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FORCE DRIVEN VIBRATIONS OF NONLINEAR PLATES ON A VISCOELASTIC WINKLER FOUNDATION UNDER THE HARMONIC MOVING LOAD

Marina V. Shitikova, Anastasiya I. Krusser
Voronezh State Technical University, Voronezh, RUSSIA

Abstract: In the present paper, the forced driven nonlinear vibrations of an elastic plate in a viscoelastic medium and resting on a viscoelastic Winkler-type foundation are studied. The damping features of the surrounding medium and foundation are described by the Kelvin-Voigt model and standard linear solid model with fractional derivatives, respectively. The dynamic response of the plate is described by the set of nonlinear differential equations with due account for the fact that the plate is being under the conditions of the internal resonance accompanied by the external resonance. The expressions for the stress function and nonlinear coefficients for different types of boundary conditions are presented.

Keywords: nonlinear vibrations of thin plates, viscoelastic Winkler-type foundation, fractional derivative model, boundary conditions, combination of internal and external resonances

АНАЛИЗ ВЫНУЖДЕННЫХ НЕЛИНЕЙНЫХ КОЛЕБАНИЙ ПЛАСТИНКИ НА ВЯЗКОУПРУГОМ ОСНОВАНИИ ВИНКЛЕРА ОТ ДЕЙСТВИЯ ПОДВИЖНОЙ НАГРУЗКИ

М.В. Шитикова, А.И. Круссер
Воронежский государственный технический университет, г. Воронеж, РОССИЯ

Аннотация: Исследованы нелинейные вынужденные колебания упругой пластины в вязкоупругой среде и на вязкоупругом основании Винклера. Демпфирующие свойства окружающей среды и основания описываются при помощи моделей Кельвина-Фойгта и стандартного линейного твердого тела с дробной производной соответственно. Колебания пластинки описываются системой нелинейных дифференциальных уравнений, с учетом того, что пластинка находится в условиях сочетания внутреннего и внешнего резонансов. Получены выражения для функции напряжений и коэффициентов при нелинейных членах для различных типов граничных условий.

Ключевые слова: нелинейные колебания пластинок, вязкоупругое основание типа Винклера, модель с дробной производной, граничные условия, сочетание внутреннего и внешнего резонансов

1. INTRODUCTION

Nonlinear dynamic response of elastic plates on the viscoelastic foundation is of great interest among researchers in the recent years. The analysis of free and force driven vibrations of nonlinear systems is of great importance for defining the dynamic parameters dependent on the amplitude-phase relationships and modes of vibration [1-3]. Moreover, nonlinear vibrations could be accompanied by such a phenomenon as the internal resonance, resulting in strong coupling between the modes of vibrations involved, and hence in the energy exchange between the interacting modes. The described type of resonance can be characterized as a structural resonance, since external resonance, for example, could be eliminated by the simplest change in the frequency of the harmonic force. Both types of resonances separately are extremely unfavorable phenomena, and their combination can lead to the distraction of not only one element, but the entire structure as a whole. The process of the nonlinear vibrations of the plates under conditions of internal resonances has been widely studied in [4-7].

Depending on the applications of the considered foundation, several models have been proposed to describe their properties [8]. The viscoelastic Winkler-type or Pasternak-type foundations, damping features of which are described by the fractional derivative approach, are becoming increasingly widespread nowadays due to the important role of fractional calculus in solving problems of structural mechanics [9].

In the literature there are mainly reports on the analysis of vibrations of plates on a viscoelastic foundation in the linear formulation of the problem [10-17]. Nevertheless, there are papers in which nonlinear vibrations of plates on the foundation are considered. Thus, large deflection dynamic response of isotropic thin rectangular plates resting on Winkler, Pasternak and nonlinear Winkler elastic foundations was investigated in [18]. The dynamic response of a rectangular nonlinear plate resting on a viscoelastic Winkler-type foundation, the damping features of which are described by the fractional derivative Kelvin-Voigt model, for the first time was studied in [19]. The standard linear solid model with fractional derivatives for defining the viscoelastic properties of the Winkler-type foundation was applied in [20] for the analysis of free vibrations of the plate.

The problems of dynamic response of the rectangular plate supported by viscoelastic foundations and subjected to moving loads are studied in [16, 21-23]. This problem could find many engineering applications, such as aircraft–runway interaction or vehicle-road interaction, pavement-foundation system, dynamics of the helipad system, ship deck (especially aircraft carriers), soil-foundation system of offshore structures, railway track system, reinforced warehouse floor, etc [1].

However, the dynamic problems considering time-dependent properties of the material of the plate or foundation are usually restricted to simply supported plates. But in the literature there are solutions for rectangular plates with different combinations of simple boundary conditions (i.e., either clamped (C), simply supported (SS), or free (F)) [24]. Thus, nonlinear frequencies of vibrations of rectangular plates for three different types of boundary conditions (B.Cs) have been calculated in [1]. The linear dynamic response of thin plates resting on a fractional derivative Kelvin-Voigt viscoelastic foundation subjected to a moving point load is investigated in [16] for four types of boundary conditions. Semi-analytical solutions and comparative analysis of natural frequencies and midpoint displacements for vibration of the viscoelastic Kirchhoff–Love plate on the Kelvin-Voigt viscoelastic foundation with various B.Cs are presented in [25].

In the present paper, the nonlinear vibrations of the “elastic plate – viscoelastic foundation” system is studied for the case of combination of one-to-one internal and external resonances. The properties of the foundation and of the surrounding medium are described by the fractional derivative standard linear solid model and Kelvin–Voigt model, respectively. The nonlinear force driven vibrations of the plate on the viscoelastic foundation are studied under the harmonic moving load for different types of boundary conditions.

2. PROBLEM FORMULATION

Let us consider nonlinear vibrations of a simply supported elastic plate rested on a viscoelastic Winkler-type foundation (Fig.1), the dynamic response of which is described by the von Karman equation in terms of plate’s lateral deflection $w = w(x, y, t)$ and Airy’s stress function ϕ:

$$D\nabla^4 w + \rho h \frac{\partial^2 w}{\partial t^2} - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 \phi}{\partial y^2} - \frac{\partial^2 w}{\partial y^2}\frac{\partial^2 \phi}{\partial x^2} + +2\frac{\partial^2 w}{\partial x \partial y}\frac{\partial^2 \phi}{\partial x \partial y} = q - F_1 - F_2, \quad (1)$$

$$\nabla^4 \phi = Eh\left[\left(\frac{\partial w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right], \quad (2)$$

where $D = Eh^3/12(1-\nu^2)$ is the plate's cylindrical rigidity, E and ν are the elastic modulus and Poisson's ratio of the plate's material, respectively, h and ρ are its thickness and density, t is the time, $q = P\delta(x - g(t))\delta(y - b/2)\sin\Omega_F t$ is external load, P and Ω_F are the magnitude and the frequency of the applied force, respectively, δ is the Dirac delta function, and $g(t)$ is the function defining the position of moving load, $g(t) = Vt$ for a load moving with a constant velocity. Besides, $g(t)$ must satisfy $0 \le g(t) \le a$.

In equation (1), F_2 is the reaction force of the viscoelastic Winkler-type foundation, $F_1 = æ_1 \tau_1^{\gamma_1} D_{0+}^{\gamma_1} w$ is the damping force of the viscoelastic medium possessing the retardation time τ_1 and damping coefficient $æ_1$, which is modeled by the viscoelastic Kelvin-Voigt model with the Riemann-Liouville derivative $D_{0+}^{\gamma_1}$ of the fractional order γ_1 $(0 < \gamma_1 \le 1)$ [9,26]

$$D_{0+}^{\gamma} x(t) = \frac{d}{dt}\int_0^t \frac{x(t-t')dt'}{\Gamma(1-\gamma)t'^{\gamma}} \quad (0 < \gamma = \gamma_1 \le 1), \quad (3)$$

and $\Gamma(1-\gamma)$ is the Gamma function.

Let us assume, following [27], that the compliance operator of a viscoelastic foundation is described by the standard linear solid model with the Riemann-Liouville fractional derivative D_{0+}^{γ} (3) at $\gamma = \gamma_2$:

$$\tilde{\lambda} = \lambda_\infty \left[1 - \nu_\varepsilon \frac{1}{1 + \tau_2^{\gamma} D_{0+}^{\gamma_2}}\right], \quad (4)$$

where λ_∞ is the coefficient of instantaneous compliance of the foundation, $\nu_\varepsilon = \Delta\lambda\lambda_\infty^{-1}$, $\Delta\lambda = \lambda_\infty - \lambda_0$ is the defect of the compliance, i.e., the value characterizing the decrease in the compliance operator from its non-relaxed value to its relaxed value.

