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FINITE ELEMENTS FOR THE ANALYSIS OF REISSNER-MINDLIN PLATES WITH JOINT INTERPOLATION OF DISPLACEMENTS AND ROTATIONS (JIDR)

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Abstract: This paper proposes a method for creating finite elements with simultaneous approximation of functions corresponding to displacements and rotations. New triangular and quadrangular finite elements have been created, which can have additional nodes on the sides. No locking effect is observed for all the created elements. All created elements retain the existing symmetry of the design models. The results of numerical experiments are presented.

Keywords: finite elements; Reissner–Mindlin; plate problem; triangular element; rectangular element; quadrangular element

КОНЕЧНЫЕ ЭЛЕМЕНТЫ ДЛЯ РАСЧЕТА ПЛАСТИН РЕЙССНЕРА–МИНДЛИНА С СОВМЕСТНОЙ ИНТЕРПОЛЯЦИЕЙ ПЕРЕМЕЩЕНИЙ И УГЛОВ ПОВОРОТА (JIDR)

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Аннотация: Предложен метод построения конечных элементов с одновременной аппроксимацией функций, соответствующих перемещениям и углам поворота. Построены новые треугольные и четырехугольные элементы конечные элементы, которые могут иметь дополнительные узлы на сторонах. Для всех построенных элементов отсутствует эффект запирания. Все построенные элементы сохраняют существующую симметрию расчетных схем. Приведены результаты численных экспериментов.

Ключевые слова: конечные элементы; Рейсснер-Миндлин; изгиб плит; треугольный элемент; прямоугольный элемент; четырехугольный элемент

1. INTRODUCTION

Write down the Lagrange functional for Reissner-Mindlin plates as follows [1-3]:

$$\Pi(\mathrm{u}) = \frac{1}{2}\int_{\Omega}(Au)^T DAud\,\Omega - \int_{\Omega} f^{\mathrm{T}} ud\,\Omega \quad (1)$$

where: Ω – plate of thickness h: solid body with a midplane ***XOY***;

$u(x) = \{w(x), \theta_x(x), \theta_y(x)\}^T$ is the vertical displacement and rotations,

$\boldsymbol{x} = \begin{Bmatrix} x \\ y \end{Bmatrix}$, $f(\boldsymbol{x}) = \begin{Bmatrix} f_z(\boldsymbol{x}) \\ m_x(\boldsymbol{x}) \\ m_y(\boldsymbol{x}) \end{Bmatrix}$ is the area load.

The geometric operator ***A*** and the elasticity matrix ***D*** (for an isotropic material):

$$\boldsymbol{A} = \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0 & 0 & 0 \\ 0 & -1 & 0 & -\frac{\partial}{\partial y} & -\frac{\partial}{\partial x} \\ 1 & 0 & \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial y} \end{bmatrix}^T, \quad (2)$$

$$D=\begin{bmatrix}\lambda &&&&\\ &\lambda&&&\\ &&1&\nu&\\ &&\nu&1&\\ &&&&0.5(1-\nu)\end{bmatrix},\quad \lambda=\frac{5(1-\nu)}{h^2}$$

E is the Young's modulus, ν is the Poisson's ratio.
Equations of equilibrium:

$$-\Delta w+\frac{\partial\theta_x}{\partial y}-\frac{\partial\theta_y}{\partial x}=\frac{q}{\lambda D}$$
$$\lambda(\theta_x-\frac{\partial w}{\partial y})-\frac{\partial^2\theta_x}{\partial y^2}-\frac{1-\nu}{2}\frac{\partial^2\theta_x}{\partial x^2}+\frac{1+\nu}{2}\frac{\partial^2\theta_y}{\partial x\partial y}=0$$
$$\lambda(\frac{\partial w}{\partial x}+\theta_y)-\frac{\partial^2\theta_y}{\partial x^2}-\frac{1-\nu}{2}\frac{\partial^2\theta_y}{\partial y^2}+\frac{1+\nu}{2}\frac{\partial^2\theta_x}{\partial x\partial y}=0 \quad (3)$$

Classic finite elements have three degrees of freedom in each node: vertical displacement w_i and rotations θ_{xi}, θ_{yi}, i=1,2,…,N, where N is the number of element nodes. Finite elements have $3N$ unknowns, which are arranged in the following order during the generation of a stiffness matrix of the element:

$$\{w_1,\theta_{x1},\theta_{y1},...,w_N,\theta_{xN},\theta_{yN}\}, \quad (4)$$

which has a corresponding system of approximating functions

$$\{\varphi_{1,1},\varphi_{1,2},\varphi_{1,3},\dots,\varphi_{N,1},\varphi_{N,2},\varphi_{N,3}\} \quad (5)$$

We introduce a generalized displacement vector

$$\tilde{\boldsymbol{u}}=\left\{w,\theta_x,\theta_y,\gamma_{xz},\gamma_{yz}\right\}^T \quad (6)$$

where :
γ_{xz}, γ_{yz} are shear characteristics depending on the displacement w and rotations θ_x, θ_y.
Then the functional (1) can be written as follows:

$$\Pi(\boldsymbol{u})=\frac{1}{2}\int_\Omega(B\tilde{u})^T DB\tilde{u}d\Omega-\int_\Omega \boldsymbol{f}^T\boldsymbol{u}d\Omega \quad (7)$$

$$\boldsymbol{B}=\begin{bmatrix}\frac{\partial}{\partial x}&0&1&1&0\\ \frac{\partial}{\partial y}&-1&0&0&1\\ 0&0&\frac{\partial}{\partial x}&0&0\\ 0&-\frac{\partial}{\partial y}&0&0&0\\ 0&-\frac{\partial}{\partial x}&\frac{\partial}{\partial y}&0&0\end{bmatrix} \quad (8)$$

Represent the approximating functions (5) of the element in a five-dimensional space for the generalized displacement vector (6):

$$\boldsymbol{\varphi}_{ij}^r(x,y)=\left\{\varphi_{ij}^1,\varphi_{ij}^2,\varphi_{ij}^3,\varphi_{ij}^4,\varphi_{ij}^5\right\}^T, i=1\div N, j=1,2,3 \quad (9)$$

where i is the node number, and j is the number of its degree of freedom.
Elements with:

- $\varphi_{i1}^1=\varphi_{i2}^2=\varphi_{i3}^3$ – the corresponding approximations of the elements of the plane problem of the theory of elasticity;
- $\varphi_{i1}^2=\varphi_{i1}^3=\varphi_{i2}^1=\varphi_{i2}^3=\varphi_{i3}^1=\varphi_{i3}^2=0$, $i=1\div N$
- $\varphi_{ij}^4=\varphi_{ij}^5=0$, $i=1\div N$, $j=1,2,3$,

as a rule, provide convergence of the method only for medium thickness plates. The so-called *locking effect* often occurs during the analysis of thin plates, when the calculation results differ significantly from the analytical ones.
The main reason for the locking effect is that it is impossible to set such values of the degrees of freedom of an element so as to ensure constant moments in its area for the corresponding tasks.
There are many methods for eliminating the locking mechanism. The most common elements use:

- *Mixed Interpolation of Tensorial Components, MITC* [2,4];
- *Discrete Shear Gap, DSG* [5];
- hybrid models based on Reissner's functional [6];

and others.

