

LOCALIZATION OF SOLUTION OF THE PROBLEM FOR POISSON'S EQUATION WITH THE USE OF B-SPLINE DISCRETE-CONTINUAL FINITE ELEMENT METHOD

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Abstract: Localization of solution of the problem for Poisson's equation with the use of B-spline discrete-continual finite element method (specific version of wavelet-based discrete-continual finite element method) is under consideration in the distinctive paper. The original operational continual and discrete-continual formulations of the problem are given, some actual aspects of construction of normalized basis functions of a B-spline are considered, the corresponding local constructions for an arbitrary discrete-continual finite element are described, some information about the numerical implementation and an example of analysis are presented.

Keywords: localization, wavelet-based discrete-continual finite element method, B-spline discrete-continual finite element method, discrete-continual finite element method, finite element method, B-spline, numerical solution, Poisson's equation

ЛОКАЛИЗАЦИЯ РЕШЕНИЯ ЗАДАЧИ ДЛЯ УРАВНЕНИЯ ПУАССОНА НА ОСНОВЕ ВЕЙВЛЕТ-РЕАЛИЗАЦИИ ДИСКРЕТНО-КОНТИНУАЛЬНОГО МЕТОДА КОНЕЧНЫХ ЭЛЕМЕНТОВ С ИСПОЛЬЗОВАНИЕМ В-СПЛАЙНОВ

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Аннотация: В настоящей статье рассматривается локализация решения задачи для уравнения Пуассона на основе вейвлет-реализации дискретно-континуального метода конечных элементов с использованием В-сплайнов. Приведены исходные операторные континуальная и дискретно-континуальная постановки задачи, рассмотрены некоторые актуальные вопросы построения нормализованных базисных функций В-сплайна, описаны соответствующие локальные построения для произвольного дискретно-континуального конечного элемента, представлены некоторые сведения о численной реализации и пример расчета.

Ключевые слова: локализация, вейвлет-реализация метода конечных элементов, дискретно-континуальный метод конечных элементов, метод конечных элементов, В-сплайны, численное решение, уравнение Пуассона

INTRODUCTION

Various problems of continuum mechanics are reduced to the Poisson equation and other similar equations of elliptic type [1-6]. As is known, boundary value problems with the Poisson equation describe, in particular, a stationary

temperature field, a stress state during torsion of a rod, membrane deflection, etc. In addition, the operator of the corresponding problem (the Laplace operator) is part of other problems that determine the state of structures under stationary and non-stationary actions. From a mathematical point of view, it is the simplest qualita-

tive analogue of other problems and an equivalent operator in iterative processes [7]. In many numerical models, at different time steps, it becomes necessary to solve (numerically) one or several boundary value problems for the Poisson equation, and in some applications the number of time steps during one analysis of the model can be of the order of thousands to millions or more [8]. In this regard, the objective of the distinctive paper is devoted to the semi-analytical method of analysis of corresponding structures with constant physical and geometric parameters in one of the directions (the so-called “basic direction”) [7, 9, 10]. This objective seems to be very relevant. The considering method is semi-analytical in the sense that along the basic direction of the structure the problem remains continual and its exact analytical solution is constructed, while in another, non-basic direction, a numerical approximation is performed. In general, this paper continues a series of papers devoted to the research and development of various wavelet-based versions of the discrete-continuous finite element method.

In the theory of boundary value problems for the Poisson and Laplace equations, several classical well-tested solution methods are normally used [1, 11-13], which, in particular, include method of separation of variables or Fourier method, Green's function method and a method of reducing boundary value problems for the Laplace equation to integral equations using potential theory.

Besides, numerical methods (finite element method, boundary element method, finite difference method, variational-difference method, finite volume method, method of point field sources, fast Fourier transform method using parallel computations (with the implementation on the cores of the central processor and on graphic processors (GPU), etc.) for solving the Poisson equation are normally used [8, 14, 15].

1. FORMULATIONS OF THE PROBLEM

Formulation of the problem has the form:

$$Lu = F, \quad 0 \leq x_1 \leq \ell_1, \quad 0 \leq x_2 \leq \ell_2; \quad (1.1)$$

$$\ell u|_{\Gamma} = g, \quad (1.2)$$

where L is the operator of the problem within the initial domain;

$$L = -\partial_1^2 - \partial_2^2; \quad \partial_1 = \partial / \partial x_1; \quad \partial_2 = \partial / \partial x_2; \quad (1.3)$$

ℓ is the operator of boundary conditions.

Let x_2 be direction along which parameters of the problem are constant (so-called “main direction”). Let us introduce the following notations

$$\bar{v} = \partial_2 \bar{u} = \bar{u}'; \quad \bar{v}' = \partial_2 \bar{v}. \quad (1.4)$$

Then we can rewrite (1.1) in the following form:

$$L_{uu}u - L_{vv}v' = F \quad \text{or} \quad L_{vv}v' = L_{uu}u - F, \quad (1.5)$$

where we have

$$L_{uu} = -\partial_1^2 = \partial_1^* \partial_1; \quad L_{vv} = 1. \quad (1.6)$$

Finally we obtain system of differential equations with operational coefficients:

$$\bar{U}' = A\bar{U} + \bar{F}, \quad (1.7)$$

where

$$A = \begin{bmatrix} 0 & 1 \\ L_{vv}^{-1}L_{uu} & 0 \end{bmatrix}; \quad \bar{F} = \begin{bmatrix} 0 \\ -L_{vv}^{-1}F \end{bmatrix}; \quad \bar{U} = \begin{bmatrix} u \\ v \end{bmatrix}. \quad (1.8)$$

The system of equations (1.7) is supplemented by boundary conditions, which are set in sections with coordinates $x_2^1 = 0$ and $x_2^2 = \ell_2$.

2. SOME ASPECTS OF THE CONSTRUCTION OF NORMALIZED BASIS FUNCTIONS OF THE B-SPLINE

The construction of B-spline basic functions is determined by the recursive Cox-de Boer formulas [16-21]:

$$k=1: \quad \varphi_{i,1}(t) = \begin{cases} 1, & x_i \leq t < x_{i+1}, \\ 0, & t < x_i \vee t \geq x_{i+1}, \end{cases} \quad (2.1)$$

$$k \geq 2: \quad \varphi_{i,k}(t) = \frac{(t-x_i)\varphi_{i,k-1}(t)}{x_{i+k-1}-x_i} + \frac{(x_{i+k}-t)\varphi_{i+1,k-1}(t)}{x_{i+k}-x_{i+1}}. \quad (2.2)$$