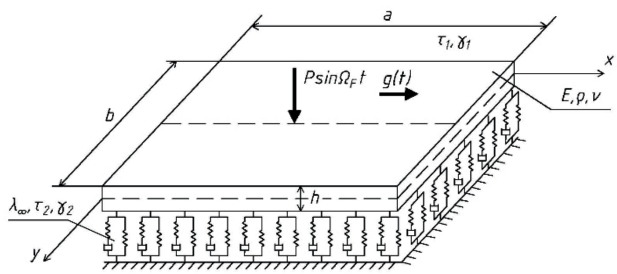

Figure 1. Plate on a viscoelastic foundation subjected to a moving harmonic load

The following boundary conditions could be added to the set of equations (1) and (2) at each edge:

1) Simply supported edges (S)

$$\text{at } x = 0 \text{ and } a, \quad w = \frac{\partial^2 w}{\partial x^2} = 0,$$
$$\text{at } y = 0 \text{ and } b, \quad w = \frac{\partial^2 w}{\partial y^2} = 0; \quad (5)$$

2) Clamped edges (C)

$$\text{at } x = 0 \text{ and } a, \quad w = \frac{\partial w}{\partial x} = 0,$$
$$\text{at } y = 0 \text{ and } b, \quad w = \frac{\partial w}{\partial y} = 0. \quad (6)$$

Therefore, the following four types of boundary conditions (B.Cs) for the plate will be considered: all edges are simply-supported (SSSS), all edges are clamped (CCCC), two opposite edges are clamped and other two edges are simply supported (CSCS), and one edge is simply supported and others are clamped (CCSC). In the abbreviation of B.Cs the letter symbols are used, for example, CSCS means a plate with edges $x=0$ and $x=a$ clamped (C), $y=0$ and $y=b$ simply supported (S).

In order to identify the possibility of the occurrence of the internal resonance during nonlinear vibrations of a plate rested on a viscoelastic foundation and to carry out its subsequent analysis, suppose that only two natural modes of vibrations with numbers m_1n_1 и m_2n_2 are excited. Then the deflection of the plate could be represented in the following form:

$$w(x,y,t)=x_1(t)W_{m_1n_1}(x,y)+x_2(t)W_{m_2n_2}(x,y), \quad (7)$$

where $x_i(t)$ $(i=1,2)$ are generalized displacements, and $W_{m_in_i}(x,y)$ are the eigen functions. The mode shape functions for various B.Cs are presented in Table1 [25].

<u>Table 1.</u> Plate mode shapes and natural frequencies for different B.Cs.

B.Cs.	$W_{m_in_j}(x,y)$	Ω_i^2
SSSS	$\sin\frac{\pi m_i x}{a}\sin\frac{\pi n_i y}{b}$	$\Omega_i^2=\frac{E\pi^4h^2}{12\rho(1-\nu^2)b^4}\left(\xi^2m_i^2+n_i^2\right)^2$
CCCC	$(1-\cos\frac{2\pi m_i x}{a})(1-\cos\frac{2\pi n_i y}{b})$	$\Omega_i^2=\frac{4E\pi^4h^2}{27\rho(1-\nu^2)b^4}\left(3\xi^4m_i^4+2\xi^2m_i^2n_i^2+3n_i^4\right)$
CSCS	$(1-\cos\frac{2\pi m_i x}{a})\sin\frac{\pi n_i y}{b}$	$\Omega_i^2=\frac{E\pi^4h^2}{9\rho(1-\nu^2)b^4}\left(4\xi^4m_i^4+2\xi^2m_i^2n_i^2+0.75n_i^4\right)$
CCSC	$(\cos\frac{3\pi m_i x}{2a}-\cos\frac{\pi m_i x}{2a})(1-\cos\frac{2\pi n_i y}{b})$	$\Omega_i^2=\frac{E\pi^4h^2}{18\rho(1-\nu^2)b^4}\left(3.85\xi^4m_i^4+5\xi^2m_i^2n_i^2+8n_i^4\right)$

Substituting the proposed solution (7) in (2), taking into account the boundary conditions to be considered for each specific case and integrating with account for the orthogonality conditions of sines, we obtain the expressions for the stress function for different types of boundary conditions, which are presented in Appendix A.

Substituting the assumed two-term expansion for the deflection function of the plate (7) and the corresponding stress function in the equation of motion of the plate (1) resting on the viscoelastic Winkler-type foundation yields the following set of nonlinear differential equations with respect to the generalized displacements:

$$\ddot{x}_1+\omega_1^2x_1+\alpha_1x_1^3+\alpha_2x_1x_2^2+\varepsilon^2\mu_1D_{0+}^{\gamma_1}x_1- \\ -\varepsilon^2\mu_2\,\mathfrak{Э}_\gamma^*(\tau_2^{\gamma_2})x_1=P_1(t), \quad (8)$$

$$\ddot{x}_2+\omega_2^2x_2+\alpha_3x_2^3+\alpha_4x_2x_1^2+\varepsilon^2\mu_1D_{0+}^{\gamma_1}x_2- \\ -\varepsilon^2\mu_2\,\mathfrak{Э}_\gamma^*(\tau_2^{\gamma_2})x_2=P_2(t), \quad (9)$$

where

$$P_i(t) = \left(\int_0^a \int_0^b P \sin \Omega_F t \, \delta(x - g(t)) \delta(y - 0.5b) \times \right. \\ \left. \times W_{m_i n_i}(x, y) \, dxdy \right) / \rho h \int_0^a \int_0^b \left[W_{m_i n_i}(x, y) \right]^2 dxdy, \tag{10}$$

α_i are the coefficients depending on numbers of the vibration modes which are given in Appendix B, ε is a small dimensionless parameter, μ_i are finite values, $\varepsilon^2 \mu_1 = \dfrac{E_0 \tau_1^{\gamma_1}}{\rho h}$, $\varepsilon^2 \mu_2 = \dfrac{\lambda_\infty \nu_\varepsilon}{\rho h}$, ω_1^2 and ω_2^2 are vibration frequencies of the mechanical system "plate + viscoelastic foundation"

$$\omega_i^2 = \Omega_i^2 + \frac{\lambda_\infty}{\rho h}, \tag{11}$$

and Ω_i^2 are the natural frequencies of the linear vibration of the plate presented in Table 1, and $\vartheta_\gamma^* (\tau_2^{\gamma_2})$ is the Rabotnov dimensionless fractional operator defined as follows [28]

$$\vartheta_\gamma^* (\tau_2^{\gamma_2}) = \frac{1}{1 + \tau_2^{\gamma_2} D_{0+}^{\gamma_2}}. \tag{12}$$

When deriving relations (8) and (9), the filtering property of the delta function should be taken into account:

$$\iint \delta(x - x_0) \delta(y - y_0) f(x, y) \, dxdy = f(x_0, y_0). \tag{13}$$

Considering (13), equation (10) for the case of a simply supported plate is reduced as

$$P_i(t) = P \sin \Omega_F t \iint \delta(x - Vt) \delta(y - b/2) \times \\ \times \sin\left(\frac{\pi m_i x}{a}\right) \sin\left(\frac{\pi n_i y}{b}\right) dxdy = \\ = P \sin\left(\frac{\pi m_i V t}{a}\right) \sin\left(\frac{\pi n_i}{2}\right) \sin \Omega_F t. \tag{14}$$

Then with due account for (14) the governing equations (8) and (9) could be written as

$$\ddot{x}_1 + \omega_1^2 x_1 + \alpha_1 x_1^3 + \alpha_2 x_1 x_2^2 + \varepsilon^2 \mu_1 D_{0+}^{\gamma_1} x_1 - \\ -\varepsilon^2 \mu_2 \, \vartheta_\gamma^* (\tau_2^{\gamma_2}) x_{11} - 4\varepsilon^3 f_1 \sin \omega_{f_1} t \sin \Omega_F t = 0, \tag{15}$$

$$\ddot{x}_2 + \omega_2^2 x_2 + \alpha_3 x_2^3 + \alpha_4 x_2 x_1^2 + \varepsilon^2 \mu_1 D_{0+}^{\gamma_1} x_2 - \\ -\varepsilon^2 \mu_2 \, \vartheta_\gamma^* (\tau_2^{\gamma_2}) x_2 - 4\varepsilon^3 f_2 \sin \omega_{f_2} t \sin \Omega_F t = 0, \tag{16}$$

where $\omega_{f_i} = \dfrac{\pi m_i V}{a}$ are frequencies, and $\varepsilon^3 f_i = \dfrac{P}{ab\rho h} \sin\left(\dfrac{\pi n_i}{2}\right)$.

3. METHOD OF SOLUTION

In order to solve the set of Eqs. (15)-(16), the method of multiple time scales [29,30] could be utilized, according to which the generalized displacements $x_i(t)$ could be represented via the following expansion in two time scales T_0 and T_2:

$$x_i(t) = \varepsilon X_{i1}(T_0, T_2) + \varepsilon^2 X_{i2}(T_0, T_2) + \\ + \varepsilon^3 X_{i3}(T_0, T_2) + \ldots, \tag{17}$$

where $T_n = \varepsilon^n t$ are new independent variables, among them: $T_0 = t$ is a fast scale characterizing motions with the natural frequencies, and $T_2 = \varepsilon t^2$ is a slow scale characterizing the modulation of the amplitudes and phases of the modes with nonlinearity.

Recall that the first and the second time derivatives, as well as fractional derivative could be expanded in terms of the new time scales, respectively, as follows [29]:

$$\begin{aligned} &\frac{d}{dt} = D_0 + \varepsilon^2 D_2 + ..., \\ &\frac{d^2}{dt^2} = D_0^2 + 2\varepsilon^2 D_0 D_2 + ... \end{aligned} \tag{18}$$

$$\begin{aligned} D_+^\gamma &= \left(\frac{d}{dt}\right)^\gamma = \left(D_0 + \varepsilon^2 D_2 + ...\right)^\gamma = \\ &= D_0^\gamma + \varepsilon^2 \gamma D_0^{\gamma-1} D_2 + ..., \end{aligned} \tag{19}$$

where $D_0 = \partial / \partial T_0$, and $D_2 = \partial / \partial T_2$.