This paper proposes another method for creating finite elements for Reissner-Mindlin plates without the locking effect: *Joint Interpolation of Displacements and Rotations* (*JIDR*). In this method:

- $\varphi_{i1}^1,\ \varphi_{i2}^2,\ \varphi_{i3}^3$ – not necessarily correspond-ding approximations of the elements of the plane problem of the theory of elasticity. These can be, for example, approximations of finite elements for Kirchhoff-Love plates;
- $\varphi_{i2}^1 = \varphi_{i3}^1$ – nonzero functions depending on $\varphi_{i2}^2,\ \varphi_{i3}^3$;
- $\varphi_{i1}^2 = \varphi_{i1}^3 = \varphi_{i2}^3 = \varphi_{i3}^2 = 0,\quad \varphi_{ij}^4 = \varphi_{ij}^5 = 0$, j=1,2,3
- in addition to approximating functions (5), *up to four* specially constructed functions are introduced corresponding to some internal degrees of freedom of the element:

$$\boldsymbol{\mu}_k = \left\{0, 0, 0, \mu_k^4, \mu_k^5\right\}^T,\ k \le 4 \tag{10}$$

We will assume that the functions $\varphi_{i1}^1,\ \varphi_{i2}^2,\ \varphi_{i3}^3$ are compatible for the constructed ***JIDR*** elements. Incompatible functions are (10), which may have discontinuities on the element sides.

2. COMPLETENESS AND INCOMPATIBILITY CRITERIA

Consider the residual:

$$\zeta(\boldsymbol{x}) = \begin{Bmatrix} w \\ \theta_x \\ \theta_y \end{Bmatrix} - A_1 \sum_{i=1}^{N}\left(w_i \boldsymbol{\varphi}_{i1} + \theta_{x,i}\boldsymbol{\varphi}_{i2} + \theta_{y,i}\boldsymbol{\varphi}_{i3}\right), \tag{11}$$

where $\boldsymbol{A}_1$ is the matrix operator that transforms functions in a five-dimensional space into three-dimensional ones:

$$\boldsymbol{A}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

It follows from the equilibrium quations (3) that:

$$\theta_x = \frac{\partial w}{\partial y} + \frac{1}{\lambda}\left(\frac{\partial^2\theta_x}{\partial y^2} + \frac{1-\nu}{2}\frac{\partial^2\theta_x}{\partial x^2} - \frac{1+\nu}{2}\frac{\partial^2\theta_y}{\partial x\partial y}\right) =$$
$$\frac{\partial w}{\partial y} + \frac{1}{\lambda}\left(\frac{\partial^3 w}{\partial y^3} + \frac{1-\nu}{2}\frac{\partial^3 w}{\partial x^2\partial y} - \frac{1+\nu}{2}\frac{\partial^3 w}{\partial x\partial y^2}\right) + \ldots$$
$$\theta_y = -\frac{\partial w}{\partial x} + \frac{1}{\lambda}\left(\frac{\partial^2\theta_y}{\partial x^2} + \frac{1-\nu}{2}\frac{\partial^2\theta_y}{\partial y^2} - \frac{1+\nu}{2}\frac{\partial^2\theta_x}{\partial x\partial y}\right) =$$
$$-\frac{\partial w}{\partial x} - \frac{1}{\lambda}\left(\frac{\partial^3 w}{\partial x^3} + \frac{1-\nu}{2}\frac{\partial^3 w}{\partial x\partial y^2} - \frac{1+\nu}{2}\frac{\partial^3 w}{\partial x^2\partial y}\right) + \ldots \tag{12}$$

Substituting (12) into (11) and expanding the values of displacement w with respect to the origin, we obtain:

$$\boldsymbol{\zeta}(\boldsymbol{x}) = \begin{Bmatrix} w|_{\boldsymbol{x}=0} + xw_x|_{\boldsymbol{x}=0} + yw_y|_{\boldsymbol{x}=0} + \frac{1}{2}x^2 w_{xx}|_{\boldsymbol{x}=0} + xyw_{xy}|_{\boldsymbol{x}=0} + \frac{1}{2}y^2 w_{yy}|_{\boldsymbol{x}=0} + \ldots \\ w_y|_{\boldsymbol{x}=0} + xw_{xy}|_{\boldsymbol{x}=0} + yw_{yy}|_{\boldsymbol{x}=0} + (\frac{1}{2}x^2 + \frac{1}{\lambda})w_{xxy}|_{\boldsymbol{x}=0} + xyw_{xyy}|_{\boldsymbol{x}=0} + (\frac{1}{2}y^2 + \frac{1}{\lambda})w_{yyy}|_{\boldsymbol{x}=0} + \ldots \\ -w_x|_{\boldsymbol{x}=0} - xw_{xx}|_{\boldsymbol{x}=0} - yw_{xy}|_{\boldsymbol{x}=0} - (\frac{1}{2}x^2 + \frac{1}{\lambda})w_{xxx}|_{\boldsymbol{x}=0} - xyw_{xxy}|_{\boldsymbol{x}=0} - (\frac{1}{2}y^2 + \frac{1}{\lambda})w_{xyy}|_{\boldsymbol{x}=0} + \ldots \end{Bmatrix} -$$
$$\sum_{i=1}^{N}\left(w + x_i w_x + y_i w_y + \frac{1}{2}x_i^2 w_{xx} + x_i y_i w_{xy} + \frac{1}{2}y_i^2 w_{yy} + \ldots\right)\Big|_{\boldsymbol{x}=0} \boldsymbol{A}_1\boldsymbol{\varphi}_{i1} -$$
$$\sum_{i=1}^{N}\left(w_y + x_i w_{xy} + y_i w_{yy} + (\frac{1}{2}x_i^2 + \frac{1}{\lambda})w_{xxy} + x_i y_i w_{xyy} + (\frac{1}{2}y_i^2 + \frac{1}{\lambda})w_{yyy}\right)\Big|_{\boldsymbol{x}=0} \boldsymbol{A}_1\boldsymbol{\varphi}_{i2} +$$
$$\sum_{i=1}^{N}\left(w_x + x_i w_{xx} + y_i w_{xy} + (\frac{1}{2}x_i^2 + \frac{1}{\lambda})w_{xxx} + x_i y_i w_{xxy} + (\frac{1}{2}y_i^2 + \frac{1}{\lambda})w_{yyy}\right)\Big|_{\boldsymbol{x}=0} \boldsymbol{A}_1\boldsymbol{\varphi}_{i3} + \ldots \tag{13}$$

Let us equate to zero the coefficients of the corresponding derivatives of w. We obtain the identities of the *completeness criterion* [7-8]:

- of order p=1:

$$\boldsymbol{A}_1\sum_{i=1}^{N}\boldsymbol{\varphi}_{i1}\equiv\begin{Bmatrix}1\\0\\0\end{Bmatrix},\quad \boldsymbol{A}_1\sum_{i=1}^{N}(x_i\boldsymbol{\varphi}_{i1}-\boldsymbol{\varphi}_{i3})\equiv\begin{Bmatrix}x\\0\\-1\end{Bmatrix},$$

$$\boldsymbol{A}_1\sum_{i=1}^{N}(y_i\boldsymbol{\varphi}_{i1}+\boldsymbol{\varphi}_{i2})\equiv\begin{Bmatrix}y\\1\\0\end{Bmatrix} \tag{14}$$

- of order p=2:

$$\boldsymbol{A}_1\sum_{i=1}^{N}\left(\frac{x_i^2}{2}\boldsymbol{\varphi}_{i1}-x_i\boldsymbol{\varphi}_{i3}\right)\equiv\left\{\frac{x^2}{2},0,-x\right\}^T,$$

$$\boldsymbol{A}_1\sum_{i=1}^{N}\left(\frac{y_i^2}{2}\boldsymbol{\varphi}_{i1}+y_i\boldsymbol{\varphi}_{i2}\right)\equiv\left\{\frac{y^2}{2},y,0\right\}^T$$

$$\boldsymbol{A}_1\sum_{i=1}^{N}(x_iy_i\boldsymbol{\varphi}_{i1}+x_i\boldsymbol{\varphi}_{i2}-y_i\boldsymbol{\varphi}_{i3})\equiv\{xy,x,-y\}^T \tag{15}$$

