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LOCALIZATION OF SOLUTION OF THE PROBLEM OF TWO-DIMENSIONAL THEORY OF ELASTICITY WITH THE USE OF B-SPLINE DISCRETE-CONTINUAL FINITE ELEMENT METHOD

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Abstract: Localization of solution of the problem of two-dimensional theory of elasticity with the use of B-spline discrete-continual finite element method (specific version of wavelet-based discrete-continual finite element method) is under consideration in the distinctive paper. The original operational continual and discrete-continual formulations of the problem are given, some actual aspects of construction of normalized basis functions of a B-spline are considered, the corresponding local constructions for an arbitrary discrete-continual finite element are described, some information about the numerical implementation and an example of analysis are presented.

Keywords: localization, wavelet-based discrete-continual finite element method, B-spline discrete-continual finite element method, discrete-continual finite element method, finite element method, B-spline, numerical solution, two-dimensional theory of elasticity, structural analysis.

ЛОКАЛИЗАЦИЯ РЕШЕНИЯ ДВУМЕРНОЙ ЗАДАЧИ ТЕОРИИ УПРУГОСТИ НА ОСНОВЕ ВЕЙВЛЕТ-РЕАЛИЗАЦИИ ДИСКРЕТНО-КОНТИНУАЛЬНОГО МЕТОДА КОНЕЧНЫХ ЭЛЕМЕНТОВ С ИСПОЛЬЗОВАНИЕМ B-СПЛАЙНОВ

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Аннотация: В настоящей статье рассматривается локализация решения двумерной задачи теории упругости на основе вейвлет-реализации дискретно-континуального метода конечных элементов с использованием B-сплайнов. Приведены исходные операторные континуальная и дискретно-континуальная постановки задачи, рассмотрены некоторые актуальные вопросы построения нормализованных базисных функций B-сплайна, описаны соответствующие локальные построения для произвольного дискретно-континуального конечного элемента, представлены некоторые сведения о численной реализации и пример расчета.

Ключевые слова: локализация, вейвлет-реализация метода конечных элементов, дискретно-континуальный метод конечных элементов, метод конечных элементов, B-сплайны, численное решение, двумерная задача теории упругости, расчеты конструкций.

INTRODUCTION

As we have already mentioned [1, 2], the B-spline in a given simple knot sequence can be constructed by employing piecewise polynomials between the knots and joining them together at the knots [1-3].

For instance, compared with commonly used Daubechies wavelets [4-8] B-spline wavelet on interval (BSWI) has explicit expressions, facilitating the calculation of coefficient integration and differentiation [1-3]. Besides, the multiresolution and localization properties of BSWI can also supply some superiority for engineering structural analysis [1-3]. The early applications of spline can be found in papers of H. Antes [9], J.G. Han [10, 11, 27], Y. Huang [10, 11], W.X. Ren [10, 11]. The spline wavelet finite element method was further developed in papers of D.P. Chen [28], X.F. Chen [12, 13, 15-18, 23, 24, 26], H.B. Dong [23], J.G. Han [25], Y.M. He [17], Z.H. He [18], Z.J. He [12, 13, 15-17, 23, 24, 26], Y. Huang [25, 27], Z.S. Jiang [22], B. Li [13, 15, 17, 23], M. Liang [19, 21], J.Q. Long [20], G. Ma [20], T. Matsumoto [20, 22], S.T. Mau [30], H.H. Miao [15], Q,M. Mo [18], T.H.H. Pian [28-30], K.Y. Qi [23], W.X. Ren [25, 27], K. Sumihara [29], P. Tong [30], Y.W. Wang [22], J.W. Xiang [12-14, 17-22], Z.B. Yang [15, 16, 24], X.W. Zhang [16, 24, 26], Y.H. Zhang [12], Y.T. Zhong [14].

