M., Ahmadi A.** Size exclusion mechanism, suspension flow through porous medium // International Journal of Modern Nonlinear Theory and Application, 2012, vol. 1, pp. 113-117.
16. **Santos A., Bedrikovetsky P.** Size exclusion during particle suspension transport in porous media: stochastic and averaged equations // Computational and Applied Mathematics, 2004, vol. 23(2-3), pp. 259-284.
17. **You Z., Bedrikovetsky P., Kuzmina L.** Exact solution for long-term size exclusion suspension-colloidal transport in porous media // Hindawi Publishing Corporation Abstract and Applied Analysis, 2013, vol. 2013, 680693.
18. **Kuzmina L.I., Osipov Yu.V.** Exact solution for 1D deep bed filtration with particle capture by advection and dispersion // International Journal of Non-Linear Mechanics, 2021, vol. 137, 103830.
19. **Nazaikinskii V.E., Bedrikovetsky P.G., Kuzmina L.I., Osipov Y.V.** Exact solution for deep bed filtration with finite blocking time // SIAM Journal on Applied Mathematics, 2020, vol. 80 (5), pp. 2120-2143.
20. **Kuzmina L.I., Osipov Y.V. and Zheglova Y.G.** Analytical model for deep bed filtration with multiple mechanisms of particle capture // International Journal of Non-linear Mechanics, 2018, vol. 105, pp. 242-248.
21. **Kuzmina L.I., Osipov Y.V., Gorbunova T.N.** Asymptotics for filtration of polydisperse suspension with small impurities // Applied Mathematics and Mechanics (English Edition), 2021, vol. 42(1), pp. 109-126.

22. **Osipov Y.V.** Calculation of filtration of polydisperse suspension in a porous medium // MATEC Web of Conferences, 2017, vol. 117, 00131.
23. **Kuzmina L.I., Osipov Yu.V.** Deep bed filtration asymptotics at the filter inlet // Procedia Engineering, 2016, vol. 153, pp. 366-370.
24. **Safina G.L.** Calculation of retention profiles in porous medium // Lecture Notes in Civil Engineering, 2021, vol. 170, pp. 21-28.
25. **Safina G.L.** Numerical solution of filtration in porous rock // E3S Web of Conferences, 2019, vol. 97, 05016.
26. **Osipov Yu., Safina G., Galaguz Yu.** Filtration model with multiple particle capture // Journal of Physics Conference Series, 2019, vol. 1425(1), 012110.
27. **Yang S., Russell T., Badalyan A., Schacht U., Woolley M., Bedrikovetsky P.** Characterisation of fines migration system using laboratory pressure measurements // Journal of Natural Gas Science and Engineering, 2019, vol. 65, pp. 108-124.
28. **Osipov Yu., Safina G., Galaguz Yu.** Calculation of the filtration problem by finite differences methods // MATEC Web of Conferences, 2018, vol. 251, 04021.
29. **You Z., Osipov Y., Bedrikovetsky P., Kuzmina L**. Asymptotic model for deep bed filtration // Chemical Engineering Journal, 2014, vol. 258, pp. 374-385.
30. **Dev Burman G.K., Das P.K.** Groundwater exploration in hard rock terrain: An experience from eastern India // The Hydrological Basis for Water Resources Management, 1990, vol. 197, pp. 19-30.

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