- of order p=3:

$$\boldsymbol{A}_1\sum_{i=1}^{N}\left(\frac{x_i^3}{6}\boldsymbol{\varphi}_{i1}-\left(\frac{x_i^2}{2}+\frac{1}{\lambda}\right)\boldsymbol{\varphi}_{i3}\right)\equiv\left\{\frac{x^3}{6},0,-\frac{x^2}{2}-\frac{1}{\lambda}\right\}^T,$$

$$\boldsymbol{A}_1\sum_{i=1}^{N}\left(\frac{x_i^2y_i}{2}\boldsymbol{\varphi}_{i1}+\left(\frac{x_i^2}{2}+\frac{1}{\lambda}\right)\boldsymbol{\varphi}_{i2}-x_iy_i\boldsymbol{\varphi}_{i3}\right)\equiv$$
$$\left\{\frac{x^2y}{2},\frac{x^2}{2}+\frac{1}{\lambda},-xy\right\}^T$$

$$\boldsymbol{A}_1\sum_{i=1}^{N}\left(\frac{x_iy_i^2}{2}\boldsymbol{\varphi}_{i1}+x_iy_i\boldsymbol{\varphi}_{i2}-\left(\frac{y_i^2}{2}+\frac{1}{\lambda}\right)\boldsymbol{\varphi}_{i3}\right)\equiv$$
$$\left\{\frac{1}{2}xy^2,xy,-\frac{1}{2}y^2-\frac{1}{\lambda}\right\}^T$$

$$\boldsymbol{A}_1\sum_{i=1}^{N}\left(\frac{y_i^3}{6}\boldsymbol{\varphi}_{i1}+\left(\frac{y_i^2}{2}+\frac{1}{\lambda}\right)\boldsymbol{\varphi}_{i2}\right)\equiv\left\{\frac{y^3}{6},\frac{y^2}{2}+\frac{1}{\lambda},0\right\}^T \tag{16}$$

It should be noted that the identities of the completeness criterion (14) and (15) coincide with the corresponding identities of the Kirchhoff-Love thin plate elements. See [7,8,10]. Identities of order p=3 coincide as well if

$$\sum_{i=1}^{N}\varphi_{ij}^{j}=1,\ j=2,3 \tag{17}$$

The completeness criterion identities of order p=1 are the equations of the rigid body motion of a finite element.

Failure to satisfy the completeness criterion identities of order p = 2, as a rule, leads to the so-called locking effect, when the method does not converge to an analytical solution during the analysis of thin plates.

If the completeness criterion identities of order p=2 are not satisfied, then it is impossible to implement the constant moment tests, and for p=3 – the constant shear tests.

For all the created elements, incompatibility is allowed only for functions (10). Since when constructing the stiffness matrix of a finite element the functional includes the following expressions:

$$\mu_k^4\left(\frac{\partial}{\partial y}\varphi_{ij}^1-\varphi_{ij}^2\right),\quad \mu_k^5\left(\frac{\partial}{\partial x}\varphi_{ij}^1+\varphi_{ij}^3\right), \tag{18}$$

then *the incompatibility criterion* [7-9] of the minimum order that provides piecewise testing [11] is reduced for this problem to the following equalities:

$$\int_{\Omega_\kappa}\mu_k^j d\Omega=0,\ \text{k}\le 4. \tag{19}$$

Due to the fact that functions (10) correspond to internal degrees of freedom, they can only increase the order of fulfillment of the completeness criterion identities of the system of functions (5).

According to [7-9], if the completeness criterion identities (14), (15) and the incompatibility criterion equalities (19) are satisfied, the convergence of the method will be ensured.

3. CONSTRUCTION OF APPROXIMATING FUNCTIONS

We will assume that in (9)

$$\varphi_{i1}^1=\varphi_{i2}^2=\varphi_{i3}^3=\chi_i\ ,i=1\div N \tag{20}$$

where χ_i are classic approximations of the elements of the plane problem of the theory of elasticity, for which the completeness criterion identities of order p=1 are satisfied:

$$\sum_{i=1}^{N}\chi_i \equiv 1,\ \ \sum_{i=1}^{N}x_i\chi_i \equiv x,\ \ \sum_{i=1}^{N}y_i\chi_i \equiv y \tag{21}$$

Transform the coordinate system for isoparametric elements:

$$x=\sum_{i=1}^{N}x_i\chi_i,\ \ y=\sum_{i=1}^{N}y_i\chi_i, \tag{22}$$

Hence, (21) is satisfied as well.
Suppose that the last identity (15) is satisfied. Then, according to (21):

$$\sum_{i=1}^{N}\left(x_i y_i\chi_i + x_i\varphi_{i2}^1 - y_i\varphi_{i3}^1\right)\equiv \sum_{i,j=1}^{N} x_i y_j \chi_i\chi_j \tag{23}$$

Let us set, keeping symmetry:

$$\varphi_{i2}^1=\frac{1}{2}\chi_i\left(y-y_i\right),\ \ \varphi_{i3}^1=-\frac{1}{2}\chi_i\left(x-x_i\right) \tag{24}$$

Check that all identities (14) and (15) are satisfied for (24), since:

$$\sum_{i=1}^{N}\varphi_{i2}^1=\frac{1}{2}\sum_{i\in\Omega_r}\chi_i\left(y-y_i\right)=0,$$
$$\sum_{i=1}^{N}\varphi_{i3}^1=-\frac{1}{2}\sum_{i\in\Omega_r}\chi_i\left(x-x_i\right)=0 \tag{25}$$

The fulfillment of (24) ensures the convergence of the method without the locking effect, which is confirmed by the numerical experiments.
Functions (10) are designed to improve the accuracy of elements. To construct them, we use the residuals of the completeness criterion (16) of order p=3:

$$\boldsymbol{\zeta}_1=\left\{\frac{x^3}{6},0,-\frac{x^2}{2}-\frac{1}{\lambda}\right\}^T-\sum_{i=1}^{N}\boldsymbol{A}_1\left(\frac{x_i^3}{6}\boldsymbol{\varphi}_{i1}-\left(\frac{x_i^2}{2}+\frac{1}{\lambda}\right)\boldsymbol{\varphi}_{i3}\right),$$
$$\boldsymbol{\zeta}_2=\left\{\frac{x^2y}{2},\frac{x^2}{2}+\frac{1}{\lambda},-xy\right\}^T-\sum_{i=1}^{N}\boldsymbol{A}_1\left(\frac{x_i^2y_i}{2}\boldsymbol{\varphi}_{i1}+\left(\frac{x_i^2}{2}+\frac{1}{\lambda}\right)\boldsymbol{\varphi}_{i2}-x_iy_i\boldsymbol{\varphi}_{i3}\right)$$
$$\boldsymbol{\zeta}_3=\left\{\frac{xy^2}{2},xy,-\frac{y^2}{2}-\frac{1}{\lambda}\right\}^T-\sum_{i=1}^{N}\boldsymbol{A}_1\left(\frac{x_iy_i^2}{2}\boldsymbol{\varphi}_{i1}+x_iy_i\boldsymbol{\varphi}_{i2}-\left(\frac{y_i^2}{2}+\frac{1}{\lambda}\right)\boldsymbol{\varphi}_{i3}\right)$$
$$\boldsymbol{\zeta}_4=\left\{\frac{y^3}{6},\frac{y^2}{2}+\frac{1}{\lambda},0\right\}^T-\sum_{i=1}^{N}\boldsymbol{A}_1\left(\frac{y_i^3}{6}\boldsymbol{\varphi}_{i1}+\left(\frac{y_i^2}{2}+\frac{1}{\lambda}\right)\boldsymbol{\varphi}_{i2}\right) \tag{26}$$

