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NUMERICAL SOLUTION OF THE PROBLEM FOR POISSON'S EQUATION WITH THE USE OF DAUBECHIES WAVELET DISCRETE-CONTINUAL FINITE ELEMENT METHOD

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Abstract: Numerical solution of the problem for Poisson's equation with the use of Daubechies wavelet discrete-continual finite element method (specific version of wavelet-based discrete-continual finite element method) is under consideration in the distinctive paper. The operational initial continual and discrete-continual formulations of the problem are given, several aspects of finite element approximation are considered. Some information about the numerical implementation and an example of analysis are presented.

Keywords: Daubechies wavelet discrete-continual finite element method, wavelet-based discrete-continual finite element method, discrete-continual finite element method, finite element method, Daubechies wavelet, numerical solution, Poisson's equation.

ЧИСЛЕННОЕ РЕШЕНИЕ КРАЕВОЙ ЗАДАЧИ ДЛЯ УРАВНЕНИЯ ПУАССОНА НА ОСНОВЕ ДИСКРЕТНО-КОНТИНУАЛЬНОГО МЕТОДА КОНЕЧНЫХ ЭЛЕМЕНТОВ С ИСПОЛЬЗОВАНИЕМ МАСШТАБИРУЮЩИХ ФУНКЦИЙ ДОБЕШИ

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Аннотация: В настоящей статье рассматривается численное решение краевой задачи для уравнения Пуассона на основе дискретно-континуального метода конечных элементов с использованием масштабирующих функций Добеши. Приведены (в операторном виде) исходная континуальная и дискретно-континуальные постановки задачи, рассмотрены некоторые вопросы конечноэлементной аппроксимации. Представлены некоторые сведения о численной реализации и пример расчета.

Ключевые слова: вейвлет-реализация дискретно-континуального метода конечных элементов, дискретно-континуальный метод конечных элементов, метод конечных элементов, функции Добеши, численное решение, уравнение Пуассона.

INTRODUCTION

As is known [1], various problems of continuum mechanics are reduced to the Poisson equation and other similar equations of elliptic type [2-7]. Boundary value problems with the Poisson equation describe, in particular, a stationary temperature field, a stress state during torsion of a rod, membrane deflection, etc. In addition, the operator of the corresponding problem (the Laplace operator) is part of other problems that determine the state of structures under stationary and non-stationary actions.

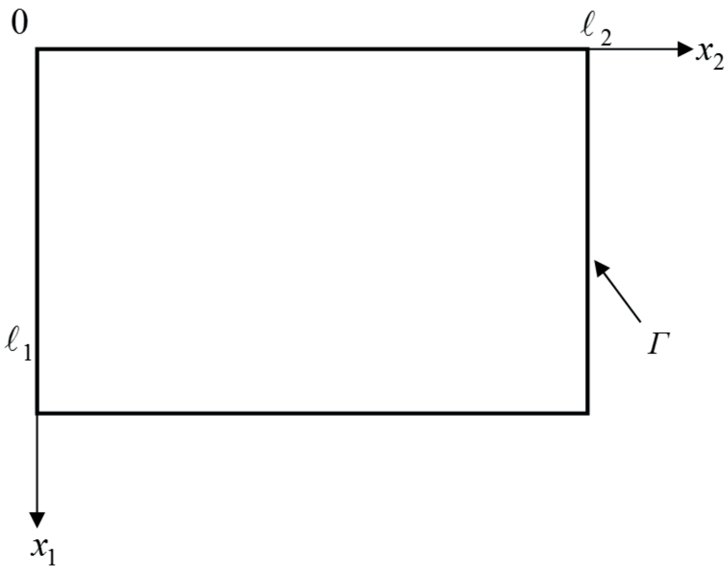

Figure 1.1. About formulation of the problem (initial domain).

From a mathematical point of view, it is the simplest qualitative analogue of other problems and an equivalent operator in iterative processes [8]. In many numerical models, at different time steps, it becomes necessary to solve (numerically) one or several boundary value problems for the Poisson equation, and in some applications the number of time steps during one analysis of the model can be of the order of thousands to millions or more [9]. In this regard, the objective of the distinctive paper is devoted to the semi-analytical method of analysis of corresponding structures with constant physical and geometric parameters in one of the directions (the so-called "basic direction") [8, 10, 11]. This objective seems to be very relevant. The considering method is semi-analytical in the sense that along the basic direction of the structure the problem remains continual and its exact analytical solution is constructed, while in another, non-basic direction, a numerical approximation is performed. In general, this paper continues a series of papers devoted to the research and development of various wavelet-based versions of the discrete-continuous finite element method.