Expansion of the Rabotnov dimensionless fractional operator in a Taylor series in terms of a small parameter has the form [20]:

$$\begin{aligned} \text{э}_\gamma^* (\tau^\gamma) &= \frac{1}{1+\tau^\gamma D_{0+}^\gamma} = (1+\tau^\gamma D_0^{\ \gamma})^{-1} - \\ &-\varepsilon^2 (1+\tau^\gamma D_0^{\ \gamma})^{-2} \tau^\gamma \gamma D_0^{\ \gamma-1} D_2 + ... \end{aligned} \tag{20}$$

Substituting expansion (17) with account for relationships (18)-(19), after equating the coefficients at like powers of ε to zero, we are led for the case of forced vibrations to the following set of recurrence equations to various orders:

to order ε

$$D_0^2 X_{11} + \omega_1^2 X_{11} = 0\,, \tag{21}$$

$$D_0^2 X_{21} + \omega_2^2 X_{21} = 0\,, \tag{22}$$

to order ε^3

$$\begin{aligned} &D_0^2 X_{13} + \omega_1^2 X_{13} = -2D_0 D_2 X_{11} - X_{11}\left(\bar{\mu}_1 D_0^{\gamma_1} - \right. \\ &\left. -\bar{\mu}_2 (1+\tau_2^{\gamma_2} D_0^{\gamma_2})^{-1}\right) - \alpha_1 X_{11}^3 - \alpha_2 X_{11} X_{21}^2 + \\ &+2 f_1 \cos\left((\omega_{f_1} - \Omega_F) T_0\right) + 2 f_1 \cos\left((\omega_{f_1} + \Omega_F) T_0\right), \end{aligned} \tag{23}$$

$$\begin{aligned} &D_0^2 X_{23} + \omega_2^2 X_{23} = -2D_0 D_2 X_{21} - X_{21}\left(\bar{\mu}_1 D_0^{\gamma_1} - \right. \\ &\left. -\bar{\mu}_2 (1+\tau_2^{\gamma_2} D_0^{\gamma_2})^{-1}\right) - \alpha_3 X_{21}^3 - \alpha_4 X_{21} X_{11}^2 + \\ &+2 f_2 \cos\left((\omega_{f_2} - \Omega_F) T_0\right) + 2 f_2 \cos\left((\omega_{f_2} + \Omega_F) T_0\right), \end{aligned} \tag{24}$$

where $\bar{\mu}_i = \dfrac{\mu_i}{\rho h}$ $(i = 1, 2)$.

The solution of linear equations (21) and (22) has the form

$$\begin{aligned} &X_{j1} = A_j\left(T_2\right)\exp\left(i\omega_j T_0\right) + \\ &+\bar{A}_j\left(T_2\right)\exp\left(-i\omega_j T_0\right), \end{aligned} \tag{25}$$

where $A_j\left(T_2\right)\left(j = 1, 2\right)$ are yet unknown functions, and $\bar{A}_j\left(T_2\right)$ are conjugate functions with $A_j\left(T_2\right)$.

Substituting relationships (25) in equations (23) and (24) yields

$$\begin{aligned} &D_0^2 X_{13} + \omega_1^2 X_{13} = -2i\omega_1 D_2 A_1 \exp\left(i\omega_1 T_0\right) - \\ &-\left[\bar{\mu}_1 \left(i\omega_1\right)^{\gamma_1} - \bar{\mu}_2 (1+\tau_2^{\gamma_2} \left(i\omega_1\right)^{\gamma_2})^{-1}\right] A_1 \exp\left(i\omega_1 T_0\right) - \\ &-\alpha_1 \left[A_1 \exp\left(3i\omega_1 T_0\right) + 3\bar{A}_1 \exp\left(i\omega_1 T_0\right)\right] A_1^2 - \\ &-\alpha_2 \left\{A_2^2 \exp\left[\left(\omega_1 + 2\omega_2\right) T_0\right] + 2A_2\bar{A}_2 \exp\left(i\omega_1 T_0\right) + \right. \\ &\left. +\bar{A}_2^2 \exp\left[i\left(\omega_1 - 2\omega_2\right) T_0\right]\right\} A_1 + f_1 \exp\left(i(\omega_{f_1} - \Omega_F) T_0\right) + \\ &+ f_1 \exp\left(i(\omega_{f_1} + \Omega_F) T_0\right) + cc, \end{aligned} \tag{26}$$

$$\begin{aligned} &D_0^2 X_{23} + \omega_2^2 X_{23} = -2i\omega_2 D_2 A_2 \exp\left(i\omega_2 T_0\right) - \\ &-\left[\bar{\mu}_1 \left(i\omega_2\right)^{\gamma_1} - \bar{\mu}_2 (1+\tau_2^{\gamma_2} \left(i\omega_2\right)^{\gamma_2})^{-1}\right] A_2 \exp\left(i\omega_2 T_0\right) - \\ &-\alpha_3 \left[A_2 \exp\left(3i\omega_2 T_0\right) + 3\bar{A}_2 \exp\left(i\omega_2 T_0\right)\right] A_2^2 - \\ &-\alpha_4 \left\{A_1^2 \exp\left[\left(2\omega_1 + \omega_2\right) T_0\right] + 2A_1\bar{A}_1 \exp\left(i\omega_2 T_0\right) + \right. \\ &\left. +\bar{A}_1^2 \exp\left[i\left(\omega_2 - 2\omega_1\right) T_0\right]\right\} A_2 + f_2 \exp\left(i(\omega_{f_2} - \Omega_F) T_0\right) + \\ &+ f_2 \exp\left(i(\omega_{f_2} + \Omega_F) T_0\right) + cc. \end{aligned} \tag{27}$$

The analysis of relations (26)-(27) shows that the case of the occurrence of the one-to-one internal resonance is possible, when any two vibration frequencies of the mechanical system “plate+viscoelastic foundation” are close to each other, namely:

$$\omega_1 = \omega_2, \quad \text{and therefore,} \quad \Omega_1 = \Omega_2 . \tag{28}$$

From equations (26) and (27) it follows that the internal resonance could be accompanied by the external resonance when one of the following conditions is fulfilled:

$$(1)\ \omega_i = \omega_{fi} - \Omega_F ,$$

$$(2)\ \omega_i = \omega_{fi} + \Omega_F . \tag{29}$$

The condition for eliminating secular terms in equations (26) and (27) with account for relationships (28)-(29) leads to a set of two governing equations:

$$\begin{aligned} &2i\omega_1 D_2 A_1 + \left[\bar{\mu}_1 (i\omega_1)^{\gamma_1} - \bar{\mu}_2 (1+\tau_2^{\gamma_2} (i\omega_1)^{\gamma_2})^{-1} \right] A_1 + \\ &+3\alpha_1 A_1^2 \bar{A}_1 + \alpha_2 \bar{A}_1 A_2^2 + 2\alpha_2 A_1 A_2 \bar{A}_2 - f_1 = 0, \end{aligned} \tag{30}$$

$$\begin{aligned} &2i\omega_2 D_2 A_2 + \left[\bar{\mu}_1 (i\omega_2)^{\gamma_1} - \bar{\mu}_2 (1+\tau_2^{\gamma_2} (i\omega_2)^{\gamma_2})^{-1} \right] A_2 + \\ &+3\alpha_3 A_2^2 \bar{A}_2 + \alpha_4 \bar{A}_2 A_1^2 + 2\alpha_4 A_2 A_1 \bar{A}_1 - f_2 = 0. \end{aligned} \tag{31}$$

Multiplying (30) by $\bar{A}_1$ and (31) by $\bar{A}_2$, adding and subtracting the equations conjugate to them, and representing functions A_i in the polar form

$$A_i = a_i e^{i\varphi_i} \quad (i = 1, 2), \tag{32}$$

where $a_i = a_i(T_2)$ and $\varphi_i = \varphi_i(T_2)$ are the functions of amplitudes and phases of vibrations, yield the following set of equations:

$$\begin{aligned} &\left(a_1^2\right)^{\square} + s_1 a_1^2 + \omega_1^{-1} \alpha_2 a_1^2 a_2^2 \sin\delta + \\ &+ f_1 \omega_1^{-1} a_1 \sin\varphi_1 = 0, \end{aligned} \tag{33}$$

$$\begin{aligned} &\left(a_2^2\right)^{\square} + s_2 a_2^2 - \omega_2^{-1} \alpha_4 a_1^2 a_2^2 \sin\delta + \\ &+ f_2 \omega_2^{-1} a_2 \sin\varphi_2 = 0, \end{aligned} \tag{34}$$

$$\begin{aligned} &\dot{\varphi}_1 - \frac{1}{2}\lambda_1 - \frac{3}{2}\alpha_1 \omega_1^{-1} a_1^2 - \alpha_2 \omega_1^{-1} a_2^2 - \\ &-\frac{1}{2}\alpha_2 \omega_1^{-1} a_2^2 \cos\delta + \frac{1}{2} f_1 \left(\omega_1 a_1\right)^{-1} \cos\varphi_1 = 0, \end{aligned} \tag{35}$$

$$\begin{aligned} &\dot{\varphi}_2 - \frac{1}{2}\lambda_2 - \frac{3}{2}\alpha_3 \omega_2^{-1} a_2^2 - \alpha_4 \omega_2^{-1} a_1^2 - \\ &-\frac{1}{2}\alpha_4 \omega_2^{-1} a_1^2 \cos\delta + \frac{1}{2} f_2 \left(\omega_2 a_2\right)^{-1} \cos\varphi_2 = 0, \end{aligned} \tag{36}$$