We will consider such a construction for the case $x_i = i$ are integers. Let us note that,

$$\varphi_{i,k}(t) = \varphi_{0,k}(t-i)$$

and therefore, recursive formulas (2.1)-(2.2) can be represented in the form

$$k=1: \quad \varphi_{0,1}(t) = \begin{cases} 1, & 0 \leq t < 1 \\ 0, & t < 0 \vee t \geq 1; \end{cases} \quad (2.3)$$

$$k \geq 2: \quad \varphi_{0,k}(t) = \frac{1}{k-1} [t \cdot \varphi_{0,k-1}(t) + (k-t)\varphi_{0,k-1}(t-1)]. \quad (2.4)$$

The function $\varphi_{0,1}(t)$ can be represented by formula

$$\varphi_{0,1}(t) = \frac{1}{2} [\text{sign}(t) - \text{sign}(t-1)]. \quad (2.5)$$

Let us denote by Δ_1 the operator of the first difference. Then we have

$$\varphi_{0,1}(t) = -\frac{1}{2} \Delta_1 \text{sign}(t). \quad (2.6)$$

We can substitute formula (2.5) into (2.4) in order to determine $\varphi_{0,2}(t)$:

$$\begin{aligned} \varphi_{0,2}(t) &= 1 \cdot [t \cdot \varphi_{0,1}(t) + (2-t)\varphi_{0,1}(t-1)] = \\ &= \frac{1}{2} [t \cdot [\text{sign}(t) - \text{sign}(t-1)] + \\ & (2-t)[\text{sign}(t-1) - \text{sign}(t-2)]] = \\ &= \frac{1}{2} [t \text{sign}(t) - 2(t-1)\text{sign}(t-1) + \\ & (t-2)\text{sign}(t-2)] = \frac{1}{2} [|t| - 2|t-1| + |t-2|]. \end{aligned}$$

Let us denote by Δ_2 the operator of the second difference. Then we have

$$\varphi_{0,2}(t) = \frac{1}{2} [|t| - 2|t-1| + |t-2|] = \frac{1}{2} \Delta_2 |t-1|. \quad (2.7)$$

We can define function $\varphi_{0,3}(t)$:

$$\varphi_{0,3}(t) = \frac{1}{2} [t \cdot \varphi_{0,2}(t) + (3-t)\varphi_{0,2}(t-1)].$$

Omitting intermediate calculations, we get

$$\begin{aligned} \varphi_{0,3}(t) &= \frac{1}{4} [t \cdot |t| - 3(t-1)|t-1| + \\ & + 3(t-2)|t-2| - (t-3)|t-3|] = \\ &= -\frac{1}{2!} \frac{1}{2} \Delta_1 \Delta_2 (|t-1|). \end{aligned} \quad (2.8)$$

Based on formulas (2.8) and (2.4), we can define the function

$$\varphi_{0,4}(t) = \frac{1}{3} [t \cdot \varphi_{0,3}(t) + (4-t)\varphi_{0,3}(t-1)].$$

Omitting intermediate calculations, as a result we get

$$\begin{aligned} \varphi_{0,4}(t) &= \\ &= \frac{1}{2 \cdot 3} \cdot \frac{1}{2} [t^2 \cdot |t| - 4(t-1)^2 |t-1| + \\ & + 6(t-2)^2 |t-2| - 4(t-3)^2 |t-3| + \\ & + (t-4)^2 |t-4|] = \\ &= \frac{1}{3!} \frac{1}{2} (\Delta_2)^2 (|t-2|). \end{aligned} \quad (2.9)$$

It can be proved that for even $k = 2m$ we have

$$\varphi_{0,k}(t) = \frac{1}{(2m-1)!} \frac{1}{2} (\Delta_2)^m (|t-m|) \quad (2.10)$$

and for odd (uneven) $k = 2m+1$ we have

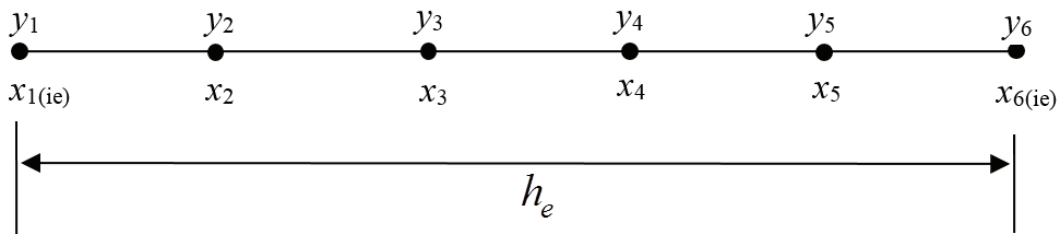


Figure 3.1. Finite element discretization for $N_k = 5$ (sample).

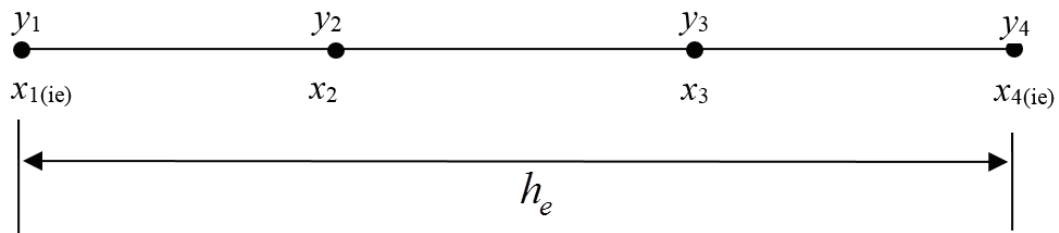


Figure 3.2. Finite element discretization for $N_k = 3$ (sample).

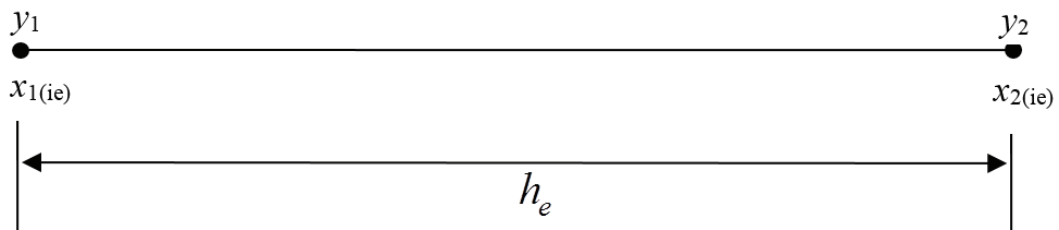


Figure 3.3. Finite element discretization for $N_k = 1$ (sample).

$$\varphi_{0,k}(t) = -\frac{1}{(2m)!} \frac{1}{2} \Delta_1(\Delta_2)^m ((t-m)^{2m-1} |t-1|). \quad (2.11)$$

Note that $\varphi_{0,k}(t)$ is a polynomial of degree $k-1$ with bounded support and, as follows from the difference operator, this support is equal to the interval $[0, k]$.