In the theory of boundary value problems for the Poisson and Laplace equations, several classical well-tested solution methods are normally used [2, 12-14], which, in particular, include method of separation of variables or Fourier method, Green's function method and a method of reducing boundary value problems for the Laplace equation to integral equations using potential theory.

Besides, numerical methods (finite element method, boundary element method, finite difference method, variational-difference method, finite volume method, method of point field sources, fast Fourier transform method using parallel computations (with the implementation on the cores of the central processor and on graphic processors (GPU), etc.) for solving the Poisson equation are normally used [9, 15, 16].

1. FORMULATIONS OF THE PROBLEM

Formulation of the problem has the form (Figure 1.1):

$$L\,u = \tilde{F}\,, \quad 0 \le x_1 \le \ell_1\,, \quad 0 \le x_2 \le \ell_2\;; \quad (1.1)$$

$$L = -(\partial_1 \theta\, \partial_1 + \partial_2 \theta\, \partial_2)\,; \quad \tilde{F} = \theta F + \delta_\Gamma f\,; \quad (1.2)$$

$$F(x_1, x_2) = P\delta(x_1 - 0.5\cdot\ell_1)\delta(x_2 - 0.5\cdot\ell_2)\,; \quad (1.3)$$

$$f(x_1, x_2) = 0\,; \quad (1.4)$$

$$\partial_1 = \partial / \partial x_1\,; \quad \partial_2 = \partial / \partial x_2\,; \quad (1.5)$$

$$\theta(x_1, x_2) = \begin{cases} 1, & 0 < x_1 < \ell_1 \wedge 0 < x_2 < \ell_2 \\ 0\,, & \neg(0 < x_1 < \ell_1 \wedge 0 < x_2 < \ell_2), \end{cases} \quad (1.6)$$

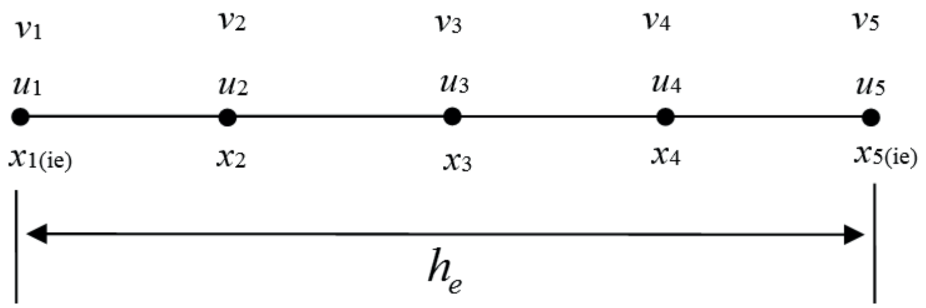

Figure 2.1. Finite element discretication for $N_k = 4$ (sample).

where L is the operator of the problem within the initial domain; $\theta(x_1, x_2)$ is the characteristic function of the domain; $\delta_\Gamma(x_1, x_2)$ is the delta-function of the boundary.

Let x_2 be direction along which parameters of the problem are constant (so-called “main direction”). Let us introduce the following notations

$$L_1 = \theta\,; \quad L_2 = -\partial_1 \theta\, \partial_1\,; \tag{1.7}$$

$$\bar{v} = \partial_2 \bar{u} = \bar{u}'\,; \quad \bar{v}' = \partial_2 \bar{v}\,. \tag{1.8}$$

Then we can rewrite (1.1) in the following form:

$$-L_1 v' + L_2 u = F\,. \tag{1.9}$$

Thus, we have

$$\begin{bmatrix} E & 0 \\ 0 & L_1 \end{bmatrix} \begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 0 & E \\ L_2 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} - \begin{bmatrix} 0 \\ \tilde{F} \end{bmatrix}; \tag{1.10}$$

or

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 0 & E \\ L_1^{-1} L_2 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} - \begin{bmatrix} 0 \\ L_1^{-1} \tilde{F} \end{bmatrix}, \tag{1.11}$$

where E is identity operator.