(31)

where $\delta = 2(\varphi_2 - \varphi_1)$ is the phase difference,

$$\begin{aligned} &s_i = \bar{\mu}_1 \omega_i^{\gamma_1 - 1} \sin\psi_1 + \bar{\mu}_2 \omega_i^{-1} R_i \sin\Phi_i + \\ &+\bar{\mu}_3 \omega_i^{\gamma_2 - 1} \sin\psi_2 , \\ &\lambda_i = \bar{\mu}_1 \omega_i^{\gamma_1 - 1} \cos\psi_1 - \bar{\mu}_2 \omega_i^{-1} R_i \cos\Phi_i + \\ &+\bar{\mu}_3 \omega_i^{\gamma_2 - 1} \cos\psi_2 , \\ &\psi_i = \frac{1}{2}\pi\gamma_i \quad (i = 1, 2), \\ &R_i = \sqrt{1 + 2(\tau_2 \omega_i)^{\gamma_2} \cos\psi_2 + (\tau_2 \omega_i)^{2\gamma_2}} , \\ &\tan\Phi_i = \frac{(\tau_2 \omega_i)^{\gamma_2} \sin\psi_2}{1 + (\tau_2 \omega_i)^{\gamma_2} \cos\psi_2} . \end{aligned} \tag{37}$$

The set of equations (33)-(36) is the governing one for the amplitudes and phases of nonlinear force driven vibrations of the elastic simply supported plate on a nonlinear viscoelastic Winkler-type foundation, damping features of which are defined by the fractional derivative standard linear solid model (4), when vibrations occur in a viscoelastic surrounding medium, properties of which are described by the fractional derivative Kelvin-Voigt model.

For other types of boundary conditions, the governing set of equations could be obtained in a similar way by changing the vibration frequencies (see Table 1) and coefficients $\alpha_1 - \alpha_4$ (presented in Appendix B) in the expressions (33)-(36), as well as the terms depending on the external load.

4. NUMERICAL EXAMPLE

Equations (33)-(36) were solved numerically in the «Mathcad 15» system by using the method suggested in [31] for the cases of free and forced driven vibrations of the SSSS plate. A quadratic plate was considered as an example with the following geometric parameters: $a = b = 10m$, $h = 0,3m$, $m_1 = n_2 = 1$, $m_2 = n_1 = 3$ and material parameters $E = 3,25 \cdot 10^7 \, kPa$, $\rho = 2400 \, kg/m^3$, and $\nu = 0.3$. The harmonic load is moving with the constant velocity $V = 30 \, m/s$ and frequency $\Omega_F = 95 \, s^{-1}$ along the *x*-axis. The vibrations of the plate are studied for three cases of external load: $P = 2140N$ (Fig. 2b), $P = 5000N$ (Fig. 2c), and $P = 7140N$ (Fig. 2d).

The plate is subjected to the conditions of the internal resonance 1:1 at $\omega_1 = \omega_2 = 104,42 \, s^{-1}$, accompanied by the external resonance:

$$\omega_1 = \omega_{f_1} + \Omega_F = \frac{3,14 \cdot 1 \cdot 30}{10} + 95 = 104,42 \, s^{-1}.$$

Figure 2 clearly shows the energy exchange between interacting modes of nonlinear free vibrations and force driven vibrations of the simply supported plate on the elastic ($\gamma_2 = 0$) and viscoelastic ($\gamma_2 \neq 0$) foundation via the fractional calculus standard linear solid for different values of external load. It is seen that an increase in the magnitude of the external force results in the increase in dimensionless amplitudes of vibrations of the plate. The dependence of the amplitudes of nonlinear vibrations on the values of fractional parameters γ_1 and γ_2 is shown in Figure 3. With the appearance of the damping properties of the viscoelastic medium, the damping of vibrations increases.

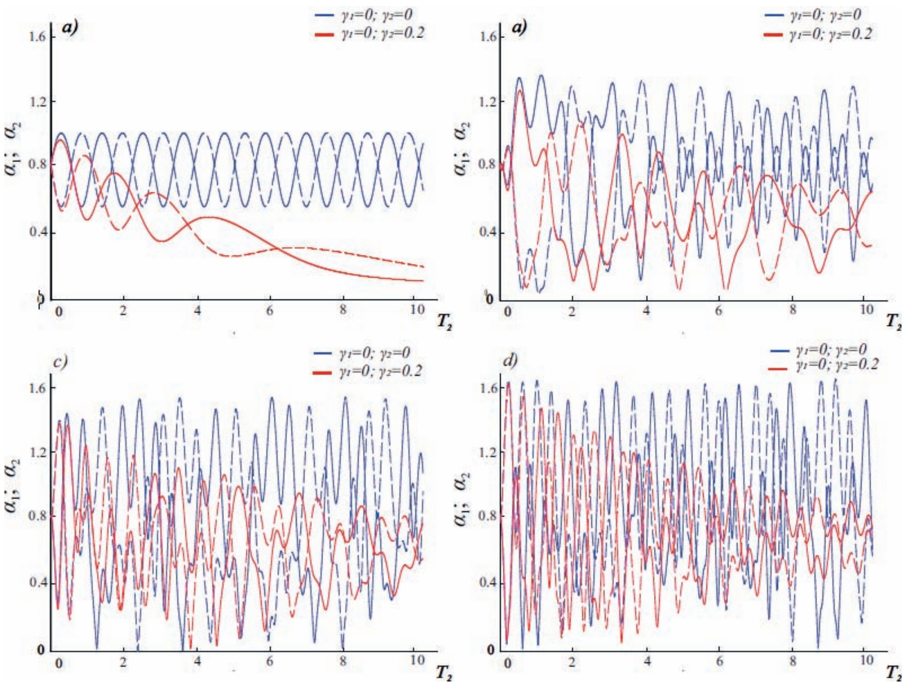

Figure 2. The dimensionless T_2-dependence of the dimensionless amplitudes of nonlinear vibrations for SSSS-plate for $m_1 = n_2 = 1$, $m_2 = n_1 = 3$: a) free vibrations, and force driven vibrations at b) $f_1 = -3$, $f_2 = 3$; c) $f_1 = -7$, $f_2 = 7$; d) $f_1 = -10$, $f_2 = 10$; solid line – a_2, dashed line – a_1

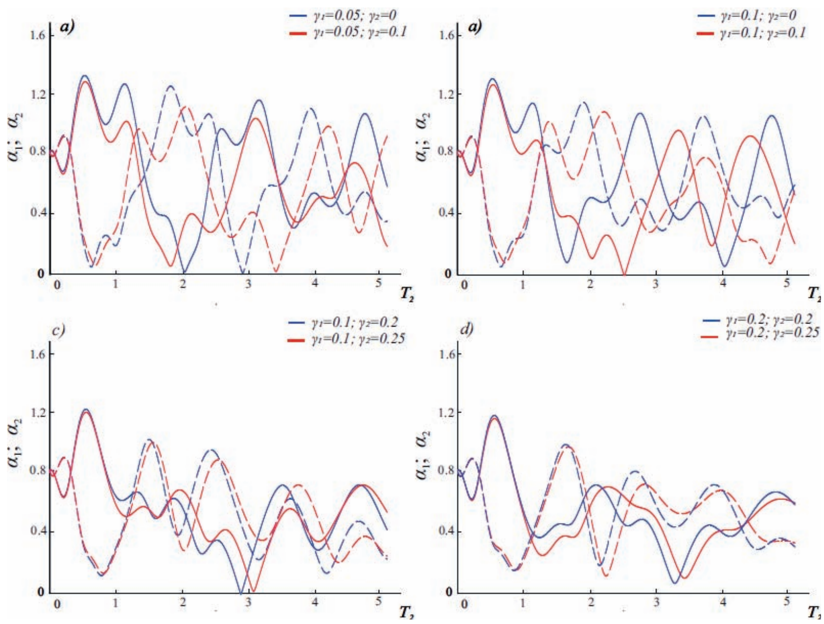

Figure 3. The dimensionless T_2-dependence of the dimensionless amplitudes of nonlinear vibrations of SSSS-plate for different values of fractional parameters at $f_1 = -3,\ f_2 = 3;\ m_1 = n_2 = 1,$ $m_2 = n_1 = 3;$ *solid line – a_2, dashed line – a_1*

5. CONCLUSION

In the present paper, the problem of nonlinear vibrations of a von Karman elastic plate based on a viscoelastic Winkler-type foundation and subjected to moving load is solved. The damping features of the viscoelastic foundation are described by the fractional derivative standard linear solid model, while the damping properties of the environment in which the vibrations occur are described by the Kelvin-Voigt model with the Riemann-Liouville fractional derivative. The expressions for the stress function and nonlinear coefficients for SSSS, CCCC and CSCS types of boundary conditions are presented. The governing equations are obtained for determining nonlinear amplitudes and phases in the case of forced driven vibrations, when the natural frequencies of the two dominant vibration modes are close to each other and to the frequency of the external load. The resulting set of equations allows one to control the damping properties of the external environment and the foundation by changing the fractional parameters from zero, what corresponds to an elastic medium and/or elastic foundation, to unit, what conforms to the traditional standard linear solid model, resulting in the expansion of the range of applicability of the solution obtained.