As is known, generally the structural analysis normally require accurate computer-intensive calculations using numerical (discrete) methods. The field of application of discrete-continual finite element method (DCFFEM), proposed by A.B. Zolotov [33] and P.A. Akimov [31-33] comprises structures with regular (in particular, constant or piecewise constant) physical and geometrical parameters in some dimension (so-called “basic” direction (dimension)). Considering problems remain continual along “basic” direction while along other directions DCFEM presupposes finite element approximation. Solution of corresponding resultant multipoint boundary problems [34] for systems of ordinary differential equations with piecewise constant coefficients and immense number of unknowns is the most time-consuming stage of the computing, especially if we take into account the limitation in performance of personal computers, contemporary software and necessity to obtain correct semianalytical solution in a reasonable time.

High-accuracy solution at all points of the model is not required normally, it is necessary to find only the most accurate solution in some pre-known domains. Generally the choice of these domains is a priori data with respect to the structure being modelled. Designers usually choose domains with the so-called edge effect (with the risk of significant stresses that could potentially lead to the destruction of structures, etc.) and regions which are subject to specific operational requirements. It is obvious that the stress-strain state in such domains is of paramount importance. Specified factors along with the obvious needs of the designer or researcher to reduce computational costs by application of DCFEM cause considerable urgency of constructing of special algorithms for obtaining local solutions (in some domains known in advance) of boundary problems. Wavelet analysis provides effective and popular tool for such researches. Solution of the considering problem within multilevel wavelet analysis is represented as a composition of local and global components. Wavelet-based DCFEM is presented in papers of P.A. Akimov [35-42], M. Aslami [38-40], T.B. Kaytukov, M.L. Mozgaleva [35-42] and O.A. Negrozov [38-40].

The distinctive paper is devoted to numerical solution of the problem of two-dimensional theory of elasticity with the use of B-spline DCFEM.

1. FORMULATIONS OF THE PROBLEM

In accordance with [1] let the constancy of the parameters of the problem be in the direction corresponding to x_2 (main direction). The operational formulation of the problem with the use of so-called method of extended domain [43], taking into account the selection of the main direction, is determined by the equation:

$$L\,\bar{u} = \bar{\bar{F}}\,, \quad 0 \le x_1 \le \ell_1\,,\ 0 \le x_2 \le \ell_2\,, \qquad (1.1)$$

where we have

$$L = -L_{vv}\,\partial_2^2 + L_{uv}\,\partial_2 + L_{uu}\,; \qquad (1.2)$$

$$L_{vv} = D_2^T C D_2\,; \qquad (1.3)$$

$$L_{uv} = \partial_1^* D_1^T C D_2 - D_2^T C D_1 \partial_1\,; \qquad (1.4)$$

$$L_{uu} = \partial_1^* D_1^T C D_1 \partial_1\,; \qquad (1.5)$$

$$\overline{\overline{F}} = \theta\overline{F} + \delta_\Gamma \bar{f}\,; \qquad (1.6)$$

$$\theta(x_1,x_2) = \begin{cases} 1, & (x_1,x_2)\in\Omega \\ 0, & (x_1,x_2)\notin\Omega; \end{cases} \qquad (1.7)$$

$$\delta_\Gamma(x_1,x_2) = \partial\theta/\partial\overline{n}\,; \qquad (1.8)$$

Ω is the domain, occupied by structure;

$$\Omega = \{(x_1,x_2):\ 0<x_1<\ell_1;\ 0<x_2<\ell_2\}; \qquad (1.9)$$

ℓ_1,ℓ_2 are corresponding dimensions of extended domain (linear dimensions of considering structure); $x=(x_1,x_2)$; x_1,x_2 are Cartesian coordinates; $\theta(x_1,x_2)$ is characteristic function of domain Ω; $\delta_\Gamma=\delta_\Gamma(x_1,x_2)$ is the delta function of boundary $\Gamma=\partial\Omega$; $\overline{n}=[\,n_1\ \ n_2\,]^T$ is boundary normal vector; $\overline{u}$ is the vector of displacements (unknown vector function),

$$\overline{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}; \qquad (1.10)$$

$$D_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix};\quad D_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}; \qquad (1.11)$$

$$C = \begin{bmatrix} 2\mu+\lambda & \lambda & 0 \\ \lambda & 2\mu+\lambda & 0 \\ 0 & 0 & \mu \end{bmatrix}; \qquad (1.12)$$

μ and λ are Lame parameters; $\overline{F}$ is the load vector in domain Ω; $\bar{f}$ is the corresponding boundary load vector; $\partial_s=\partial/\partial x_s$, $s=1,2$; $\partial_s^*=-\partial/\partial x_s$, $s=1,2$.