Form the vectors from the residuals (26):

$$\boldsymbol{\omega}_k=\begin{Bmatrix}\dfrac{\partial}{\partial x}\zeta_k^1+\zeta_k^3\\ \dfrac{\partial}{\partial y}\zeta_k^1-\zeta_k^2\end{Bmatrix},\ k=1,2,3,4 \tag{27}$$

Specify the components of functions (10) as follows:

$$\mu_k^4=\omega_k^1+a_k,\ \ \mu_k^5=\omega_k^2+b_k,\ k=1,2,3,4 \tag{28}$$

The constants a_k, b_k in (28) are found from the incompatibility criterion equations (19).
Nonzero functions (28) usually have discontinuities on the element sides.
Analyze the constructed system of functions and supplement the system of approximating functions (5) with *nonzero and linearly independent* functions (10), correlating them with some internal degrees of freedom.
Instead of introducing the internal degrees of freedom, we can “scatter” functions (28) over

the approximations of the element, specifying in (9)

$$\varphi_{ij}^{m}=\sum_{k}c_{ij}^{k}\mu_{k}^{m}\ ,\ i=1\div N,\ j=1,2,3,\ \ m=4,5, \qquad (29)$$

where c_{ij}^{k} are coefficients which are determined by solving systems of equations based on the completeness criterion identities.

Let high-precision elements be used, for which the completeness criterion identities of order p=2 are satisfied for the approximations χ_i:

$$\sum_{i=1}^{N}x_i^2\chi_i\equiv x^2,\ \sum_{i=1}^{N}x_iy_i\chi_i\equiv xy,\ \sum_{i=1}^{N}y_i^2\chi_i\equiv y^2 \qquad (30)$$

Then, using the first and last identities (16) and keeping symmetry, we assume:

$$\varphi_{i2}^{1}=\frac{1}{3}\chi_i\left(y-y_i\right),\ \ \varphi_{i3}^{1}=-\frac{1}{3}\chi_i\left(x-x_i\right) \qquad (31)$$

It follows from (31) and (30) that:

$$\sum_{i=1}^{N}x_i\varphi_{i2}^{1}=\sum_{i=1}^{N}y_i\varphi_{i2}^{1}=\sum_{i=1}^{N}x_i\varphi_{i3}^{1}=\sum_{i=1}^{N}y_i\varphi_{i3}^{1}=0 \qquad (32)$$

Hence, identities (15) are satisfied.

All identities (16) of the completeness criterion of the 3rd order are satisfied as well:

$$\begin{aligned}
&\sum_{i=1}^{N}\left(\frac{1}{6}x_i^3\varphi_{i1}^{1}-(\frac{1}{2}x_i^2+\frac{1}{\lambda})\varphi_{i3}^{1}\right)=\frac{1}{6}x^3\\
&\sum_{i=1}^{N}\left(\frac{1}{2}x_i^2y_i\varphi_{i1}^{1}+(\frac{1}{2}x_i^2+\frac{1}{\lambda})\varphi_{i2}^{1}-x_iy_i\varphi_{i3}^{1}\right)=\frac{1}{2}x^2y\\
&\sum_{i=1}^{N}\left(\frac{1}{2}x_iy_i^2\varphi_{i1}^{1}+x_iy_i\varphi_{i2}^{1}-(\frac{1}{2}y_i^2+\frac{1}{\lambda})\varphi_{i3}^{1}\right)=\frac{1}{2}xy^2\\
&\sum_{i=1}^{N}\left(\frac{1}{6}y_i^3\varphi_{i1}^{1}+(\frac{1}{2}y_i^2+\frac{1}{\lambda})\varphi_{i2}^{1}\right)=\frac{y^3}{6}
\end{aligned} \qquad (33)$$

To construct approximations (5) and (10), we can use the *approximating functions of Kirchhoff-Love thin plate finite elements*.

Let in (9):

$$\begin{aligned}
&\varphi_{ij}^{1}=\psi_{ij},\ \ \varphi_{ij}^{2}=\varphi_{ij}^{3}=\chi_i,\\
&\varphi_{i1}^{2}=\varphi_{i1}^{3}=\varphi_{i2}^{3}=\varphi_{i3}^{2}=\varphi_{ij}^{4}=\varphi_{ij}^{5}=0,
\end{aligned} \qquad (34)$$

i=1÷N, j=1,2,3,

where ψ_{ij} is the system of approximating functions of a Kirchhoff-Love thin plate element corresponding to the degrees of freedom (4). They usually satisfy the second-order, if not third-order completeness criterion identities. In order to ensure consistency they only have to belong to the Sobolev space W_2^1, and not W_2^2;

χ_i is the system of approximating functions of the element of the plane problem of the theory of elasticity.

We construct functions (27) based on residuals (26). Next we calculate the constants in (28) from the equations (19). Then we analyze the constructed system of functions (10) and supplement the system of approximating functions (5) with *nonzero and linearly independent* functions, correlating them with some internal degrees of freedom.

It would be a mistake to define the functions corresponding to the rotations through the derivatives ψ_{ij}:

$$\varphi_{i2}^{j}=\frac{\partial}{\partial y}\psi_{ij},\ \ \varphi_{i3}^{j}=-\frac{\partial}{\partial x}\psi_{ij}\ \ i=1\div N,\ j=1,2,3 \qquad (35)$$

The relationship between the approximations of the rotation functions and vertical displacements leads to a significant narrowing of the required space for solving the variational problem. Thus, for example, in the case of analysis of a simply supported plate, we obtain *zero shear forces* with good convergence in displacements and moments.

4. FINITE ELEMENTS

4.1. Three-Node Element (JIDR3)

Let us consider a triangle in the local coordinate system shown in Fig. 1a. After changing the coordinates (36), it is transformed into a right triangle with unit legs shown in Fig. 1b.

$$\xi = \frac{1}{a}\left(x - \frac{b}{c}y\right), \quad \eta = \frac{1}{c}y \tag{36}$$

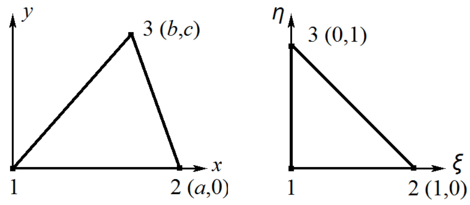

Figure 1. Triangle and its master-element

Assume that the functions χ_i in (20) are linear approximations:

$$\chi_1 = 1 - \xi - \eta, \quad \chi_2 = \xi, \quad \chi_3 = \eta, \; i = 1,2,3 \tag{37}$$

Since (21) is satisfied, then using (24):

$$\begin{aligned}
&\varphi_{i1}^1 = \varphi_{i2}^2 = \varphi_{i3}^3 = \chi_i, \quad i = 1,2,3 \\
&\varphi_{12}^1 = \frac{c}{2}\eta\chi_1, \qquad \varphi_{13}^1 = -\frac{1}{2}(a\xi + b\eta)\chi_1, \\
&\varphi_{22}^1 = \frac{c}{2}\xi\eta, \qquad \varphi_{23}^1 = -\frac{1}{2}\xi(a\xi + b\eta - a), \\
&\varphi_{32}^1 = \frac{c}{2}\eta(\eta - 1), \quad \varphi_{33} = -\frac{1}{2}\eta(a\xi + b\eta - b)
\end{aligned} \tag{38}$$