The derived set of equations has been solved numerically for the SSSS case of boundary conditions using the approach described in [31].

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APPENDIX A

1) SSSS-plate

$$\begin{aligned}\phi(x,y,t) = Eh\Bigg[&\sum_i\sum_p\sum_q \phi_{ipq} X_{ip} Y_{iq}\, x_i(t)^2 + \\ &+\frac{1}{4}\Big(B_1^2 M_{11} N_{21} + C_1^2 M_{21} N_{11} - A_1^2 M_{11} N_{11} - D_1^2 M_{21} N_{21}\Big) x_1(t) x_2(t)\Bigg],\end{aligned} \tag{A.1}$$

where $\xi = \dfrac{b}{a}$, $\quad \eta_i = \dfrac{m_i}{n_i}, i = 1, 2,\quad X_{ip} = \cos\dfrac{\pi p m_i x}{a},\quad Y_{iq} = \cos\dfrac{\pi q n_i y}{b},$

$$M_{1p} = \cos\frac{\pi p(m_1 + m_2)x}{a}, \quad M_{2p} = \cos\frac{\pi p(m_1 - m_2)x}{a}, \ \phi_{i02} = \frac{\xi^2\eta_i^2}{32}, \quad \phi_{i20} = \frac{1}{32\xi^2\eta_i^2}$$

$$N_{1q} = \cos\frac{\pi q(n_1 + n_2)y}{b}, \quad N_{2q} = \cos\frac{\pi q(n_1 - n_2)y}{b},$$

$$A_1 = \frac{\xi\left(m_1n_2 - m_2n_1\right)}{\left(m_1 + m_2\right)^2\xi^2 + \left(n_1 + n_2\right)^2}, \qquad B_1 = \frac{\xi\left(m_1n_2 + m_2n_1\right)}{\left(m_1 + m_2\right)^2\xi^2 + \left(n_1 - n_2\right)^2},$$

$$C_1 = \frac{\xi\left(m_1n_2 + m_2n_1\right)}{\left(m_1 - m_2\right)^2\xi^2 + \left(n_1 + n_2\right)^2}, \qquad D_1 = \frac{\xi\left(m_1n_2 - m_2n_1\right)}{\left(m_1 - m_2\right)^2\xi^2 + \left(n_1 - n_2\right)^2}.$$

2) CCCC-plate

$$\begin{aligned}\phi(x,y,t) = Eh\Bigg\{&\sum_i\sum_p\sum_q \phi_{ipq}X_{ip}Y_{iq}\,x_i(t)^2 + \Bigg[\frac{1}{64}\Big(A_1^2M_{12}N_{12} + D_1^2M_{22}N_{22} - \\ &-B_1^2M_{12}N_{22} - C_1^2M_{22}N_{12}\Big) - \frac{m_1^2n_2^2}{32}\Big(B_2^2M_{12}Y_{21} + C_2^2M_{22}Y_{21} - \\ &-A_2^2X_{11}N_{12} - D_2^2X_{11}N_{22} - E_2^2X_{11}Y_{21}\Big) - \frac{m_2^2n_1^2}{32}\Big(B_3^2M_{12}Y_{11} + C_3^2M_{22}Y_{11} - \\ &-A_3^2X_{21}N_{12} - D_3^2X_{21}N_{22} - E_3^2X_{21}Y_{11}\Big)\Bigg]x_1(t)x_2(t)\Bigg\},\end{aligned} \tag{A.2}$$

where $\phi_{i04} = -\frac{\xi^2\eta_i^2}{512}, \quad \phi_{i40} = -\frac{1}{512\xi^2\eta_i^2},$

$$\phi_{i42} = \frac{\xi^2\eta_i^2}{32\left[4\eta_i^2\xi^2 + 1\right]^2}, \quad \phi_{i22} = -\frac{\xi^2\eta_i^2}{32\left[\eta_i^2\xi^2 + 1\right]^2}, \quad \phi_{i24} = \frac{\xi^2\eta_i^2}{32\left[\eta_i^2\xi^2 + 4\right]^2},$$

$$A_2 = \frac{\xi}{m_1^2\xi^2 + \left(n_1 + n_2\right)^2}, \ B_2 = \frac{\xi}{\left(m_1 + m_2\right)^2\xi^2 + n_2^2},$$

$$C_2 = \frac{\xi}{\left(m_1 - m_2\right)^2\xi^2 + n_2^2}, \ D_2 = \frac{\xi}{m_1^2\xi^2 + \left(n_1 - n_2\right)^2}, \ E_2 = \frac{\xi}{m_1^2\xi^2 + n_2^2},$$

$$A_3 = \frac{\xi}{m_2^2\xi^2 + \left(n_1 + n_2\right)^2}, \ B_3 = \frac{\xi}{\left(m_1 + m_2\right)^2\xi^2 + n_1^2},$$

$$C_3 = \frac{\xi}{\left(m_1 - m_2\right)^2\xi^2 + n_1^2}, \ D_3 = \frac{\xi}{m_2^2\xi^2 + \left(n_1 - n_2\right)^2}, \ E_3 = \frac{\xi}{m_2^2\xi^2 + n_1^2}$$

3) CSCS-plate

$$\phi(x,y,t)=Eh\left\{\sum_i\sum_p\sum_q \phi_{ipq}X_{ip}Y_{iq}\,x_i(t)^2+\left[\frac{1}{4}\left(A_4{}^2M_{12}N_{11}-D_4{}^2M_{22}N_{21}-B_4{}^2M_{12}N_{21}+C_4{}^2M_{22}N_{11}\right)+\right.\right.$$
$$\left.\left.+\frac{m_1^2n_2^2}{2}\left(A_5{}^2X_{11}N_{21}-B_5{}^2X_{11}N_{11}\right)+\frac{m_2^2n_1^2}{2}\left(C_5{}^2X_{21}N_{21}-D_5{}^2X_{21}N_{11}\right)\right]x_1(t)x_2(t)\right\}, \quad (11)$$

where $A_4=\dfrac{\xi\left(m_1n_2-m_2n_1\right)}{4\left(m_1+m_2\right)^2\xi^2+\left(n_1+n_2\right)^2}$, $B_4=\dfrac{\xi\left(m_1n_2+m_2n_1\right)}{4\left(m_1+m_2\right)^2\xi^2+\left(n_1-n_2\right)^2}$,

$C_4=\dfrac{\xi\left(m_1n_2+m_2n_1\right)}{4\left(m_1-m_2\right)^2\xi^2+\left(n_1+n_2\right)^2}$, $D_4=\dfrac{\xi\left(m_1n_2-m_2n_1\right)}{4\left(m_1-m_2\right)^2\xi^2+\left(n_1-n_2\right)^2}$,

$A_5=\dfrac{\xi}{4m_1{}^2\xi^2+\left(n_1-n_2\right)^2}$, $B_5=\dfrac{\xi}{4m_1{}^2\xi^2+\left(n_1+n_2\right)^2}$,

$C_5=\dfrac{\xi}{4m_2{}^2\xi^2+\left(n_1-n_2\right)^2}$, $D_5=\dfrac{\xi}{4m_2{}^2\xi^2+\left(n_1+n_2\right)^2}$.

APPENDIX B

1) SSSS-plate

$$\alpha_1=-\frac{E}{2\rho ab}\pi^4\int_0^a\int_0^b K_{11}{}^2L_{11}{}^2\left[\frac{m_1^4}{a^4}Y_{12}+\frac{n_1^4}{b^4}X_{12}\right]dxdy, \quad (B.1)$$

$$\alpha_3=-\frac{E}{2\rho ab}\pi^4\int_0^a\int_0^b K_{21}{}^2L_{21}{}^2\left[\frac{m_2^4}{a^4}Y_{22}+\frac{n_2^4}{b^4}X_{22}\right]dxdy \quad (B.2)$$

$$\alpha_2=-\frac{E}{2\rho ab}\pi^4\int_0^a\int_0^b\left\{K_{11}L_{11}\left[\frac{m_1^2m_2^2}{a^4}Y_{22}+\frac{n_1^2n_2^2}{b^4}X_{22}\right]+\frac{2m_2^2}{a^2b^2}K_{21}L_{21}M_{11}\left[-A_1^2\left(n_1+n_2\right)^2N_{11}+B_1^2\left(n_1-n_2\right)^2N_{21}\right]+\right.$$
$$+\frac{2m_2^2}{a^2b^2}K_{21}L_{21}M_{21}\left[-D_1^2\left(n_1-n_2\right)^2N_{21}+C_1^2\left(n_1+n_2\right)^2N_{11}\right]+\frac{2n_2^2}{a^2b^2}K_{21}L_{21}N_{11}\left[B_1^2\left(m_1-m_2\right)^2M_{21}-\right.$$
$$\left.-A_1^2\left(m_1+m_2\right)^2M_{11}\right]+\frac{2n_2^2}{a^2b^2}K_{21}L_{21}N_{21}\left[-D_1^2\left(m_1-m_2\right)^2\cos M_{11}+C_1^2\left(m_1+m_2\right)^2M_{21}\right]-$$
$$-\frac{4m_2n_2}{a^2b^2}X_{21}Y_{21}\left(m_1+m_2\right)S_{11}\left[-A_1^2\left(n_1+n_2\right)T_{11}+B_1^2\left(n_1-n_2\right)T_{21}\right]-$$
$$\left.-\frac{4m_2n_2}{a^2b^2}X_{21}Y_{21}\left(m_1-m_2\right)S_{21}\left[-D_1^2\left(n_1-n_2\right)T_{21}+C_1^2\left(n_1+n_2\right)T_{11}\right]\right\}K_{11}L_{11}dxdy, \quad (B.3)$$