In addition, we should note the following property of B-spline basis functions:

$$\sum_i \varphi_{0,k}(t-i) \equiv 1 \text{ for arbitrary } t. \quad (2.12)$$

3. SOME GENERAL ASPECTS OF FINITE ELEMENT APPROXIMATION

The discrete component of the numerical solution is represented by the direction along the axis corresponding to x_1 . The fulfillment within

an element (interval) for all components of a vector functions \bar{u} and \bar{v} (see (1.8)) is the same. Therefore, let us use the following notation for simplicity:

$$x = x_1, \ell = \ell_1, y = y(x), \quad (3.1)$$

where $y = y(x)$ is unknown function (component of vector function).

Let us divide the interval $(0, \ell)$ segment into N_e parts (elements). Therefore $h_e = \ell / N_e$ is the length of the element. Besides, let us also divide each element into N_k parts. It should be noted that on the elements of the localization of the solution, parameter N_k is of greater importance than on the other elements. For example, on localization elements, we can set $N_k = 5$, i.e. unknown functions will be represented by polynomials (B-splines) of the 5th degree (Figure 3.1).

Let us use the following notation system: i_e is the element number; $N_p = N_k + 1 = 6$ is the number of nodes within the element; $x_1(i_e)$ is the coordinate of the starting point of the i_e -th element; $x_6(i_e)$ is the coordinate of the end point of the i_e -th element. Thus, the number of unknowns per element with such approximation is equal to

$$N_{ie} = N_p = 6.$$

For the elements of localization we can take reduced number of N_k . For instance, if we take $N_k = 3$ (Figure 3.2) we get $N_p = N_k + 1 = 4$ and the number of unknowns per element with such approximation is equal to

$$N_{ie} = N_p = 4;$$

$x_1(i_e)$ is the coordinate of the starting point of the i_e -th element; $x_4(i_e)$ is the coordinate of the end point of the i_e -th element.

Besides, let us consider the case with $N_k = 1$ (Figure 3.3). Therefore we have $N_p = N_k + 1 = 2$ and the number of unknowns per element with such approximation is equal to

$$N_{ie} = N_p = 2,$$

where $x_1(i_e)$ is the coordinate of the starting point of the i_e -th element; $x_2(i_e)$ is the coordinate of the end point of the i_e -th element.

4. LOCAL CONSTRUCTIONS FOR ARBITRARY FINITE ELEMENT

Let us introduce local coordinates:

$$t = (x - x_{1(ie)}) / h_e, \quad x_{1(ie)} \leq x \leq x_{N_p(ie)}, \quad 0 \leq t \leq 1. \quad (4.1)$$

In this case, we have the following relations:

$$x = x_i \Rightarrow t_i = (x_i - x_{1(ie)}) / h_e, \quad i = 1, \dots, N_p; \quad (4.2)$$

$$\frac{d^p}{dx^p} = \frac{1}{h_e^p} \frac{d^p}{dt^p}; \quad dx = h_e \cdot dt. \quad (4.3)$$

Since the number of unknowns on the element is equal to $N_{ie} = 6$, we use a B-spline of the fifth degree in order to represent the unknown deflection function.

Let us use the following notation:

$$\varphi(t) = \varphi_{0,6}(t+3);$$

$$\begin{aligned} \varphi(t) &= \frac{1}{5!} \frac{1}{2} (\Delta_2)^3 (t^4 | t |) = \\ &= \frac{1}{5! \cdot 2} [(t+3)^4 | t+3 | - 6(t+2)^4 | t+2 | + \\ &+ 15(t+1)^4 | t+1 | - 20t^4 | t | + \\ &+ 15(t-1)^4 | t-1 | - 6(t-2)^4 | t-2 | + \\ &+ (t-3)^4 | t-3 |]. \end{aligned} \quad (4.4)$$

This function is a B-spline, symmetric with respect to $t=0$ and its support is defined by an interval $[-3, 3]$ (Figure 4.1).

We take the following six functions as basis functions on the unit interval (Figures 4.2, 4.3):

$$\begin{aligned} \varphi_1(t) &= \varphi(t+2), \quad \varphi_2(t) = \varphi(t+1), \\ \varphi_3(t) &= \varphi(t), \quad \varphi_4(t) = \varphi(t-1), \\ \varphi_5(t) &= \varphi(t-2), \quad \varphi_6(t) = \varphi(t-3), \\ &0 \leq t \leq 1. \end{aligned} \quad (4.5)$$

Since the number of unknowns on the element is equal to $N_{ie} = 4$, we use a B-spline of the third degree in order to represent the unknown deflection function.

Let us use the following notation:

$$\varphi(t) = \varphi_{0,4}(t+4);$$

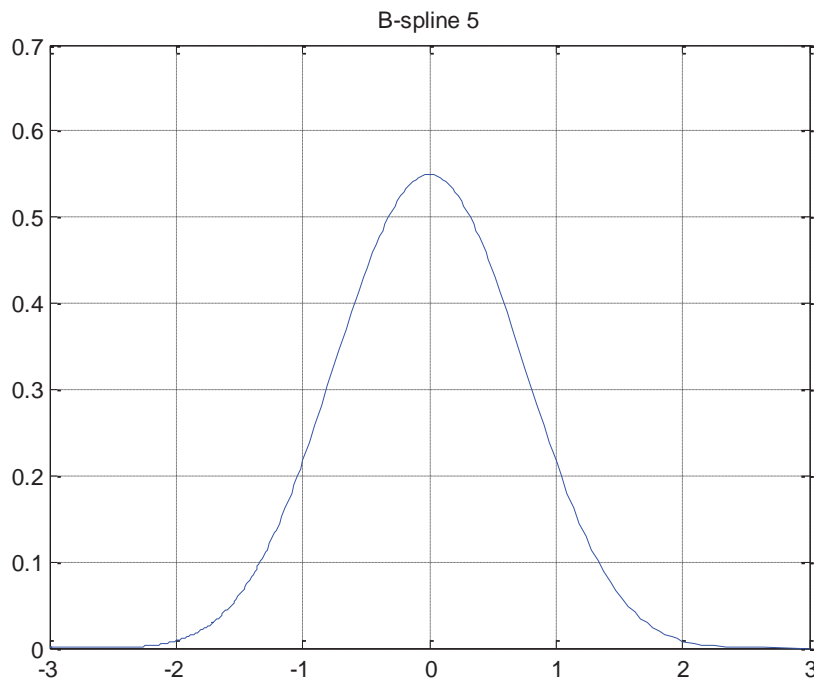


Figure 4.1. B-spline of the fifth order $\varphi(t) = \varphi_{0,6}(t + 3)$.