Finally we obtain system of differential equations with operational coefficients:

$$\bar{U}' = \tilde{L} \bar{U} + \tilde{\tilde{F}}\,, \tag{1.12}$$

where

$$\tilde{L} = \begin{bmatrix} 0 & E \\ L_1^{-1} L_2 & 0 \end{bmatrix}; \; \tilde{\tilde{F}} = \begin{bmatrix} 0 \\ -L_1^{-1} F \end{bmatrix}; \; \bar{U} = \begin{bmatrix} u \\ v \end{bmatrix}; \tag{1.13}$$

$$\bar{U}' = \partial_2 \bar{U} = \begin{bmatrix} \partial_2 u \\ \partial_2 v \end{bmatrix} = \begin{bmatrix} u' \\ v' \end{bmatrix}. \tag{1.14}$$

The system of equations (1.12) is supplemented by boundary conditions, which are set in sections with coordinates $x_2^1 = 0$ and $x_2^2 = \ell_2$.

For instance, for $\bar{U}(x_2)$ from the system of equations (1.12) we have

$$\bar{U}(0) = \bar{U}(\ell_2) = 0\,. \tag{1.15}$$

2. SOME ASPECTS OF THE FINITE ELEMENT APPROXIMATION

Let us divide the interval $(0, \ell_1)$ segment into N_e parts (elements). Therefore $h_e = \ell_1 / N_e$ is the length of the element. Besides, let us also divide each element into N_k parts (for instance, $N_k = 4$ (Figure 2.1)). Let us use the following notation system: i_e is the element number; $x_1(i_e)$ is the coordinate of the starting point of the i_e-th element; $x_5(i_e)$ is the coordinate of the end point of the i_e-th element. Let $u_i(x_2)$ and $v_i(x_2)$ ($i = 1,2,3,4,5$) be unknowns per element. Thus, the number of unknowns is equal to $2N$, where $N = 5$. The number of boundary nodes is equal to $N_b = N_e + 1$. The number of inner nodes for all elements is equal to $N_p = N_e(N_k - 1)$. Thus, the total (global) number of unknowns for such approximation is equal to $N_g = 2(N_p + N_b)$.

Let us introduce local coordinates for arbitrary element

$$t = (x - x_{1(ie)}) / h_e, \quad x_{1(ie)} \le x \le x_{5(ie)}, \ 0 \le t \le 1. \tag{2.1}$$

In this case, we have the following relations:

$$\begin{cases} x = x_{1(i_e)} \Rightarrow t = 0 \\ x = x_2 \Rightarrow t = 0.25 \\ x = x_3 \Rightarrow t = 0.5 \\ x = x_4 \Rightarrow t = 0.75 \\ x = x_{5(i_e)} \Rightarrow t = 1; \end{cases} \quad \frac{d}{dx} = \frac{d}{dt} \cdot \frac{dt}{dx} = \frac{1}{h_e} \frac{d}{dt};$$

$$dx = h_e \cdot dt. \tag{2.2}$$

In order to construct the local stiffness matrix corresponding to the operator L_2 (formula (1.7)), we consider the bilinear form taking into account relations (2.2)

$$\begin{aligned} B(y,z) &= < L_2 y, z >= -\int_{x_{1(i_e)}}^{x_{5(i_e)}} \frac{d^2 y}{dx^2} z\, dx = \\ &= \int_{x_{1(i_e)}}^{x_{5(i_e)}} \frac{dy}{dx} \cdot \frac{dz}{dx} dx = \frac{1}{h_e} \int_0^1 \frac{dw}{dt} \cdot \frac{dv}{dt} dt = \\ &= B(w,v), \end{aligned} \tag{2.3}$$

where we consider the following functions

$$y(x) = w(t) = \sum_{k=0}^{N-1} \alpha_k \varphi(t+k); \tag{2.4}$$

$$z(x) = v(t) = \sum_{k=0}^{N-1} \beta_k \varphi(t+k), \tag{2.5}$$

where $x_{1(i_e)} \le x \le x_{5(i_e)}$; $0 \le t \le 1$; $\varphi(s)$ is Daubechies scaling function, $[0, N] \subseteq \operatorname{supp} \varphi$. Let us substitute (2.4) and (2.5) into (2.3):

$$\begin{aligned} B(w,v) &= \frac{1}{h_e} \int_0^1 \frac{dw}{dt} \cdot \frac{dv}{dt} dt = \\ &= \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} \alpha_i \beta_j \frac{1}{h_e} \int_0^1 \varphi'(t+j) \varphi'(t+j)\, dt = \\ &= (K_{\alpha\beta}^{i_e} \overline{\alpha}, \overline{\beta}) = B_{i_e}(\overline{\alpha}, \overline{\beta}), \end{aligned} \tag{2.6}$$