$$\alpha_4 = -\frac{E}{2\rho ab}\pi^4 \int_0^a\int_0^b \left\{ K_{21}L_{21}\left[\frac{m_1^2 m_2^2}{a^4}Y_{12} + \frac{n_1^2 n_2^2}{b^4}X_{12}\right] + \frac{2m_1^2}{a^2b^2}K_{11}L_{11}M_{11}\left[-A_1^2\left(n_1+n_2\right)^2 N_{11} + B_1^2\left(n_1-n_2\right)^2 N_{21}\right] + \right.$$
$$+\frac{2m_1^2}{a^2b^2}K_{11}L_{11}M_{21}\left[-D_1^2\left(n_1-n_2\right)^2 N_{21} + C_1^2\left(n_1+n_2\right)^2 N_{11}\right] + \frac{2n_1^2}{a^2b^2}K_{11}L_{11}N_{11}\left[B_1^2\left(m_1-m_2\right)^2 M_{21} - \right.$$
$$\left. -A_1^2\left(m_1+m_2\right)^2 M_{11}\right] + \frac{2n_1^2}{a^2b^2}K_{11}L_{11}N_{21}\left[-D_1^2\left(m_1-m_2\right)^2 M_{21} + C_1^2\left(m_1+m_2\right)^2 M_{11}\right] -$$
$$-\frac{4m_1 n_1}{a^2b^2}X_{11}Y_{11}\left(m_1+m_2\right)S_{11}\left[-A_1^2\left(n_1+n_2\right)T_{11} + B_1^2\left(n_1-n_2\right)T_{21}\right] -$$
$$\left. -\frac{4m_1 n_1}{a^2b^2}X_{11}Y_{11}\left(m_1-m_2\right)S_{21}\left[-D_1^2\left(n_1-n_2\right)T_{21} + C_1^2\left(n_1+n_2\right)T_{11}\right]\right\} K_{21}L_{21}dxdy, \tag{B.4}$$

where $K_{ip} = \sin\frac{\pi p m_i x}{a}$, $L_{iq} = \sin\frac{\pi q n_i y}{b}$, $S_{1p} = \sin\frac{\pi p(m_1+m_2)x}{a}$, $S_{2p} = \sin\frac{\pi p(m_1-m_2)x}{a}$,
$T_{1q} = \sin\frac{\pi q(n_1+n_2)y}{b}$, $T_{2q} = \sin\frac{\pi q(n_1-n_2)y}{b}$.

2) CCCC-plate

$$\alpha_1 = \frac{16E}{9\rho ab}\pi^4\int_0^a\int_0^b\left\{\frac{1}{8}\left[\frac{m_1^4}{a^4}X_{12}(1-Y_{12})(Y_{12}-\frac{Y_{14}}{4}) + \frac{n_1^4}{b^4}Y_{12}\left(1-X_{12}\right)(X_{12}-\frac{X_{14}}{4})\right] - \right.$$
$$+\frac{m_1^4 n_1^4}{8a^2b^2}\left[A^2X_{14}Y_{12}(X_{12}+4Y_{12}-5X_{12}Y_{12}+4K_{12}L_{12}) - B^2X_{12}Y_{12}(X_{12}+Y_{12}-2X_{12}Y_{12}+2K_{12}L_{12}) + \right. \tag{B.5}$$
$$\left.\left. +C^2X_{12}Y_{14}(4X_{12}+Y_{12}-5X_{12}Y_{12}+4K_{12}L_{12})\right]\right\}(1-X_{12})(1-Y_{12})dxdy,$$

$$\alpha_3 = \frac{16E}{9\rho ab}\pi^4\int_0^a\int_0^b\left\{\frac{1}{8}\left[\frac{m_2^4}{a^4}X_{22}(1-Y_{22})(Y_{22}-\frac{Y_{24}}{4}) + \frac{n_2^4}{b^4}Y_{22}\left(1-X_{22}\right)(X_{22}-\frac{X_{24}}{4})\right] - \right.$$
$$+\frac{m_2^4 n_2^4}{8a^2b^2}\left[D^2X_{24}Y_{22}(X_{22}+4Y_{22}-5X_{22}Y_{22}+4K_{22}L_{22}) - E^2X_{22}Y_{22}(X_{22}+Y_{22}-2X_{22}Y_{22}+2K_{22}L_{22}) + \right. \tag{B.6}$$
$$\left.\left. +F^2X_{22}Y_{24}(4X_{22}+Y_{22}-5X_{22}Y_{22}+4K_{22}L_{22})\right]\right\}(1-X_{22})(1-Y_{22})dxdy,$$

$$\alpha_2 = \frac{16E}{9\rho ab}\pi^4 \int_0^a\int_0^b \left\{ \frac{1}{8}\left[\frac{m_1^2 m_2^2}{a^4} X_{12}(1-Y_{12})\left(Y_{22}-\frac{Y_{24}}{4}\right) + \frac{n_1^2 n_2^2}{b^4} Y_{12}(1-X_{12})(X_{22}-\frac{X_{24}}{4}) \right] + \right.$$
$$+\frac{m_1^2 m_2^2 n_2^4}{8a^2b^2} X_{12}(1-Y_{12})\left[D^2X_{24}Y_{22} - E^2X_{22}Y_{22} + 4F^2X_{22}Y_{24}\right] +$$
$$+\frac{n_1^2 n_2^2 m_2^4}{8a^2b^2} Y_{12}(1-X_{12})\left[4D^2X_{24}Y_{22} - E^2X_{22}Y_{22} + F^2X_{22}Y_{24}\right] +$$
$$+\frac{m_1 n_1 m_2^3 n_2^3}{4a^2b^2} K_{12}L_{12}\left[2D^2K_{24}L_{22} - E^2K_{22}L_{22} + 2F^2K_{22}L_{24}\right] -$$
$$-\frac{m_2^2}{16a^2b^2} X_{22}(1-Y_{22})\left[A_1^2(n_1+n_2)^2 M_{12}N_{12} + B_1^2(n_1-n_2)^2 M_{12}N_{22} + \right.$$
$$+C_1^2(n_1+n_2)^2 M_{22}N_{12} + D_1^2(n_1-n_2)^2 M_{22}N_{22} - 2B_2^2 m_1^2 n_2^4 M_{12}Y_{22} -$$
$$-2C_2^2 m_1^2 n_2^4 M_{22}Y_{22} + 2A_2^2 m_1^2 n_2^2 (n_1+n_2)^2 N_{12}X_{12} + 2D_2^2 m_1^2 n_2^2 (n_1-n_2)^2 N_{22}X_{12} +$$
$$+2E_2^2 m_1^2 n_2^4 X_{12}Y_{22} - 2B_3^2 m_2^2 n_1^4 M_{12}Y_{12} - 2C_3^2 m_2^2 n_1^4 M_{22}Y_{12} + 2A_3^2 m_2^2 n_1^2 (n_1+n_2)^2 N_{12}X_{22} +$$
$$\left. +2D_3^2 m_2^2 n_1^2 (n_1-n_2)^2 N_{22}X_{22} + 2E_3^2 m_2^2 n_1^4 X_{22}Y_{12} \right] - \frac{n_2^2}{16a^2b^2} Y_{22}(1-X_{22}) \times$$
$$\times\left[A_1^2(m_1+m_2)^2 M_{12}N_{12} + B_1^2(m_1+m_2)^2 M_{12}N_{22} + C_1^2(m_1-m_2)^2 M_{22}N_{12} + D_1^2(m_1-m_2)^2 M_{22}N_{22} - \right.$$
$$-2B_2^2 m_1^2 n_2^2 (m_1+m_2)^2 M_{12}Y_{22} - 2C_2^2 m_1^2 n_2^2 (m_1-m_2)^2 M_{22}Y_{22} + 2A_2^2 m_1^4 n_2^2 N_{12}X_{12} + 2D_2^2 m_1^4 n_2^2 N_{22}X_{12} +$$
$$+2E_2^2 m_1^4 n_2^2 X_{12}Y_{22} - 2B_3^2 m_2^2 n_1^2 (m_1+m_2)^2 M_{12}Y_{12} - 2C_3^2 m_2^2 n_1^2 (m_1-m_2)^2 M_{22}Y_{12} + 2A_3^2 m_2^4 n_1^2 N_{12}X_{22} +$$
$$\left. +2D_3^2 m_2^4 n_1^2 N_{22}X_{22} + 2E_3^2 m_2^4 n_1^2 X_{22}Y_{12} \right] - \frac{m_2 n_2}{16a^2b^2} K_{22}L_{22}\left[B_1^2(m_1+m_2)(n_1-n_2) S_{12}T_{22} + \right.$$
$$+A_1^2(m_1+m_2)(n_1+n_2) S_{12}T_{12} + C_1^2(m_1-m_2)(n_1+n_2) S_{22}T_{12} + D_1^2(m_1-m_2)(n_1-n_2) S_{22}T_{22} -$$
$$-2B_2^2 m_1^2 n_2^3 (m_1+m_2) S_{12}L_{22} - 2C_2^2 m_1^2 n_2^3 (m_1-m_2) S_{22}L_{22} + 2A_2^2 m_1^3 n_2^2 (n_1+n_2) T_{12}K_{12} +$$
$$+2D_2^2 m_1^3 n_2^2 (n_1-n_2) T_{22}K_{12} + 2E_2^2 m_1^3 n_2^3 K_{12}L_{22} - 2B_3^2 m_2^2 n_1^3 (m_1+m_2) S_{12}L_{12} - 2C_3^2 m_2^2 n_1^3 (m_1-m_2) \times$$
$$\left.\left. \times S_{22}L_{12} + 2A_3^2 m_2^3 n_1^2 (n_1+n_2) T_{12}K_{22} + 2D_3^2 m_2^3 n_1^2 (n_1-n_2) T_{22}K_{22} + 2E_3^2 m_2^3 n_1^3 K_{22}L_{12} \right]\right\} (1-X_{12})(1-Y_{12})\,dxdy. \tag{B.7}$$