Construct residuals (26):

$$\boldsymbol{\zeta}_1 = \begin{Bmatrix} \dfrac{x^3}{6} - \dfrac{a^3}{6}\xi - \dfrac{b^3}{6}\eta - \dfrac{a^2}{4}\xi(x-a) - \dfrac{b^2}{4}\eta(x-b) \\ 0 \\ -\dfrac{x^2}{2} + \dfrac{a^2}{2}\xi + \dfrac{b^2}{2}\eta \end{Bmatrix}$$

$$\boldsymbol{\zeta}_2 = \begin{Bmatrix} \dfrac{x^2 y}{2} - \dfrac{b^2 c}{2}\eta - \dfrac{a^2}{4}\xi y - \dfrac{b^2}{4}\eta(y-c) - \dfrac{bc}{2}\eta(x-b) \\ \dfrac{x^2}{2} - \dfrac{a^2}{2}\xi - \dfrac{b^2}{2}\eta \\ -xy + bc\eta \end{Bmatrix}$$

$$\boldsymbol{\zeta}_3 = \begin{Bmatrix} \dfrac{xy^2}{2} - \dfrac{bc^2}{2}\eta - \dfrac{bc}{2}\eta(y-c) - \dfrac{c^2}{4}\eta(x-b) \\ xy - bc\eta \\ -\dfrac{y^2}{2} + \dfrac{c^2}{2}\eta \end{Bmatrix}$$

$$\boldsymbol{\zeta}_4 = \begin{Bmatrix} \dfrac{y^3}{6} - \dfrac{c^3}{6}\eta - \dfrac{c^2}{4}\eta(y-c) \\ \dfrac{y^2}{2} - \dfrac{c^2}{2}\eta \\ 0 \end{Bmatrix} \tag{39}$$

Calculate vectors (27) according to (39):

$$\boldsymbol{\omega}_1 = \frac{1}{12c}\begin{Bmatrix} a^2 c + 3(b-a)by \\ (a-b)(-a^2 + 3bx) \end{Bmatrix}, \quad \boldsymbol{\omega}_2 = \frac{2b-a}{4}\begin{Bmatrix} y \\ -x \end{Bmatrix}$$

$$\boldsymbol{\omega}_3 = \frac{c}{4}\begin{Bmatrix} y \\ -x+b \end{Bmatrix}, \qquad \boldsymbol{\omega}_4 = \frac{c^2}{12}\begin{Bmatrix} 0 \\ -1 \end{Bmatrix} \tag{40}$$

Substituting (40) into (28), we find the values of the constants a_k, b_k from the incompatibility criterion equations (19). Discarding the zero function from the linear independence condition, we obtain only one function (10):

$$\boldsymbol{\mu}_1 = \{0, 0, 0, c(3\eta - 1), a + b - 3(a\xi + b\eta)\}^T, \tag{41}$$

corresponding to the internal degree of freedom.

4.2. Four-Node Isoparametric Element (JIDR4)

Let us consider a convex quadrangular finite element in the local coordinate system shown in Fig. 2a. After an isoparametric transformation of the coordinate system (42), it is transformed into a unit square shown in Fig. 2b.

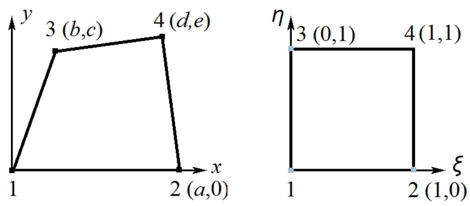

Figure 2. Quadrangle and its master-element

$$\begin{aligned} x &= a\xi(1-\eta) + b(1-\xi)\eta + d\xi\eta, \\ y &= c(1-\xi)\eta + e\xi\eta \end{aligned} \tag{42}$$

Assume that the functions χ_i in (20) are multilinear approximations:

$$\chi_1=(1-\xi)(1-\eta),\ \ \chi_2=\xi(1-\eta),\\ \chi_3=(1-\xi)\eta,\qquad \chi_4=\xi\eta \tag{43}$$

They were also used for the isoparametric transformation (42).
Since (21) is satisfied, then using (24):

$$\begin{aligned}&\varphi_{i1}^1=\varphi_{i2}^2=\varphi_{i3}^3=\chi_i,\quad i=1,2,3,4\\ &\varphi_{12}^1=\frac{1}{2}y\chi_1,\qquad \varphi_{13}^1=-\frac{1}{2}x\chi_1,\\ &\varphi_{22}^1=\frac{1}{2}y\chi_2,\qquad \varphi_{23}^1=-\frac{1}{2}(x-a)\chi_2,\\ &\varphi_{32}^1=\frac{1}{2}(y-c)\chi_3,\quad \varphi_{33}^1=-\frac{1}{2}(x-b)\chi_3,\\ &\varphi_{42}^1=\frac{1}{2}(y-e)\chi_4,\quad \varphi_{43}^1=-\frac{1}{2}(x-d)\chi_4,\end{aligned} \tag{44}$$

Construct residuals (26). To obtain functions (10), we substitute (26) into (27) and find the values of the constants a_k, b_k in (28) from the incompatibility criterion equations (19). There are only two functions left for the rectangle:

$$\begin{aligned}&\boldsymbol{\mu}_1=\{0,0,0,0,\xi(1-\xi)\}^T\\ &\boldsymbol{\mu}_2=\{0,0,0,\eta(1-\eta),0\}^T\end{aligned} \tag{45}$$

4.3. Six-Node Isoparametric Element (JIDR6)

Let us consider the triangle shown in Fig. 3.

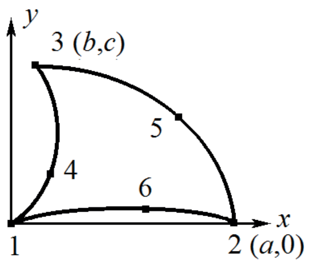

Figure 3. Isoparametric six-node element

Use the functions defined on the master-element in Fig. 1b:

$$\begin{aligned}&\chi_1=(1-\xi-\eta)(1-2\xi-2\eta);\ \chi_2=\xi(2\xi-1);\\ &\chi_3=\eta(2\eta-1);\qquad \chi_4=4\eta(1-\xi-\eta);\\ &\chi_5=4\xi(1-\xi-\eta);\qquad \chi_6=4\xi\eta.\end{aligned} \tag{46}$$

After transforming the coordinate system (22), the element is transformed into a right triangle with unit legs, shown in Fig. 1b.

Use formulas (24) to specify φ_{i2}^1, φ_{i3}^1 and construct the residuals (26). To obtain functions (10), we substitute (26) into (27) and find the values of the constants a_k, b_k in (28) from the incompatibility criterion equations (19).
If intermediate nodes are located at the midpoints of the sides of the element, then the Jacobian of the transformation (22) is a linear function. Since functions (46) satisfy the completeness criterion identities of the second order (30), formulas (31) can be used to specify φ_{i2}^1, φ_{i3}^1. All the completeness criterion identities of the third order (16) will be satisfied, and all the residuals (26) will be equal to zero.

4.4. Eight-Node Isoparametric Element (JIDR8)

Let us consider the quadrangle shown in Fig. 4.

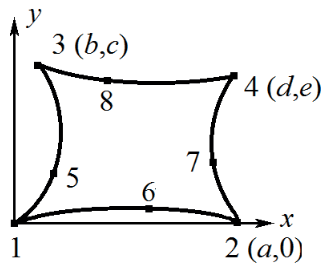

Figure 4. Isoparametric eight-node element

Use the approximations defined on the master-element in Fig. 2b:

$$\begin{aligned}&\chi_1=(1-\xi)(1-\eta)(1-2\xi-2\eta);\\ &\chi_2=\xi(1-\eta)(2\xi-2\eta-1);\\ &\chi_3=(1-\xi)\eta(2\eta-2\xi-1);\\ &\chi_4=\xi\eta(2\eta+2\xi-3);\\ &\chi_5=4(1-\xi)\eta(1-\eta);\qquad \chi_6=4\xi(1-\xi)(1-\eta);\\ &\chi_7=4\xi\eta(1-\eta);\qquad \chi_8=4\xi(1-\xi)\eta.\end{aligned} \tag{47}$$

After transforming the coordinate system (22), the element is transformed into the master-element in Fig. 2b.