Let us introduce the following notations

$$\overline{v} = \partial_2\overline{u} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}; \qquad (1.13)$$

$$\overline{u}' = \partial_2\overline{u}\,;\quad \overline{v}' = \partial_2\overline{v}\,. \qquad (1.14)$$

Thus we can rewrite (1.1):

$$L_{uu}\,\overline{u} + L_{uv}\overline{v} - L_{vv}\partial_2\overline{v} = \overline{\overline{F}}\,. \qquad (1.15)$$

Finally we obtain system of differential equations with operational coefficients:

$$\begin{cases} \overline{u}' = \overline{v} \\ \overline{v}' = L_{vv}^{-1}L_{uu}\overline{u} + L_{vv}^{-1}L_{uv}\overline{v} - L_{vv}^{-1}\overline{\overline{F}} \end{cases} \qquad (1.16)$$

or

$$\overline{z}' = \widetilde{L}\overline{z} + \overline{\widetilde{F}}\,, \qquad (1.17)$$

where

$$\widetilde{L} = \begin{bmatrix} 0 & E \\ L_{vv}^{-1}L_{uu} & L_{vv}^{-1}L_{uv} \end{bmatrix};\quad \overline{\widetilde{F}} = \begin{bmatrix} 0 \\ -L_{vv}^{-1}\overline{\overline{F}} \end{bmatrix}; \qquad (1.18)$$

$$\overline{z} = \begin{bmatrix} \overline{u} \\ \overline{v} \end{bmatrix};\quad E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \qquad (1.19)$$

The system of equations (1.17) is supplemented by boundary conditions, which are set in sections with coordinates $x_2^1=0$ and $x_2^2=\ell_2$.

2. SOME ASPECTS OF THE CONSTRUCTION OF NORMALIZED BASIS FUNCTIONS OF THE B-SPLINE

The construction of B-spline basic functions is determined by the recursive Cox-de Boer formulas [1]:

$$k=1:\quad \varphi_{i,1}(t) = \begin{cases} 1\,, & x_i\le t<x_{i+1} \\ 0\,, & t<x_i \vee t\ge x_{i+1} \end{cases}, \qquad (2.1)$$

$$k\ge 2:\quad \varphi_{i,k}(t) = \frac{(t-x_i)\varphi_{i,k-1}(t)}{x_{i+k-1}-x_i} + \frac{(x_{i+k}-t)\varphi_{i+1,k-1}(t)}{x_{i+k}-x_{i+1}}. \qquad (2.2)$$

We will consider such a construction for the case $x_i=i$ are integers. Let us note that,

$$\varphi_{i,k}(t) = \varphi_{0,k}(t-i)$$

and therefore, recursive formulas (2.1)-(2.2) can be represented in the form

$$k=1: \qquad \varphi_{0,1}(t)=\begin{cases}1\,, & 0\le t<1\\ 0\,, & t<0\vee t\ge 1;\end{cases} \qquad (2.3)$$

$$k\ge 2: \qquad \varphi_{0,k}(t)=\frac{1}{k-1}[t\cdot\varphi_{0,k-1}(t)+ +(k-t)\varphi_{0,k-1}(t-1)]. \qquad (2.4)$$

The function $\varphi_{0,1}(t)$ can be represented by formula

$$\varphi_{0,1}(t)=\frac{1}{2}[\mathrm{sign}(t)-\mathrm{sign}(t-1)]. \qquad (2.5)$$