$$\alpha_4 = \frac{16E}{9\rho ab}\pi^4 \int_0^a\int_0^b \left\{ \frac{1}{8}\left[\frac{m_1^2 m_2^2}{a^4} X_{22}(1-Y_{22})\left(Y_{12}-\frac{Y_{14}}{4}\right) + \frac{n_1^2 n_2^2}{b^4} Y_{22}(1-X_{22})(X_{12}-\frac{X_{14}}{4}) \right] + \right.$$
$$+\frac{m_1^2 m_2^2 n_1^4}{8a^2b^2} X_{22}(1-Y_{22})\left[A^2 X_{14}Y_{12} - B^2 X_{12}Y_{12} + 4C^2 X_{12}Y_{14}\right] +$$
$$+\frac{n_1^2 n_2^2 m_1^4}{8a^2b^2} Y_{22}(1-X_{22})\left[4A^2 X_{14}Y_{12} - B^2 X_{12}Y_{12} + C^2 X_{12}Y_{14}\right] +$$
$$+\frac{m_2 n_2 m_1^3 n_1^3}{4a^2b^2} K_{22}L_{22}\left[2A^2 K_{14}L_{12} - B^2 K_{12}L_{12} + 2C^2 K_{12}L_{14}\right] -$$
$$-\frac{m_1^2}{16a^2b^2} X_{12}(1-Y_{12})\left[A_1^{\,2}(n_1+n_2)^2 M_{12}N_{12} + B_1^{\,2}(n_1-n_2)^2 M_{12}N_{22} + \right.$$
$$+C_1^{\,2}(n_1+n_2)^2 M_{22}N_{12} + D_1^{\,2}(n_1-n_2)^2 M_{22}N_{22} - 2B_2^{\,2} m_1^2 n_2^4 M_{12}Y_{22} -$$
$$-2C_2^{\,2} m_1^2 n_2^4 M_{22}Y_{22} + 2A_2^{\,2} m_1^2 n_2^2 (n_1+n_2)^2 N_{12}X_{12} + 2D_2^{\,2} m_1^2 n_2^2 (n_1-n_2)^2 N_{22}X_{12} +$$
$$+2E_2^{\,2} m_1^2 n_2^4 X_{12}Y_{22} - 2B_3^{\,2} m_2^2 n_1^4 M_{12}Y_{12} - 2C_3^{\,2} m_2^2 n_1^4 M_{22}Y_{12} + 2A_3^{\,2} m_2^2 n_1^2 (n_1+n_2)^2 N_{12}X_{22} +$$
$$\left. +2D_3^{\,2} m_2^2 n_1^2 (n_1-n_2)^2 N_{22}X_{22} + 2E_3^{\,2} m_2^2 n_1^4 X_{22}Y_{12} \right] - \frac{n_1^2}{16a^2b^2} Y_{12}(1-X_{12}) \times$$
$$\times\left[A_1^{\,2}(m_1+m_2)^2 M_{12}N_{12} + B_1^{\,2}(m_1+m_2)^2 M_{12}N_{22} + C_1^{\,2}(m_1-m_2)^2 M_{22}N_{12} + D_1^{\,2}(m_1-m_2)^2 M_{22}N_{22} - \right.$$
$$-2B_2^{\,2} m_1^2 n_2^2 (m_1+m_2)^2 M_{12}Y_{22} - 2C_2^{\,2} m_1^2 n_2^2 (m_1-m_2)^2 M_{22}Y_{22} + 2A_2^{\,2} m_1^4 n_2^2 N_{12}X_{12} + 2D_2^{\,2} m_1^4 n_2^2 N_{22}X_{12} +$$
$$+2E_2^{\,2} m_1^4 n_2^2 X_{12}Y_{22} - 2B_3^{\,2} m_2^2 n_1^2 (m_1+m_2)^2 M_{12}Y_{12} - 2C_3^{\,2} m_2^2 n_1^2 (m_1-m_2)^2 M_{22}Y_{12} + 2A_3^{\,2} m_2^4 n_1^2 N_{12}X_{22} +$$
$$\left. +2D_3^{\,2} m_2^4 n_1^2 N_{22}X_{22} + 2E_3^{\,2} m_2^4 n_1^2 X_{22}Y_{12} \right] - \frac{m_1 n_1}{16a^2b^2} K_{12}L_{12}\left[B_1^{\,2}(m_1+m_2)(n_1-n_2)S_{12}T_{22} + \right.$$
$$+A_1^{\,2}(m_1+m_2)(n_1+n_2)S_{12}T_{12} + C_1^{\,2}(m_1-m_2)(n_1+n_2)S_{22}T_{12} + D_1^{\,2}(m_1-m_2)(n_1-n_2)S_{22}T_{22} -$$
$$-2B_2^{\,2} m_1^2 n_2^3 (m_1+m_2)S_{12}L_{22} - 2C_2^{\,2} m_1^2 n_2^3 (m_1-m_2)S_{22}L_{22} + 2A_2^{\,2} m_1^3 n_2^2 (n_1+n_2)T_{12}K_{12} +$$
$$+2D_2^{\,2} m_1^3 n_2^2 (n_1-n_2)T_{22}K_{12} + 2E_2^{\,2} m_1^3 n_2^3 K_{12}L_{22} - 2B_3^{\,2} m_2^2 n_1^3 (m_1+m_2)S_{12}L_{12} - 2C_3^{\,2} m_2^2 n_1^3 (m_1-m_2)\times$$
$$\left.\left.\times S_{22}L_{12} + 2A_3^{\,2} m_2^3 n_1^2 (n_1+n_2)T_{12}K_{22} + 2D_3^{\,2} m_2^3 n_1^2 (n_1-n_2)T_{22}K_{22} + 2E_3^{\,2} m_2^3 n_1^3 K_{22}L_{12} \right]\right\}(1-X_{22})(1-Y_{22})dxdy,$$

(B.8)

where $A^2 = \frac{32\varphi_{142}}{m_1^2 n_1^2}$, $B^2 = \frac{32\varphi_{122}}{m_1^2 n_1^2}$, $C^2 = \frac{32\varphi_{124}}{m_1^2 n_1^2}$, $D^2 = \frac{32\varphi_{242}}{m_2^2 n_2^2}$, $E^2 = \frac{32\varphi_{222}}{m_2^2 n_2^2}$, $F^2 = \frac{32\varphi_{224}}{m_2^2 n_2^2}$.

3) CSCS-plate

$$\alpha_1 = \frac{8E}{3\rho ab}\pi^4 \int_0^a\int_0^b \left\{ \frac{1}{4} X_{12}L_{11}\left[\frac{m_1^4}{a^4} Y_{12} + \frac{n_1^4}{4b^4}(X_{12}-1) \right] - \frac{n_1^4}{64b^4} X_{14}L_{11}(X_{12}-1) - \right.$$
$$\left. -\frac{m_1^4 n_1^4}{16a^2b^2} G^2 L_{11}X_{12}Y_{12}(5X_{12}-1) - \frac{m_1^4 n_1^4}{4a^2b^2} G^2 K_{12}^{\,2} L_{12}Y_{11} \right\}(1-X_{12})L_{11}dxdy, \qquad (B.9)$$

$$\alpha_3 = \frac{8E}{3\rho ab}\pi^4 \int_0^a\int_0^b \left\{ \frac{1}{4} X_{22} L_{21} \left[\frac{m_2^4}{a^4} Y_{22} + \frac{n_2^4}{4b^4}\left(X_{22} - 1 \right) \right] - \frac{n_2^4}{64b^4} X_{24} L_{21} \left(X_{22} - 1 \right) - \right.$$
$$\left. - \frac{m_2^4 n_2^4}{16a^2b^2} H^2 L_{21} X_{22} Y_{22} \left(5X_{22} - 1 \right) - \frac{m_2^4 n_2^4}{4a^2b^2} H^2 K_{22}{}^2 L_{22} Y_{21} \right\} (1 - X_{22}) L_{21} dxdy, \tag{B.10}$$