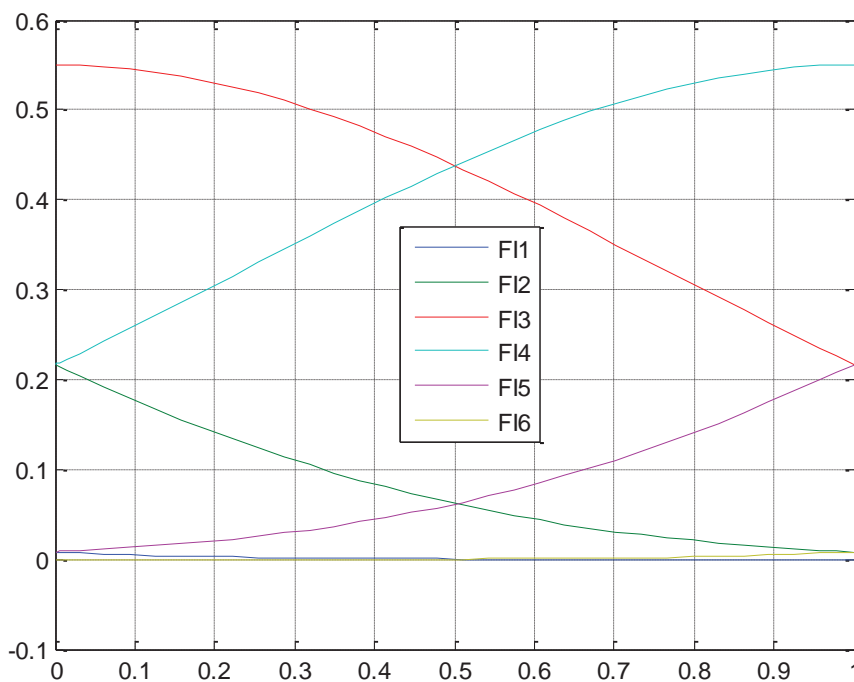


Figure 4.2. Basis functions $\varphi_k(t)$, $k = 1, 2, \dots, 6$.

$$\begin{aligned} \varphi(t) &= \frac{1}{3!} \frac{1}{2} (\Delta_2)^2 (t^2 |t|) = \\ &= \frac{1}{3! \cdot 2} [(t+2)^2 |t+2| - 4(t+1)^2 |t+1| + \\ &+ 6t^2 |t| - 4(t-1)^2 |t-1| + \\ &+ (t-2)^2 |t-2|]. \end{aligned}$$

(4.6)

This function is a B-spline, symmetric with respect to $t = 0$ and its support is defined by an interval $[-2, 2]$ (Figure 4.4).

We take the following four functions as basis functions on the unit interval (Figures 4.5):

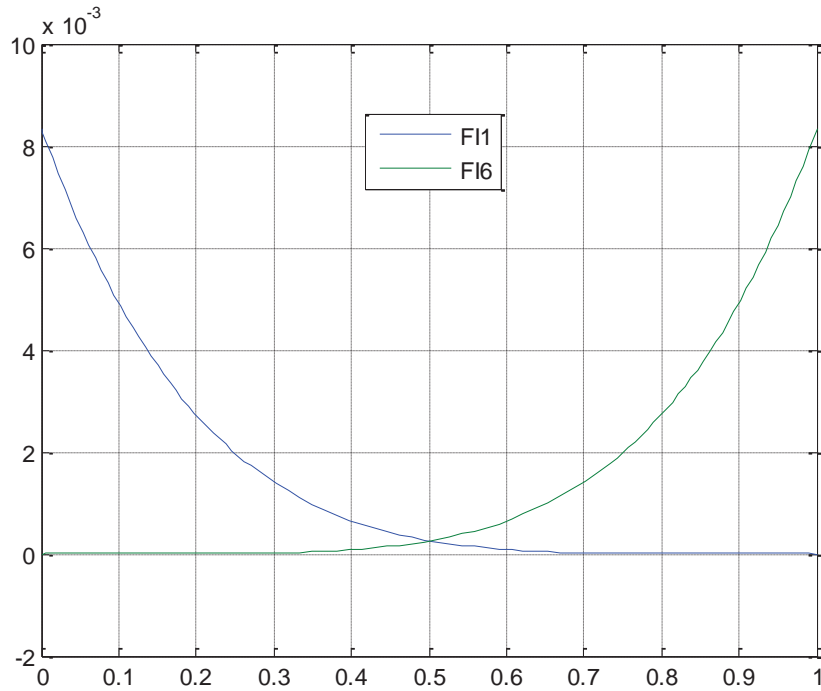


Figure 4.3. Basis functions $\varphi_1(t)$ and $\varphi_6(t)$.

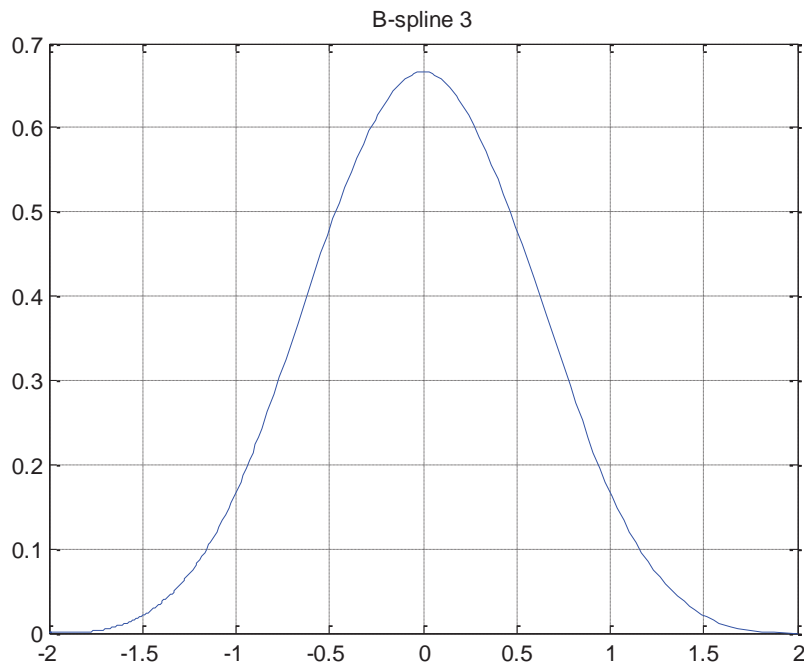


Figure 4.4. B-spline of the third order $\varphi(t) = \varphi_{0,4}(t+2)$.

$$\begin{aligned} \varphi_1(t) &= \varphi(t+1), & \varphi_2(t) &= \varphi(t), \\ \varphi_3(t) &= \varphi(t-1), & \varphi_4(t) &= \varphi(t-2), \\ & & 0 \leq t \leq 1. & \quad (4.7) \end{aligned}$$

Since the number of unknowns on the element is equal to $N_{ie} = 2$, we use a B-spline of the first degree in order to represent the unknown deflection function.