where

$$K_{\alpha\beta}^{i_e}(i,j) = \frac{1}{h_e} \int_0^1 \varphi'(t+j) \varphi'(t+j)\, dt; \tag{2.7}$$

$$\varphi' = \frac{d\varphi}{dt}. \tag{2.8}$$

Let us define the parameters α_k through the nodal unknowns on the element:

$$\begin{aligned} y_1 &= w(0) = \sum_{k=0}^{N-1} \alpha_k \varphi(k) \\ y_2 &= w(0.25) = \sum_{k=0}^{N-1} \alpha_k \varphi(k+0.25) \\ y_3 &= w(0.5) = \sum_{k=0}^{N-1} \alpha_k \varphi(k+0.5) \\ y_4 &= w(0.75) = \sum_{k=0}^{N-1} \alpha_k \varphi(k+0.75) \\ y_5 &= w(1) = \sum_{k=0}^{N-1} \alpha_k \varphi(k+1). \end{aligned} \tag{2.9}$$

We can rewrite (2.9) in matrix form:

$$\overline{y}^{i_e} = T\overline{\alpha}, \tag{2.10}$$

where we have

$$\overline{y}^{i_e} = [y_1 \ y_2 \ y_3 \ y_4 \ y_5]^T; \tag{2.11}$$

$$\overline{\alpha} = [\alpha_0 \ \alpha_1 \ \alpha_2 \ \alpha_3 \ \alpha_4]^T; \tag{2.12}$$

$$T = \\ = \begin{bmatrix} \varphi(0) & \varphi(1) & \varphi(2) & \varphi(3) & \varphi(4) \\ \varphi(0.25) & \varphi(1.25) & \varphi(2.25) & \varphi(3.25) & \varphi(4.25) \\ \varphi(0.5) & \varphi(1.5) & \varphi(2.5) & \varphi(3.5) & \varphi(4.5) \\ \varphi(0.75) & \varphi(1.75) & \varphi(2.75) & \varphi(3.75) & \varphi(4.75) \\ \varphi(1) & \varphi(2) & \varphi(3) & \varphi(4) & \varphi(5) \end{bmatrix}. \tag{2.13}$$

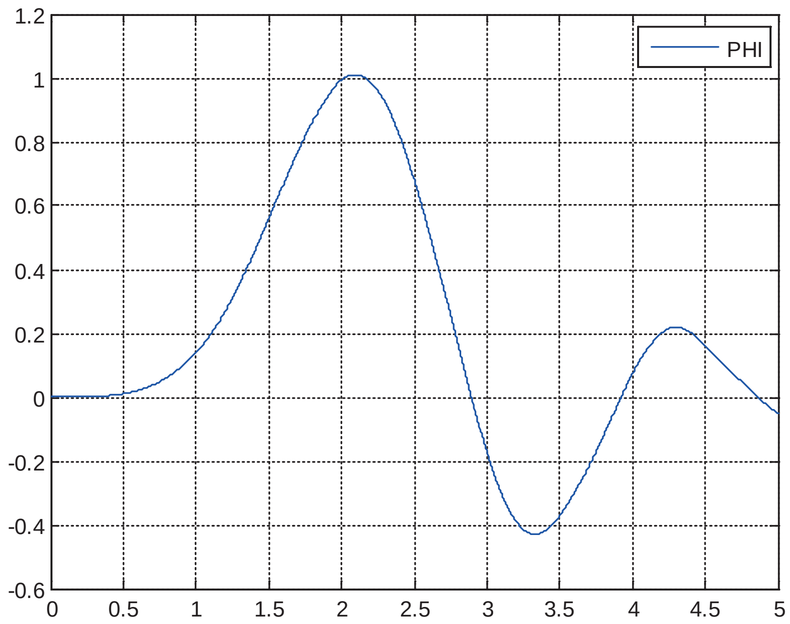

Figure 3.1. Daubechies scaling function.

We can also get

$$\bar{z}^{i_e} = T\bar{\beta}\,. \tag{2.14}$$

Using (2.10) and (2.14) we can obtain

$$\bar{\alpha} = T^{-1}\bar{y}^{i_e}\,; \quad \bar{\beta} = T^{-1}\bar{z}^{i_e}\,. \tag{2.15}$$

Thus, we can rewrite (2.6) in the following form:

$$\begin{aligned}(K^{i_e}_{\alpha\beta}\bar{\alpha}, \bar{\beta}) &= (K^{i_e}_{\alpha\beta}T^{-1}\bar{y}^{i_e}, T^{-1}\bar{z}^{i_e}) = \\ &= ((T^{-1})^{\mathrm{T}} K^{i_e}_{\alpha\beta}T^{-1}\bar{y}^{i_e}, \bar{z}^{i_e}) = \\ &= (K^{i_e}\bar{y}^{i_e}, \bar{z}^{i_e}),\end{aligned} \tag{2.16}$$

where

$$K^{i_e} = (T^{-1})^{\mathrm{T}} K^{i_e}_{\alpha\beta} T^{-1} \tag{2.17}$$

is the local stiffness matrix.