Use formulas (24) to specify φ_{i2}^{1}, φ_{i3}^{1} and construct the residuals (26). To obtain functions (10), we substitute (26) into (27) and find the values of the constants a_k, b_k in (28) from the incompatibility criterion equations (19).

Suppose that the Jacobian of the transformation (22) is a linear function (rectangle with the nodes at the midpoints of the sides of the element). Since functions (47) satisfy the completeness criterion identities of the second order (30), formulas (31) can be used to specify φ_{i2}^{1}, φ_{i3}^{1}. All the completeness criterion identities of the third order (16) will be satisfied, and all the residuals (26) will be equal to zero.

4.5. Four-Node Element with a Piecewise Polynomial Approximation (JIDR4SubAreas)

Let us consider a quadrangular finite element in the local coordinate system shown in Fig. 2a. It is transformed into a quadrangle shown in Fig. 5 by transforming the coordinate system (48). ***A*** is the intersection point of the diagonals of the element.

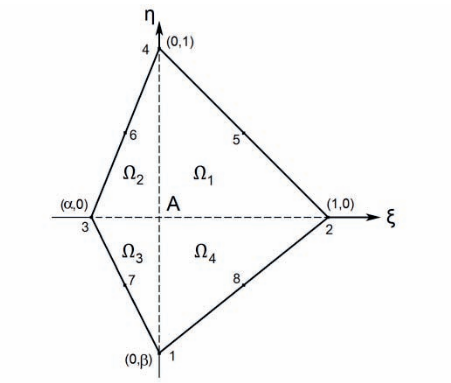

Figure 5. Four-node element in a special coordinate system

$$\begin{cases} x = x_A + (a - x_A)\xi + (d - x_A)\eta \\ y = y_A(1-\xi) + (e - y_A)\eta \end{cases} \tag{48}$$

Consider the functions from [12], which are second degree polynomials in each subdomain Ω_i, i=1,2,3,4 and are *continuous together with their first derivatives* on the diagonals of the element:

$$\chi_i,\ i=1,2,3,4 \tag{49}$$

Use formulas (24) to specify φ_{i2}^{1}, φ_{i3}^{1} and construct the residuals (26). To obtain functions (10), we substitute (26) into (27) and find the values of the constants a_k, b_k in (28) from the incompatibility criterion equations (19).

4.6. Eight-Node Element with a Piecewise Polynomial Approximation (JIDR8SubAreas)

Let us consider a quadrangular finite element in the local coordinate system shown in Fig. 2a. It is transformed into a quadrangle shown in Fig. 5 by transforming the coordinate system (46).

Consider the functions from [12], which are second degree polynomials in each subdomain Ω_i, i=1,2,3,4 and are *continuous together with their first derivatives* on the diagonals of the element:

$$\chi_i,\ i=1\div 8 \tag{50}$$

Since functions (50) satisfy the completeness criterion identities of the second order (30), formulas (31) can be used to specify φ_{i2}^{1}, φ_{i3}^{1}. All the completeness criterion identities of the third order (16) will be satisfied, and all the residuals (26) will be equal to zero.

5. TESTS

All calculations were performed in SCAD, which is a part of SCAD Office®.

5.1. Patch Tests

A rectangular plate is shown in Fig. 6. The plate sizes are proportional to those in the Patch Tests considered in [13].

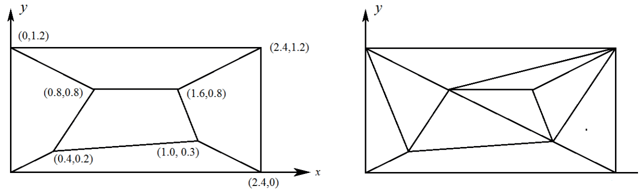

Figure 6. Rectangular plate

Two groups of kinematic loadings with known theoretical values were considered.

The first three load cases are a check of the displacement of a rectangle as a rigid body when moments and shear forces over the entire area of the plate are zero:

- displacement along the OZ axis: $w|_\Gamma=1$, $\theta_x|_\Gamma, \theta_y|_\Gamma=0$;
- rotation about the OX axis: $w|_\Gamma=y$, $\theta_x|_\Gamma=1, \theta_y|_\Gamma=0$;
- rotation about the OY axis: $w|_\Gamma=y$, $\theta_y|_\Gamma=-1, \theta_x|_\Gamma=0$.

The following three load cases provide non-zero constant moments and zero shear forces over the entire area of the plate:

- $w|_\Gamma=x^2$, $\theta_x|_\Gamma=0$, $\theta_y|_\Gamma=-2x$;
- $w|_\Gamma=y^2$, $\theta_x|_\Gamma=2y$, $\theta_y|_\Gamma=0$;
- $w|_\Gamma=xy$, $\theta_x|_\Gamma=x$, $\theta_y|_\Gamma=-y$.

Patch tests are performed in order to check whether the completeness criterion identities (15) are satisfied for all the considered elements:

- stiffness matrices of all the considered finite elements have three eigenvectors corresponding to their *displacement as rigid bodies*;
- the results for plates subjected to constant moments were obtained with an accuracy up to a computational error.

These tests serve only as a criterion for the correctness of the program code.

5.2. Rectangular Plate Simply Supported along the Perimeter Subjected to the Transverse Uniformly Distributed Load

Let us consider a rectangular plate simply supported along the perimeter subjected to the transverse uniformly distributed load shown in Fig. 6. Specify:

$$E = 30000\ kPa,\ \nu = 0.3,\ h = 0.2\ m,$$
$$a = 2.4\ m,\ b = 4.8\ m,\ p = 1.0\ kPa.$$

Specify the boundary conditions:

$$w|_\Gamma=0,\ \theta_x(0,y)=\theta_x(a,y)=\theta_y(x,0)=\theta_y(x,b)=0$$

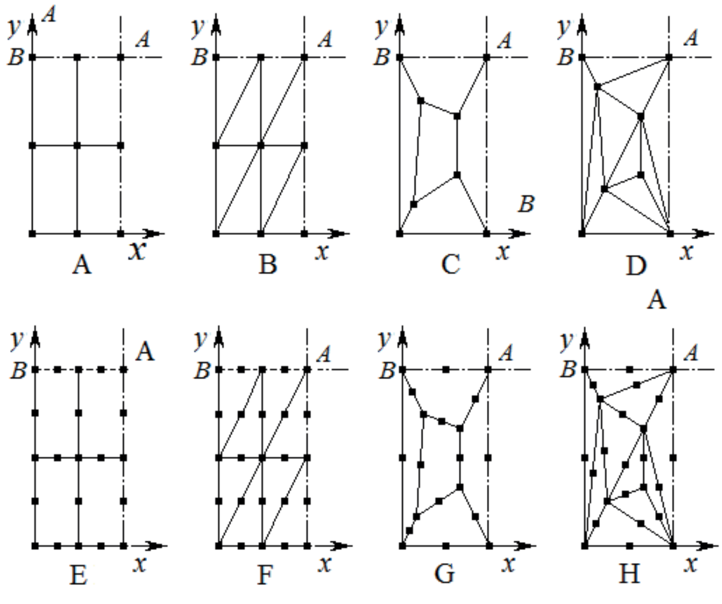

Figure 7. Design models of a rectangular plate

To study the locking effect, the plate thickness varied from h=0.001*m*=*a*/2400 to h=1.2=*a*/2. Experiment design models taking into account the symmetry axes are shown in Fig. 7. Table 1 presents the calculation results for a thin plate when h=0.08*m*=*a*/30. Analytical solution of this problem in the center of the plate (point *A*) and in the middle of the larger side (point *B*) (Analytical solution according to the spatial theory: $w|_A$ = 0.239663*m*; according to the Kirchhoff-Love theory: $w|_A$ = 0.238907 *m*, the values of the moments and shear forces coincide):