Let us use the following notation:

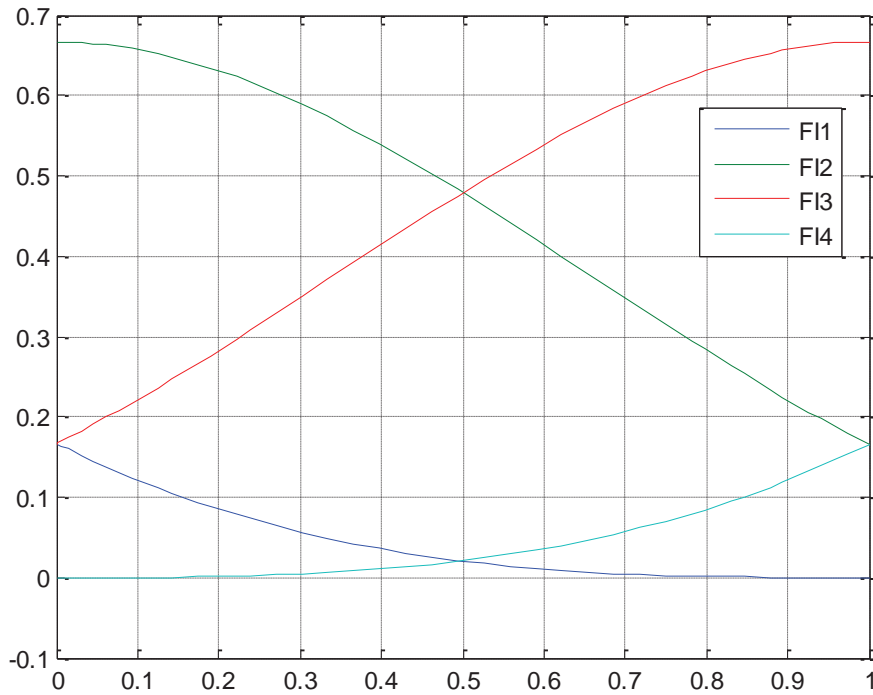


Figure 4.5. Basis functions $\varphi_k(t)$, $k = 1, 2, 3, 4$.

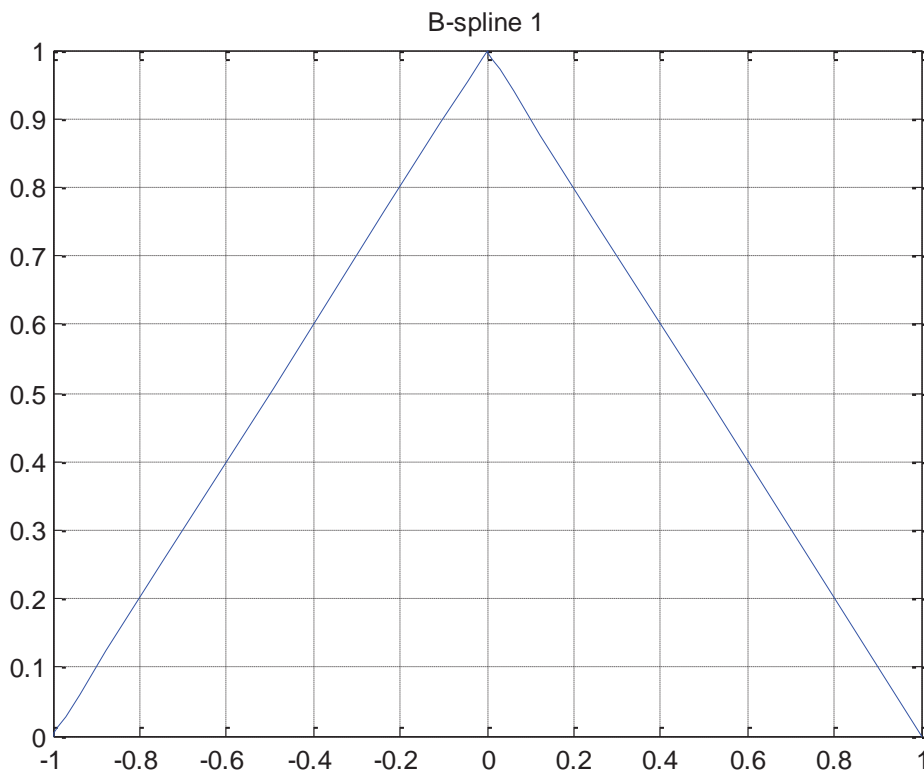


Figure 4.6. B-spline of the first order $\varphi(t) = \varphi_{0,2}(t+1)$.

$$\varphi(t) = \varphi_{0,2}(t+1);$$

$$\varphi(t) = \frac{1}{2} \Delta_2 |t| = \frac{1}{2} [|t+1| - 2|t| + |t-1|]. \quad (4.8)$$

This function is a B-spline, symmetric with respect to $t=0$ and its support is defined by an interval $[-1,1]$ (Figure 4.6).

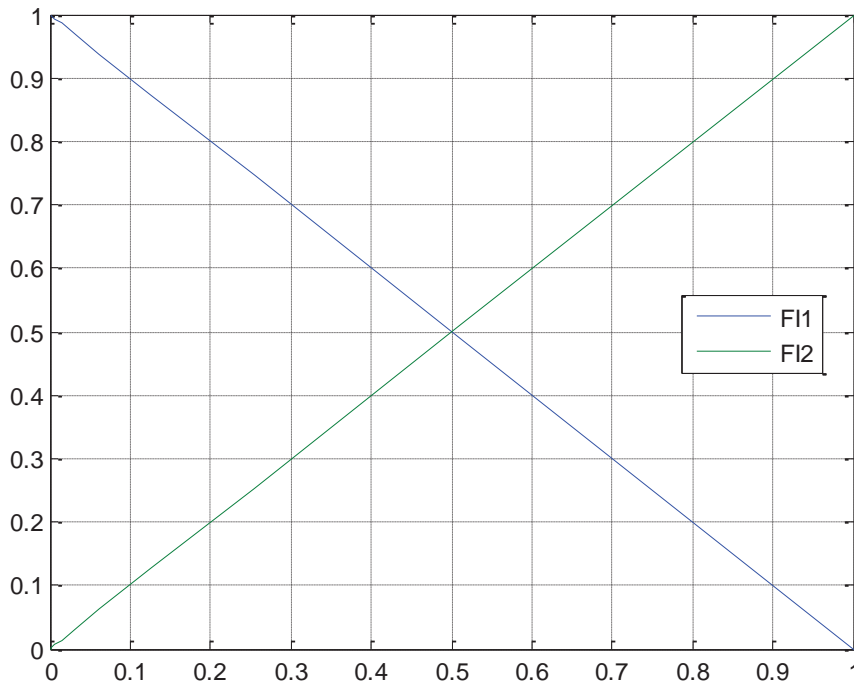


Figure 4.7. Basis functions $\varphi_k(t)$, $k = 1, 2$.