3. NUMERICAL IMPLEMENTATION

The presented algorithm can be implemented using MATLAB software tools. In particular, the call to the standard function

```
wavefun('db8',0)
```

allows to obtain the values of the Daubechies scaling function [17-32] φ (Figure 3.1) on the interval (segment) $[0,\,15] = \sup\mathrm{p}\varphi$ with a step $h_t = 1/256 = 2^{-8}$. Let us denote $N_t = 256 = 2^8$. For the value under consideration ($N = 5$) we can use the first $N_l = N_t \cdot N + 1$ values determined on the interval $[0,\,N] = [0,\,5]$. With such a small step, we find it will natural to compute the derivatives (Figure 3.2) in the form of finite differences:

$$\varphi'(t_k) \approx d\varphi_k = \frac{\varphi_{k+1} - \varphi_{k-1}}{2h_t}, \quad k = 1, 2, \ldots, N_l\,, \tag{3.1}$$

where we have $\varphi_k = \varphi(t_k)$ and $t_k = k \cdot h_t$. If $t_k \notin [0,19]$ then $\varphi_k = \varphi(t_k) = 0$.

When computing the coefficients of the local stiffness matrix (formula (2.7)), one can use the simplest quadrature formulas for numerical integration, in particular, the formula for "mean" rectangles with a step $2h_t$.

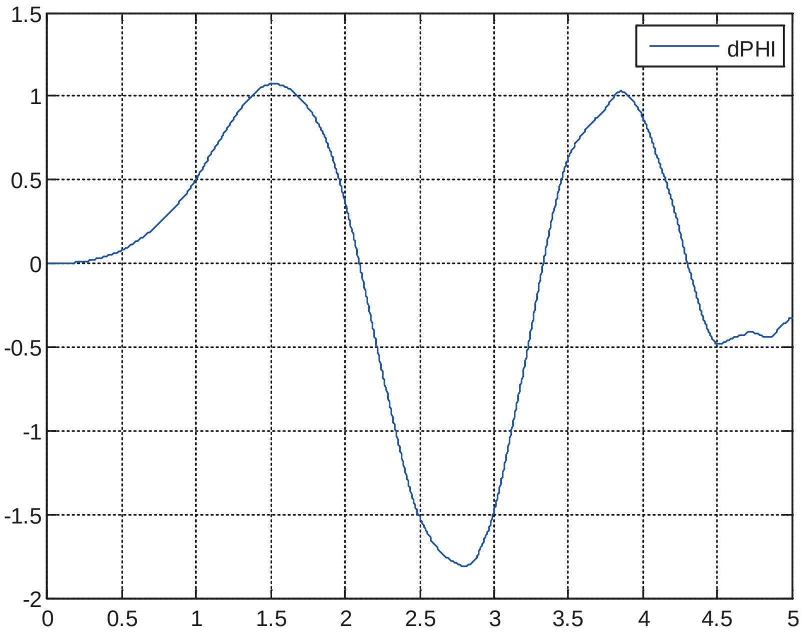

Figure 3.2. The first direvative (finite-difference) of Daubechies scaling function.

4. EXAMPLE OF ANALYSIS

For the numerical implementation, let us set, in particular, the following numerical parameters:

$$P = 100\,;\quad \ell_1 = 2.0\,;\quad \ell_2 = 2.6\,.$$

Let $N_e = 16$ be the number of elements (finite elements). Then the total number of nodal points in the discrete direction is equal to

$$N_1 = N_p + N_b = 3 \cdot 16 + 17 = 65\,.$$

Then the total number of unknowns is equal to

$$N_g = 2N_1 = 130\,.$$

The length of the element is equal to

$$h_e = \ell_1 / N_e = 2/16 = 0.125\,.$$

The distance between the coordinates of the nodes is equal to

$$h_p = h_e / 4 = 0.125/4 = 0.03125\,.$$

For comparison (verification purpose), we can use the variational-difference discrete-continual method with a step of discretization h_p along the discrete direction x_1.