$w|_A$ = 0.239759*m*,
$M_x|_A$, = 0.585695(*kNm/m*),
$M_y|_A$ = 0.266978(*kNm*/m),
$Q_x|_B$,= 1.11602(*kN/m*)

Table 1. Displacements, moments and shear forces in the plate

Mesh type	Element	Displacement w_A (*m*)				Moment $M_{x,A}$ (*kNm/m*)				Shear force $Q_{xz,B}$ (*kN/m*)			
		Mesh				Mesh				Mesh			
		2x2	4x4	8x8	16x16	2x2	4x4	8x8	16x16	2x2	4x4	8x8	16x16
A	*MTC4*	-0.2363	-0.2384	-0.9394	0.9394	0.454	0.5809	0.5845	0.5854	0.8226	0.9676	1.0415	1.0787
	JIDR4	-0.2302	-0.2382	-0.2393	-0.2396	0.585	0.5847	0.5852	0.5856	1.1523	0.9224	1.0376	1.0782
	JIDR4SA	-0.2379	-0.2397	-0.2397	-0.2397	0.6001	0.5616	0.5871	0.586	2.2426	1.3454	1.0611	1.0678
B	*DSG3*	-0.0879	-0.2171	-0.2367	-0.2393	0.2172	0.5315	0.5765	0.5838	1.2419	1.0958	1.0861	1.1369
	JIDR3	-0.1838	-0.2311	-0.2377	-0.2393	0.4276	0.5581	0.5788	0.5841	0.9327	1.0717	1.108	1.1274
C	*MTC4*	-0.2335	-0.2385	-0.2389	-0.2395	0.454	0.5872	0.5852	0.5867	0.7938	0.9263	0.9666	1.0028
	JIDR4	-0.2097	-0.237	-0.2388	-0.2395	0.4737	0.5695	0.5819	0.5854	1.1062	1.1911	0.9985	1.0187
	JIDR4SA	-0.2118	-0.2374	-0.2395	-0.2397	0.4604	0.574	0.5864	0.5862	1.0494	1.9069	1.2488	1.0785
D	*DSG3*	-0.1627	-0.222	-0.2375	-0.2393	0.3379	0.5152	0.5815	0.5861	0.8049	0.9572	1.1442	1.2105
	JIDR3	-0.1941	-0.2317	-0.2379	-0.2393	0.4239	0.5473	0.5802	0.5854	1.5143	0.8798	0.8864	0.9576
E	*JIDR4*	-0.24	-0.2398	-0.2398	-0.2398	0.6077	0.5907	0.5869	0.586	0.8362	0.9553	1.0397	1.0783
	JIDR4SA	-0.2398	-0.2398	-0.2398	-0.2398	0.617	0.5938	0.5878	0.5862	0.8685	0.966	1.0403	1.0784
F	*JIDR6*	-0.2414	-0.2399	-0.2398	-0.2398	0.659	0.6019	0.5894	0.5866	0.8142	1.0022	1.0659	1.0912
	JIDR4	-0.2417	-0.2396	-0.2397	-0.2398	0.6671	0.5985	0.5874	0.5861	0.5796	0.8685	1.0046	1.0616
	JIDR4SA	-0.2425	-0.2398	-0.2398	-0.2398	0.682	0.6056	0.5905	0.587	0.7586	0.9254	1.0077	1.0611
H	*JIDR6*	-0.2424	-0.2398	-0.2398	-0.2398	0.6849	0.604	0.5902	0.587	0.8495	0.9604	1.0436	1.082

5.3. Stress-Strain State of a Clamped Hexagonal Plate Subjected to the Uniformly Distributed Load

Let us consider a regular hexagonal plate clamped along the perimeter subjected to the transverse uniformly distributed load shown in Fig. 8.

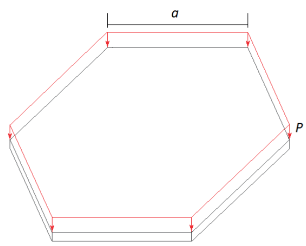

Figure 8. Hexagonal plate

Specify:

$$E = 30000\ kPa,\ \nu = 0.3,\ h = 0.1m,$$
$$a = 1m,\ p = 10\ kPa.$$

and boundary conditions:

$$w|_\Gamma = \theta_n|_\Gamma = \theta_\tau|_\Gamma = 0\,.$$

A numerical solution of this problem was obtained according to the Reissner-Mindlin theory at the center of the plate at point *A* with a high degree of accuracy:

$$w_A = -38.749(mm),\ M_x|_A = 0.6511(kNm/m).$$

The solutions were obtained for various types of finite elements. The maximum order of the system of equations for which the solution is obtained is 2747925.
The solution of this problem according to thin plate theory is given in [14] and is:

$$w_A = -36.324(mm),\ M_x = 0.64786(kNm/m).$$

The calculation results for the design models in Fig. 9 are given in Table 2.

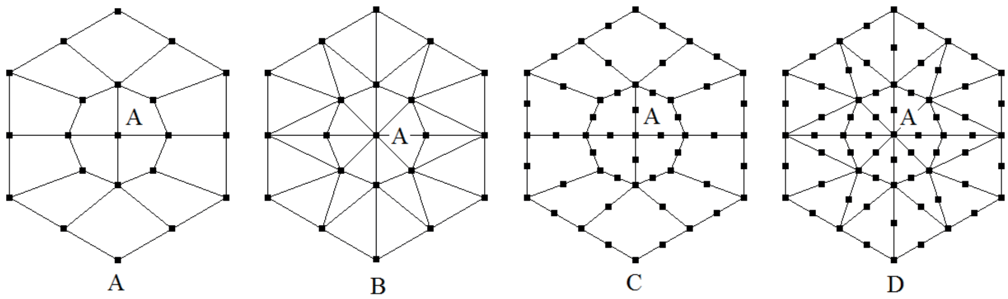

Figure 9. Design models 1x1 of a hexagonal plate

Table 2. Displacements and moments in the clamped plate

Mesh type	Element	Displacement w_A (mm)				Moment $M_{x,A}$ (kHm/m)			
		Mesh				Mesh			
		1x1	2x2	4x4	8x8	1x1	2x2	4x4	8x8
A	*MITC4*	-36.619	-37.916	-38.57	-38.737	0.7259	0.6573	0.656	0.6514
	JIDR4	-27.752	-35.899	-38.081	-38.584	0.5335	0.6405	0.6489	0.6504
	JIDR4SA	-29.89	-35.932	-38.09	-38.586	0.5236	0.6439	0.6478	0.6504
B	*DSG3*	-21.032	-35.274	-38.132	-38.663	0.3751	0.6101	0.6293	0.6429
	JIDR3	-24.194	-35.77	-38.035	-38.565	0.5076	0.6184	0.642	0.6486
C	*JIDR8*	-36.853	-38.559	-38.717	-38.743	0.7235	0.6602	0.654	0.6518
	JIDR8SA	-37.37	-38.614	-38.734	-38.747	0.7317	0.6657	0.6549	0.6521
D	*JIDR6*	-37.47	-38,66	-38.739	-38.747	0.6842	0.6613	0.6538	0.6518

6. CONCLUSIONS

All the created elements:
• have passed all Patch Tests;
• numerical experiments have confirmed that there is no locking effect;
• a good approximation of the numerical solution to the analytical results and the results of high-precision calculations has been obtained.
Elements based on thin plate elements were created. Approximating functions were used for:
• triangular elements [7,15];
• quadrangular elements [16].
Numerical experiments did not increase the accuracy of calculations in comparison with the approximations given in the paper with a significant complication of the algorithm.