We take the following two functions as basis functions on the unit interval (Figures 4.7):

$$\varphi_1(t) = \varphi(t), \quad \varphi_2(t) = \varphi(t-1), \quad 0 \leq t \leq 1. \quad (4.9)$$

We represent the unknown function $y(x)$ within the element number i_e in the form

$$y(x) = w(t) = \sum_{k=1}^{N_p} \alpha_k \varphi_k(t), \quad x_{1(i_e)} \leq x \leq x_{N_p(i_e)}, \quad 0 \leq t \leq 1. \quad (4.10)$$

We have to consider bilinear forms with allowance for relations (4.2)-(4.3) in order to construct local stiffness matrices corresponding to the operators L_{uu} , L_{vv} (see (1.6)):

$$B_{uu}(y, z) = \ll L_{uu} y, z \gg = - \int_{x_{1(i_e)}}^{x_{N_p(i_e)}} \frac{d^2 y}{dx^2} \cdot z \, dx = \int_{x_{1(i_e)}}^{x_{N_p(i_e)}} \frac{dy}{dx} \cdot \frac{dz}{dx} \, dx = \frac{1}{h_e} \int_0^1 \frac{dw}{dt} \cdot \frac{dq}{dt} \, dt = B_{vv}(w, q); \quad (4.10)$$

$$B_{vv}(y, z) = \ll L_{vv} y, z \gg = \int_{x_{1(i_e)}}^{x_{N_p(i_e)}} y \cdot z \, dx = h_e \int_0^1 w \cdot q \, dt = B_{vv}(w, q). \quad (4.11)$$

for the following type of functions

$$y(x) = w(t) = \sum_{k=1}^{N_{ie}} \alpha_k \varphi_k(t), \quad z(x) = q(t) = \sum_{k=1}^{N_{ie}} \beta_k \varphi_k(t), \quad (4.12)$$

where we have $x_{1(i_e)} \leq x \leq x_{N_p(i_e)}$, $0 \leq t \leq 1$.

Let us substitute (4.12) into (4.9)-(4.10):

$$B_{uu}(w, q) = \frac{1}{h_e} \int_0^1 \frac{dw}{dt} \cdot \frac{dq}{dt} \, dt = \frac{1}{h_e} \sum_{i=1}^{N_{ie}} \sum_{j=1}^{N_{ie}} \alpha_i \beta_j \int_0^1 \varphi_i'(t) \varphi_j'(t) \, dt = \frac{1}{h_e} (K_{\alpha\beta}^{uu} \bar{\alpha}, \bar{\beta}), \quad (4.16)$$

where

where

$$K_{\alpha\beta}^{uu}(i, j) = \int_0^1 \varphi_i'(t) \varphi_j'(t) dt; \quad \varphi' = \frac{d\varphi}{dt}; \quad (4.17)$$

$$B_{vv}(w, q) = h_e \int_0^1 w q dt =$$

$$= h_e \sum_{i=1}^{N_{ie}} \sum_{j=1}^{N_{ie}} \alpha_i \beta_j \int_0^1 \varphi_i(t) \varphi_j(t) dt = h_e (K_{\alpha\beta}^{vv} \bar{\alpha}, \bar{\beta}), \quad (4.18)$$

where

$$K_{\alpha\beta}^{vv}(i, j) = \int_0^1 \varphi_i(t) \varphi_j(t) dt; \quad (4.19)$$

$$T_6 = \begin{bmatrix} \varphi_1(0) & \varphi_2(0) & \varphi_3(0) & \varphi_4(0) & \varphi_5(0) & \varphi_6(0) \\ \varphi_1(0.2) & \varphi_2(0.2) & \varphi_3(0.2) & \varphi_4(0.2) & \varphi_5(0.2) & \varphi_6(0.2) \\ \varphi_1(0.4) & \varphi_2(0.4) & \varphi_3(0.4) & \varphi_4(0.4) & \varphi_5(0.4) & \varphi_6(0.4) \\ \varphi_1(0.6) & \varphi_2(0.6) & \varphi_3(0.6) & \varphi_4(0.6) & \varphi_5(0.6) & \varphi_6(0.6) \\ \varphi_1(0.8) & \varphi_2(0.8) & \varphi_3(0.8) & \varphi_4(0.8) & \varphi_5(0.8) & \varphi_6(0.8) \\ \varphi_1(1) & \varphi_2(1) & \varphi_3(1) & \varphi_4(1) & \varphi_5(1) & \varphi_6(1) \end{bmatrix}, \quad (4.24)$$

For the case $N_{ie} = 4$ we have

$$\bar{y}^{ie} = T_4 \bar{\alpha}, \quad (4.25)$$

$$\bar{y}^{ie} = [y_1 \ y_2 \ y_3 \ y_4]^T; \quad (4.26)$$

$$\bar{\alpha} = [\alpha_1 \ \alpha_2 \ \alpha_3 \ \alpha_4]^T; \quad (4.27)$$

where

$$T_4 = \begin{bmatrix} \varphi_1(0) & \varphi_2(0) & \varphi_3(0) & \varphi_4(0) \\ \varphi_1(1/3) & \varphi_2(1/3) & \varphi_3(1/3) & \varphi_4(1/3) \\ \varphi_1(2/3) & \varphi_2(2/3) & \varphi_3(2/3) & \varphi_4(2/3) \\ \varphi_1(1) & \varphi_2(1) & \varphi_3(1) & \varphi_4(1) \end{bmatrix}; \quad (4.28)$$

For the case $N_{ie} = 2$ we have

$$\bar{y}^{ie} = T_2 \bar{\alpha}, \quad (4.29)$$

$$T_2 = \begin{bmatrix} \varphi_1(0) & \varphi_2(0) \\ \varphi_1(1) & \varphi_2(1) \end{bmatrix}. \quad (4.32)$$

where

$$\bar{y}^{ie} = [y_1 \ y_2]^T; \quad (4.30)$$

$$\bar{\alpha} = [\alpha_1 \ \alpha_2]^T; \quad (4.31)$$

Let us define the parameters α_k and β_k through the nodal unknowns on the element:

$$y_i = w(t_i) = \sum_{k=1}^{N_{ie}} \alpha_k \varphi_k(t_i),$$

$$t_i = (x_i - x_{1(ie)}) / h_e, \quad i = 1, \dots, N_p. \quad (4.20)$$

For the case $N_{ie} = 6$ we have

$$\bar{y}^{ie} = T_6 \bar{\alpha}, \quad (4.21)$$

where

$$\bar{y}^{ie} = [y_1 \ y_2 \ y_3 \ y_4 \ y_5 \ y_6]^T; \quad (4.22)$$

$$\bar{\alpha} = [\alpha_1 \ \alpha_2 \ \alpha_3 \ \alpha_4 \ \alpha_5 \ \alpha_6]^T. \quad (4.23)$$

Similarly, we get

$$\bar{z}^{ie} = T_{N_{ie}} \bar{\beta} \quad (4.33)$$

for $N_{ie} = 6, N_{ie} = 4, N_{ie} = 2$.