$$\alpha_2 = \frac{8E}{3\rho ab}\pi^4 \int_0^a\int_0^b \left\{ \frac{1}{4} \left[\frac{m_1^2 m_2^2}{a^4} X_{12} L_{11} Y_{22} + \frac{n_1^2 n_2^2}{4b^4} L_{11} (1 - X_{12})(X_{22} - \frac{X_{24}}{4}) \right] - \right.$$
$$- \frac{m_2^2 n_2^2}{16a^2b^2} H^2 L_{11} X_{22} Y_{22} \left(4m_1^2 n_2^2 X_{12} - n_1^2 m_2^2 (1 - X_{12}) \right) - \frac{m_1 n_1 m_2^3 n_2^3}{4a^2b^2} H^2 K_{12} Y_{11} K_{22} L_{22} -$$
$$- \frac{m_2^2}{2a^2b^2} X_{22} L_{21} \left[-A_4{}^2 \left(n_1 + n_2 \right)^2 M_{12} N_{11} + B_4{}^2 \left(n_1 - n_2 \right)^2 M_{12} N_{21} - \right.$$
$$- C_4{}^2 \left(n_1 + n_2 \right)^2 M_{22} N_{11} + D_4{}^2 \left(n_1 - n_2 \right)^2 M_{21} N_{22} + 2B_5{}^2 m_1^2 n_2^2 \left(n_1 + n_2 \right)^2 X_{12} N_{11} -$$
$$\left. - 2C_5{}^2 m_2^2 n_1^2 \left(n_1 - n_2 \right)^2 X_{22} N_{21} - 2A_5{}^2 m_1^2 n_2^2 \left(n_1 - n_2 \right)^2 N_{21} X_{12} + 2D_5{}^2 m_2^2 n_1^2 \left(n_1 + n_2 \right)^2 N_{11} X_{22} \right] -$$
$$+ \frac{n_2^2}{2a^2b^2} (1 - X_{22}) L_{21} \left[-A_4{}^2 \left(m_1 + m_2 \right)^2 M_{12} N_{11} + B_4{}^2 \left(m_1 + m_2 \right)^2 M_{12} N_{21} - \right.$$
$$- C_4{}^2 \left(m_1 - m_2 \right)^2 M_{22} N_{11} + \frac{1}{4} D_4{}^2 \left(m_1 - m_2 \right)^2 M_{21} N_{22} + 2B_5{}^2 m_1^4 n_2^2 X_{12} N_{11} -$$
$$\left. - 2C_5{}^2 m_2^4 n_1^2 X_{22} N_{21} - 2A_5{}^2 m_1^4 n_2^2 N_{21} X_{12} + 2D_5{}^2 m_2^4 n_1^2 N_{11} X_{22} \right] +$$
$$+ \frac{m_2 n_2}{a^2b^2} K_{22} Y_{21} \left[-B_4{}^2 \left(m_1 + m_2 \right)\left(n_1 - n_2 \right) S_{12} T_{21} + \right.$$
$$+ A_4{}^2 \left(m_1 + m_2 \right)\left(n_1 + n_2 \right) S_{12} T_{11} + C_4{}^2 \left(m_1 - m_2 \right)\left(n_1 + n_2 \right) S_{22} T_{11} - D_4{}^2 \left(m_1 - m_2 \right)\left(n_1 - n_2 \right) S_{12} T_{22} -$$
$$- 2B_5{}^2 m_1^3 n_2^2 \left(n_1 + n_2 \right) T_{11} K_{12} + 2C_5{}^2 m_2^3 n_1^2 \left(n_1 - n_2 \right) T_{21} K_{22} + 2A_5{}^2 m_1^3 n_2^2 \left(n_1 - n_2 \right) T_{21} K_{12} -$$
$$\left.\left. - 2D_5{}^2 m_2^3 n_1^2 \left(n_1 + n_2 \right) T_{11} K_{22} \right] \right\} (1 - X_{12}) L_{11} dxdy, \tag{B.11}$$

$$\alpha_4 = \frac{8E}{3\rho ab}\pi^4 \int_0^a\int_0^b \left\{ \frac{1}{4}\left[\frac{m_1^2 m_2^2}{a^4} X_{22}L_{21}Y_{12} + \frac{n_1^2 n_2^2}{4b^4} L_{21}(1-X_{22})(X_{12}-\frac{X_{14}}{4}) \right] - \right.$$
$$-\frac{m_1^2 n_1^2}{16a^2b^2} G^2 L_{21}X_{12}Y_{12}\left(4m_2^2 n_1^2 X_{22} - n_2^2 m_1^2(1-X_{22})\right) - \frac{m_1^3 n_1^3 m_2 n_2}{4a^2b^2} G^2 K_{22}Y_{21}K_{12}L_{12}$$
$$-\frac{m_1^2}{2a^2b^2} X_{12}L_{11}\left[-A_4^{\,2}\left(n_1+n_2\right)^2 M_{12}N_{11} + B_4^{\,2}\left(n_1-n_2\right)^2 M_{12}N_{21} - \right.$$
$$-C_4^{\,2}\left(n_1+n_2\right)^2 M_{22}N_{11} + D_4^{\,2}\left(n_1-n_2\right)^2 M_{21}N_{22} + 2B_5^{\,2} m_1^2 n_2^2\left(n_1+n_2\right)^2 X_{12}N_{11} -$$
$$\left. -2C_5^{\,2} m_2^2 n_1^2\left(n_1-n_2\right)^2 X_{22}N_{21} - 2A_5^{\,2} m_1^2 n_2^2\left(n_1-n_2\right)^2 N_{21}X_{12} + 2D_5^{\,2} m_2^2 n_1^2\left(n_1+n_2\right)^2 N_{11}X_{22} \right] -$$
$$+\frac{n_1^2}{2a^2b^2}(1-X_{12})L_{11}\left[-A_4^{\,2}\left(m_1+m_2\right)^2 M_{12}N_{11} + B_4^{\,2}\left(m_1+m_2\right)^2 M_{12}N_{21} - \right.$$
$$-C_4^{\,2}\left(m_1-m_2\right)^2 M_{22}N_{11} + \frac{1}{4}D_4^{\,2}\left(m_1-m_2\right)^2 M_{21}N_{22} + 2B_5^{\,2} m_1^4 n_2^2 X_{12}N_{11} -$$
$$\left. -2C_5^{\,2} m_2^4 n_1^2 X_{22}N_{21} - 2A_5^{\,2} m_1^4 n_2^2 N_{21}X_{12} + 2D_5^{\,2} m_2^4 n_1^2 N_{11}X_{22} \right] +$$
$$+\frac{m_1 n_1}{a^2b^2} K_{12}Y_{11}\left[-B_4^{\,2}\left(m_1+m_2\right)\left(n_1-n_2\right)S_{12}T_{21} + \right.$$
$$+A_4^{\,2}\left(m_1+m_2\right)\left(n_1+n_2\right)S_{12}T_{11} + C_4^{\,2}\left(m_1-m_2\right)\left(n_1+n_2\right)S_{22}T_{11} - D_4^{\,2}\left(m_1-m_2\right)\left(n_1-n_2\right)S_{12}T_{22} - \quad \text{(B.12)}$$
$$-2B_5^{\,2} m_1^3 n_2^2\left(n_1+n_2\right)T_{11}K_{12} + 2C_5^{\,2} m_2^3 n_1^2\left(n_1-n_2\right)T_{21}K_{22} + 2A_5^{\,2} m_1^3 n_2^2\left(n_1-n_2\right)T_{21}K_{12}$$
$$\left.\left. -2D_5^{\,2} m_2^3 n_1^2\left(n_1+n_2\right)T_{11}K_{22} \right]\right\}(1-X_{22})L_{21}dxdy,$$

where $G^2 = -\dfrac{32\varphi_{122}}{m_1^2 n_1^2}$, $H^2 = -\dfrac{32\varphi_{222}}{m_2^2 n_2^2}$.

Marina V. Shitikova, Prof., Dr.Sc., Research Center on Dynamics of Solids and Structures; Voronezh State Technical University; 20-letiya Oktyabrya 84, Voronezh, 394006, Russia; Department of Structural and Theoretical Mechanics, Moscow State University of Civil Engineering, Moscow. Phone +7 (473) 271-42-20; fax +7 (473) 277-39-92; E-mail: mvs@vgasu.vrn.ru; ORCID: 0000-0003-2186-1881.

Шитикова Марина Вячеславовна, профессор, доктор физико-математических наук; руководитель международного научного центра по фундаментальным исследованиям в области естественных и строительных наук имени Заслуженного деятеля науки РФ, профессора Россихина Ю.А.; Воронежский государственный технический университет; 394006, Россия, г. Воронеж, ул. 20 лет Октября, д. 84, E-mail: mvs@vgasu.vrn.ru; ORCID: 0000-0003-2186-1881.

Anastasiya I. Krusser, Ph.D. student; Junior Researcher; Research Center on Dynamics of Solids and Structures, Voronezh State Tchnical University; E-mail: an.krusser@yandex.ru; ORCID: 0000-0001-6203-2495.

Анастасия И. Круссер, аспирант, младший научный сотрудник международного научного центра по фундаментальным исследованиям в области естественных и строительных наук имени Заслуженного деятеля науки РФ, профессора Россихина Ю.А., Воронежский государственный технический университет; E-mail: an.krusser@yandex.ru; ORCID: 0000-0001-6203-2495.