REFERENCES

1. **Washizu K.** Variational Methods in Elasticity and Plasticity. Pergamon Press, Oxford New York, 1982, 630 pages.
2. **Bathe K.J.** Finite Element Procedures. New Jersey, Prentice Hall, 1996, 1038 pages.
3. **Slivker V.I.** Stroitelnaya mekhanika. Variatsionnyye osnovy [Construction mechanics. Variational bases]. Moskow, IASV, 2005, 708 pages (in Russian).
4. **Bathe K.J. Dvorkin, E.N.** A four-node plate bending element based on Mindlin/Reissner plate theory and a mixed interpolation. Int. J. Num. Meth. Engrg., 1985, Vol. 21, pp. 367–383. DOI: doi.org/10.1002/nme.1620210213
5. **Bletzinger K.U, Bischoff M, Ramm E.** A unified approach for shear-locking-free triangular and rectangular shell finite elements. Computers & Structures, 2000, Vol. 75, pp. 321-334. DOI:10.1016/S0045-7949(99)00140-6
6. **Zienkiewicz O.C., Taylor R.L.** The Finite Element Method. 3 volumes. Planta Tree, Butterworth-Heinemann, 2000, Vol. 1, 690 pages, Vol. 2, 459 pages, Vol. 3, 334 pages.
7. **Karpilovskyi V.S**. Konstpuipovaniye nesovmestnyx konechnyx elementov [Designing Incompatible Finite Elements]. UkrNIINTI, Kyiv, 1980, 2153, 49 pages (in Russian). https://scadsoft.com/download/Karpil1980.pdf/
8. **Karpilovskyi V.S.** Issledovanie i konstruirovanie nekotorykh tipov konechnykh elementov dlya zadach stroitel`noj mekhaniki. [Study and design of some types of finite elements for the structural mechanics problems]. // PhD diss., National transport university(KADI), Kyiv, 1982, 179 pages (in Russian)
9. **Evzerov I.D.** Oczenki pogreshnosti po peremeshheniyam pri ispol`zovanii nesovmestny`kh konechny`kh e`lementov [Estimates of the error in displacements when using incompatible finite elements] // Chislennye metody mekhaniki sploshnoj sredy [Numerical methods of continuum mechanics], Novosibirsk, 1983, Vol 14(5), pp. 24-31 (in Russian)
10. **Gorodetsky A.S., Karpilovskyi V.S**. Metodicheskiye pekomendatsii po issledovaniyu i konstpuipovaniyu konechnyx elementov [Methodological recommendations for the study and construction of finite elements]. Kyiv, NIIASS, 1981. 48 pages (in Russian). https://scadsoft.com/download/GorKarpil1981.pdf
11. **Strang G., Fix G.** An Analysis of the Finite Element Method. Prentice Hall, Englewood Cliffs, N.J., 1973. 306p.
12. **Karpilovskyi V.S.** Finite Elements of the Plane Problem of the Theory of Elasticity with Drilling Degrees of Freedom. International Journal for Computational Civil and Structural Engineering, Vol. 16(1), 2020, pp. 48-72. DOI: 10.22337/2587-9618
13. **Macneal R.H., Harder, R.L.** A proposed standard set of problems to test finite element accuracy. // Finite elements in analysis and design, 1985, Vol. 1, pp. 3-20. DOI: 10.1016/0168-874X(85)90003-4

14. **Weinberg D.V.** Spravochnik po prochnosti. ustoychivosti i kolebaniyam plastin [Handbook of Strength. stability and vibration of plates]. Kyiv, Budivelnik, 1973, 488 pages (in Russian)
15. **Karpilovskyi V.S**. Tpeugolnyy shestiuzlovoy konechnyy element plity [Triangular six-node finite element of plate]. Izvestiya VUZov. Stpoitelstvo i apxitektupa [News of the Higher Educational Institutions. Construction and architecture], 1989, Vol. 6, pp. 35–39.
16. **Karpilovskyi V.S.** Chetyipexugolnyiy vosmiuzlovoy konechnyiy element plityi [Quadrangular eight-node finite element of plate]. Structural Mechanics and Analysis of Constructions], 1990, Vol. 2, pp. 13–17.

СПИСОК ЛИТЕРАТУРЫ

1. **Васидзу К.** Вариационные методы в теории упругости и пластичности. – М.: Мир, 1987. – 544с.
2. **Bathe K.J.** Finite Element Procedures. New Jersey, Prentice Hall, 1996, 1038p.
3. **Сливкер В.И.** Строительная механика. Вариационные основы. – М., ИАСВ, 2005. – 708с.
4. **Bathe K.J. Dvorkin, E.N.** A four-node plate bending element based on Mindlin/Reissner plate theory and a mixed interpolation. //Int. J. Num. Meth. Engrg., 1985, Vol. 21, pp. 367–383. DOI: doi.org/10.1002/nme.1620210213
5. **Bletzinger K.U, Bischoff M, Ramm E:** A unified approach for shear-locking-free triangular and rectangular shell finite elements. // Computers & Structures, 2000, Vol. 75, pp. 321-334. DOI:10.1016/S0045-7949(99)00140-6
6. **Zienkiewicz O.C., Taylor R.L.** The Finite Element Method. 3 volumes. Planta Tree, Butterworth-Heinemann, 2000, Vol. 1, pages 690, Vol. 2, pages 459, Vol. 3, pages 334.
7. **Карпиловский В.С.** Конструирование несовместных конечных элементов. Киев: Деп. в УкрНИИНТИ, 1980, 2153. – 49 с. – URL: https://scadsoft.com/download/Karpil1980.pdf
8. **Карпиловский В.С.** Исследование и конструирование некоторых типов конечных элементов для задач строительной механики. Диссертация на соискание ученой степени кандидата технических наук по специальности 01.02.03 – «Строительная механика». – Киев: Национальный транспортный университет (КАДИ), 1982, 179 с.
9. **Евзеров И.Д.** Оценки погрешности по перемещениям при использовании несовместных конечных элементов // Численные методы механики сплошной среды, Новосибирск, 1983, том 14, №5, с. 24-31.
10. **Городецкий А.С., Карпиловский В.С.** Методические рекомендации по исследованию и конструированию конечных элементов. Киев: NIIASS, 1981. – 48с. – URL: https://scadsoft.com/download/GorKarpil1981.pdf
11. **Стренг Г., Фикс Дж**. Теория метода конечных элементов – М.: Мир, 1977. – 349 с.
12. **Karpilovskyi V.S.** Finite Elements of the Plane Problem of the Theory of Elasticity with Drilling Degrees of Freedom. // International Journal for Computational Civil and Structural Engineering, 2020, Vol. 16(1), pp. 48–72. DOI: 10.22337/2587-9618
13. **Macneal R.H., Harder, R.L.** A proposed standard set of problems to test finite element accuracy. // Finite elements in analysis and design, 1985, Vol. 1, pp. 3-20. DOI: 10.1016/0168-874X(85)90003-4
14. **Вайнберг Д.В.** Справочник по прочности, устойчивости и колебаниям пластин. – Киев: Будiвельник, 1973. – 488 с
15. **Карпиловский В.С.** Треугольный шестиузловой конечный элемент плиты. // Известия ВУЗов. Строительство и архитектура, 1989, №6. – С. 35–39.
16. **Карпиловский В.С.** Четырехугольный восьмиузловой конечный элемент плиты // Строительная механика и расчет сооружений, 1990, №2. – С. 13–17.

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