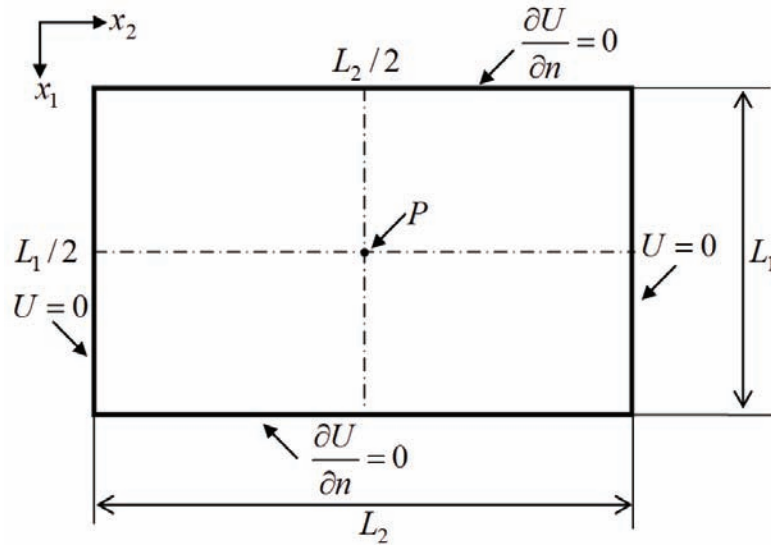


Figure 5.1. Example of analysis.

From (4.21)-(4.33) it follows

$$\bar{\alpha} = T_{N_p}^{-1} \bar{y}^{ie}; \quad \bar{\beta} = T_{N_p}^{-1} \bar{z}^{ie}, \quad (4.34)$$

where

$$T_{N_p} = \{T_{ij}\}_{i,j=1,\dots,N_p}, \quad T_{ij} = \varphi_j(t_i). \quad (4.35)$$

Generally we have the following chain of equalities

$$\begin{aligned} (K_{\alpha\beta} \bar{\alpha}, \bar{\beta}) &= (K_{\alpha\beta} T_{N_p}^{-1} \bar{y}^{ie}, T_{N_p}^{-1} \bar{z}^{ie}) = \\ &= ((T_{N_p}^{-1})^T K_{\alpha\beta} T_{N_p}^{-1} \bar{y}^{ie}, \bar{z}^{ie}). \end{aligned} \quad (4.36)$$

Therefore, substituting (4.34) sequentially in (4.16), (4.18), we obtain local stiffness matrices K_{uu}^{ie} and K_{vv}^{ie} , corresponding to the operators L_{uu} and L_{vv} .

5. EXAMPLE OF ANALYSIS

5.1. Formulation of the problem.

Let us consider the problem shown at Figure 5.1. Let us consider the following geometric parameters: $L_1 = 1.2$, $L_2 = 2.0$ is the thickness. Let external load parameter be equal to $P = 100$.

5.2. Structural analysis with allowance for localization.

Let the number of elements be equal to $N_e = 4$. Then we have the following element length:

$$h_e = \ell_1 / N_e = 1.2 / 4 = 0.3.$$

Let's define localization in the load area. For the first element and for the fourth element we have $N_k = 1$ and third-order spline; distance between the coordinates of the nodes of the first element and the sixth element is equal to

$$h_1 = h_4 = 0.3 / 1 = 0.3.$$

For the second element and for the third element we have $N_k = 3$ and fifth-order spline; distance between the coordinates of the nodes of the second element and the third element is equal to

$$h_2 = h_3 = 0.3 / 5 = 0.06.$$

The total number of nodes for all elements is equal to

$$N_x = 2 \cdot 1 + 2 \cdot 5 + 1 = 13.$$

The total number of nodal unknowns is equal to

$$N_U = 2N_x = 2 \cdot 13 = 26.$$

5.3. Structural analysis without localization.

In this case, we will consider only the standard linear fulfilment. In this case, the length of the element is taken equal to the minimum distance between the nodes, i.e. $h_e = 0.06$. Then the number of elements is equal to

$$N_e = 1.2/0.06 = 20$$

and the total number of nodes is equal to $N_x = 21$. In this case the total number of nodal unknowns is equal to