Graphical comparison of the corresponding results is presented at Figure 4.1, where we use the following notation system: `Udb` is the result obtained using the Daubechies scaling function; `Uvr` is the result obtained on the basis of the variational-difference method; `d1U` is a finite-difference analogue of the derivative with a step; `d2U` is derivative, $\partial_2 u = v$; $h_1 = h_p = 0.03125$ and $h_2 = 0.1$ are steps for visualization of results along directions x_1 and x_2 respectively.

As researcher can see, the results obtained are almost completely identical.

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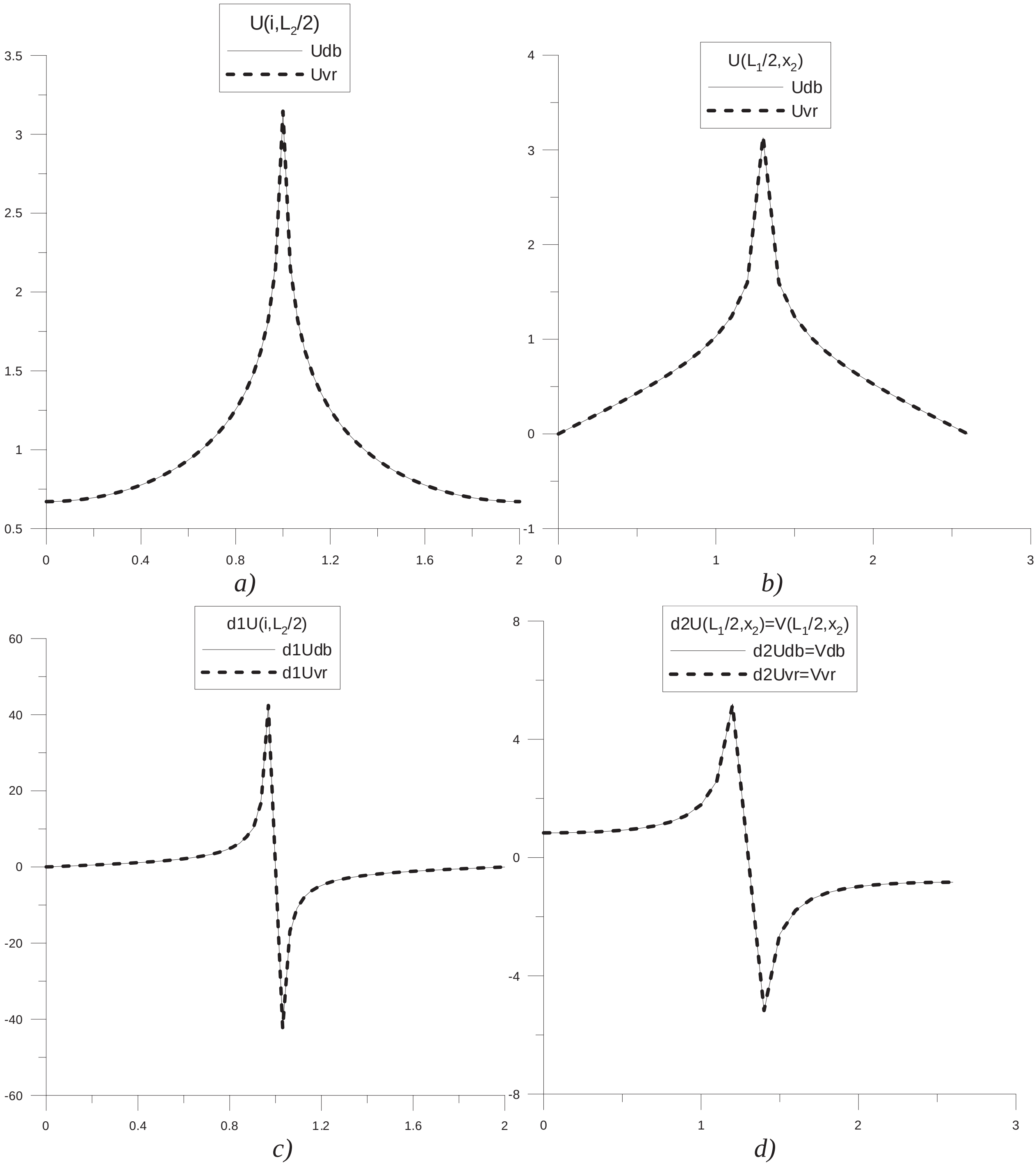

Figure 4.1. Comparison of results.

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