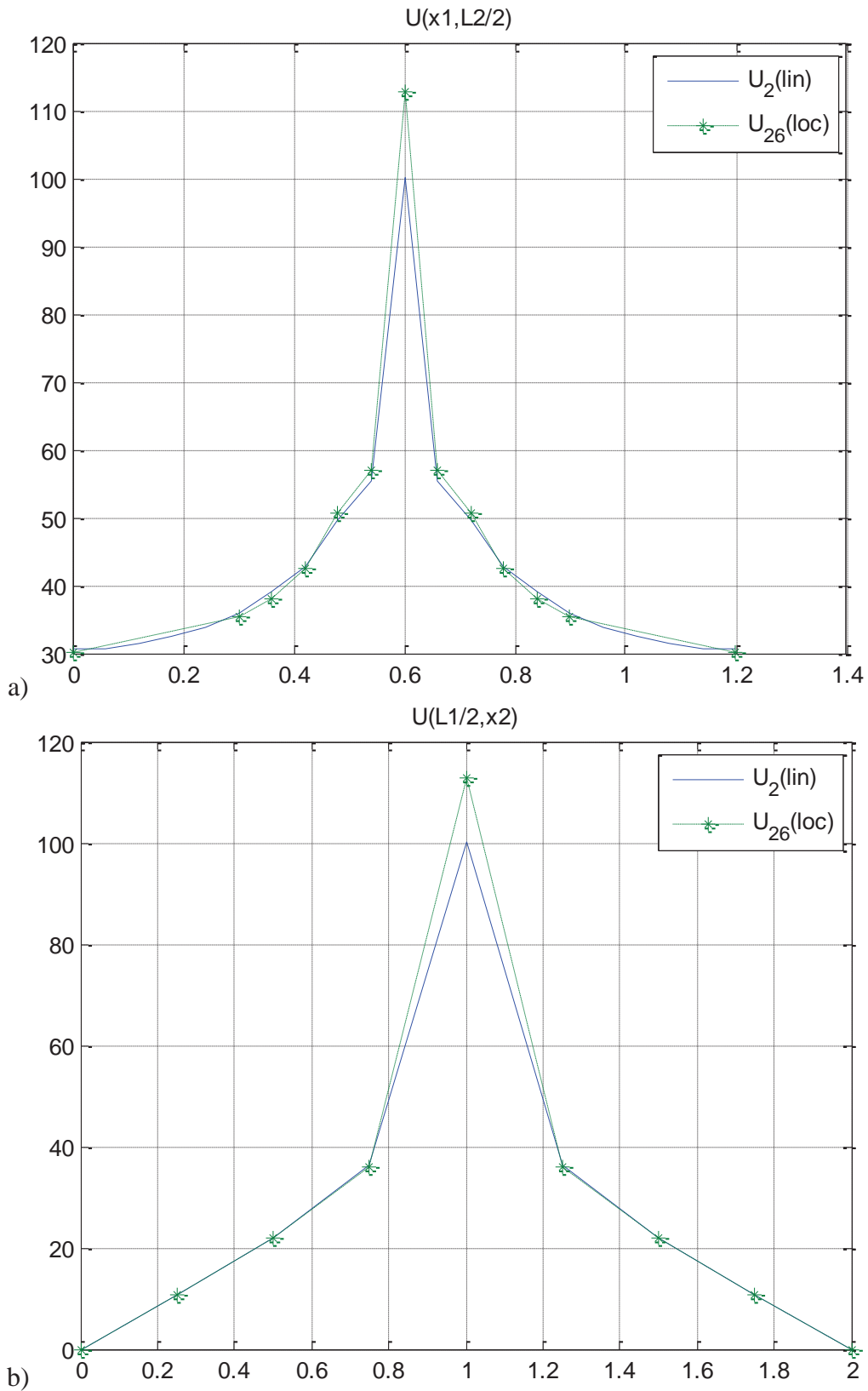
$$N_U = 2N_x = 2 \cdot 21 = 42.$$

Graphical comparison of corresponding results of analysis is presented at Figures 5.2-5.4 (U26(10c) are nodal values computed with allowance for localization; U2(1in) are nodal values computed without localization).

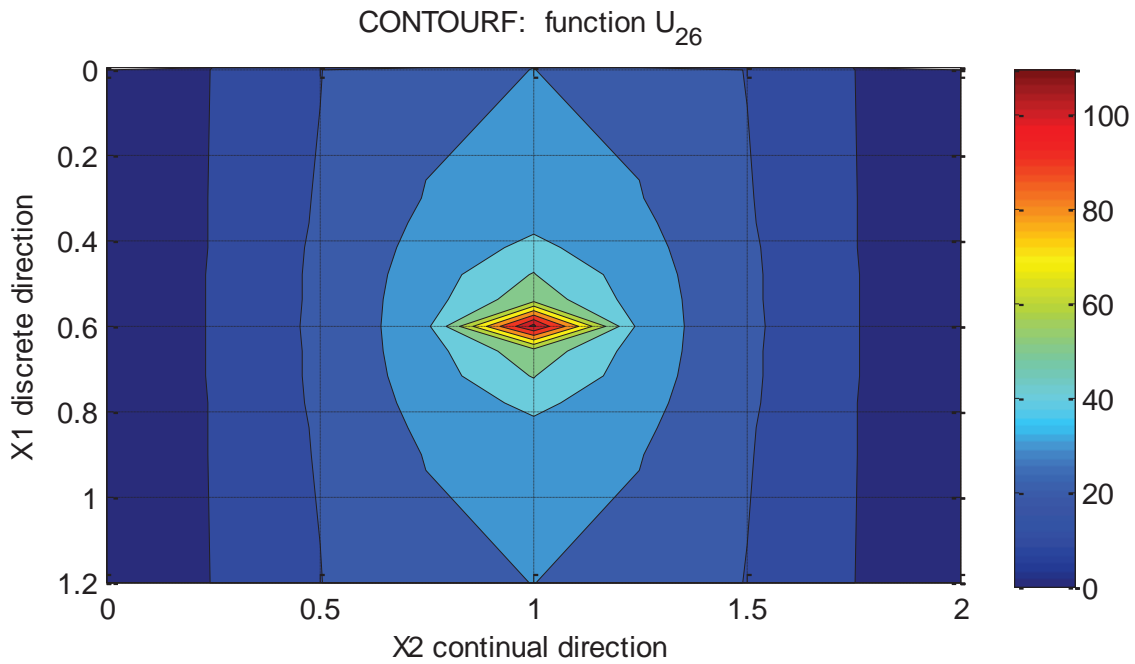
As researcher can see, the results obtained are almost completely identical. Besides, the use of localization based on application of B-splines of various degrees leads to a significant decrease in the number of unknowns.

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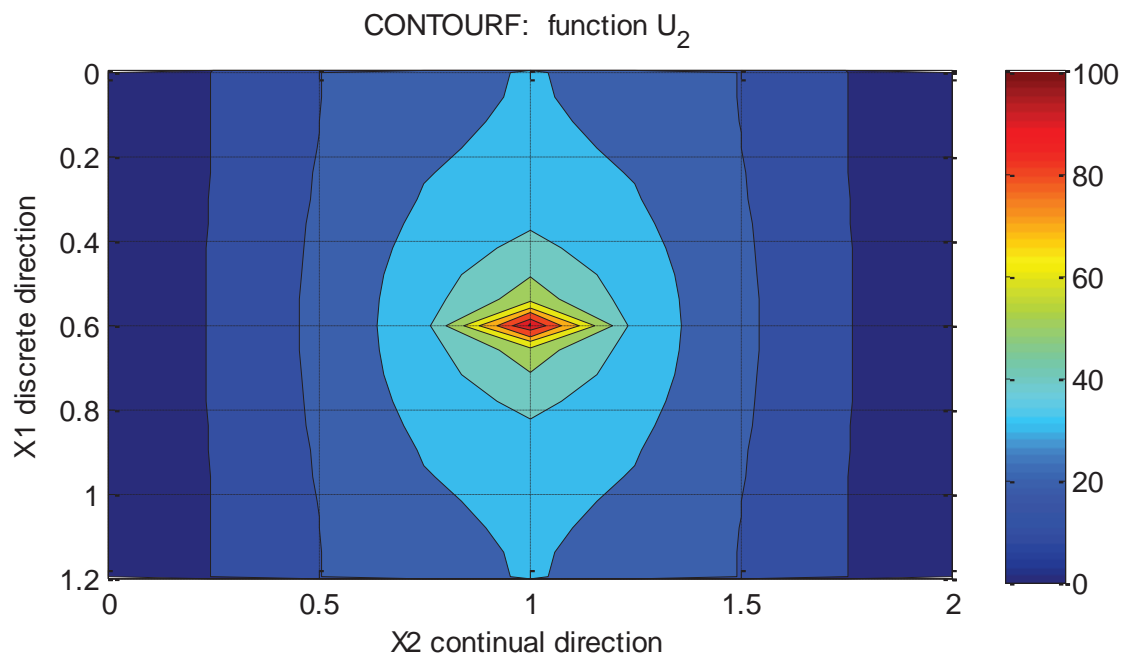
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*Figures 5.2. Comparison of the results of analysis in the middle sections:
 a) along x_1 direction (discrete direction);
 b) along x_2 direction (continual direction).*



Figures 5.3. Solution with the use of localization.



Figures 5.4. Solution with the use of standard (linear) approximation of function within the element.

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