Let us denote by Δ_1 the operator of the first difference. Then we have

$$\varphi_{0,1}(t)=-\frac{1}{2}\Delta_1\,\mathrm{sign}(t). \qquad (2.6)$$

We can substitute formula (2.5) into (2.4) in order to determine $\varphi_{0,2}(t)$:

$$\begin{aligned}\varphi_{0,2}(t)&=1\cdot[t\cdot\varphi_{0,1}(t)+(2-t)\varphi_{0,1}(t-1)]=\\ &=\frac{1}{2}\{t\cdot[\mathrm{sign}(t)-\mathrm{sign}(t-1)]+\\ &(2-t)[\mathrm{sign}(t-1)-\mathrm{sign}(t-2)]\}=\\ &=\frac{1}{2}[t\,\mathrm{sign}(t)-2(t-1)\,\mathrm{sign}(t-1)+\\ &(t-2)\,\mathrm{sign}(t-2)]=\frac{1}{2}[|t|-2|t-1|+|t-2|.\end{aligned}$$

Let us denote by Δ_2 the operator of the second difference. Then we have

$$\varphi_{0,2}(t)=\frac{1}{2}[|t|-2|t-1|+|t-2|=\frac{1}{2}\Delta_2\,|t-1|. \qquad (2.7)$$

We can define function $\varphi_{0,3}(t)$:

$$\varphi_{0,3}(t)=\frac{1}{2}[t\cdot\varphi_{0,2}(t)+(3-t)\varphi_{0,2}(t-1)].$$

Omitting intermediate calculations, we get

$$\begin{aligned}\varphi_{0,3}(t)&=\frac{1}{4}[t\cdot|t|-3(t-1)|t-1|+\\ &+3(t-2)|t-2|-(t-3)|t-3|]=\\ &=-\frac{1}{2!}\frac{1}{2}\Delta_1\Delta_2((t-1)|t-1|).\end{aligned} \qquad (2.8)$$

Based on formulas (2.8) and (2.4), we can define the function

$$\varphi_{0,4}(t)=\frac{1}{3}[t\cdot\varphi_{0,3}(t)+(4-t)\varphi_{0,3}(t-1)].$$

Omitting intermediate calculations, as a result we get

$$\begin{aligned}\varphi_{0,4}(t)&=\\ &=\frac{1}{2\cdot 3}\cdot\frac{1}{2}[t^2\cdot|t|-4(t-1)^2|t-1|+\\ &+6(t-2)^2|t-2|-4(t-3)^2|t-3|+\\ &+(t-4)^2|t-4|]=\\ &=\frac{1}{3!}\frac{1}{2}(\Delta_2)^2((t-2)^2|t-2|).\end{aligned} \qquad (2.9)$$

It can be proved that for even $k=2m$ we have

$$\varphi_{0,k}(t)=\frac{1}{(2m-1)!}\frac{1}{2}(\Delta_2)^m((t-m)^{2m-2}|t-m|) \qquad (2.10)$$

and for odd (uneven) $k=2m+1$ we have

$$\varphi_{0,k}(t)=-\frac{1}{(2m)!}\frac{1}{2}\Delta_1(\Delta_2)^m((t-m)^{2m-1}|t-1|). \qquad (2.11)$$

Note that $\varphi_{0,k}(t)$ is a polynomial of degree $k-1$ with bounded support and, as follows from the difference operator, this support is equal to the interval $[0,\,k]$.

In addition, we should note the following property of B-spline basis functions:

$$\sum_i \varphi_{0,k}(t-i)\equiv 1 \text{ for arbitrary } t. \qquad (2.12)$$

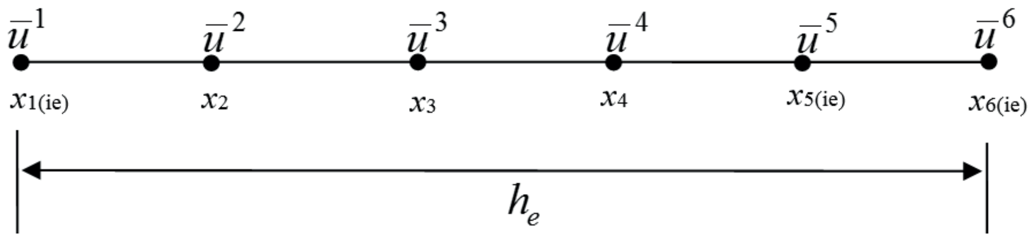


Figure 3.1. Finite element discretication for $N_k = 5$ *(sample).*

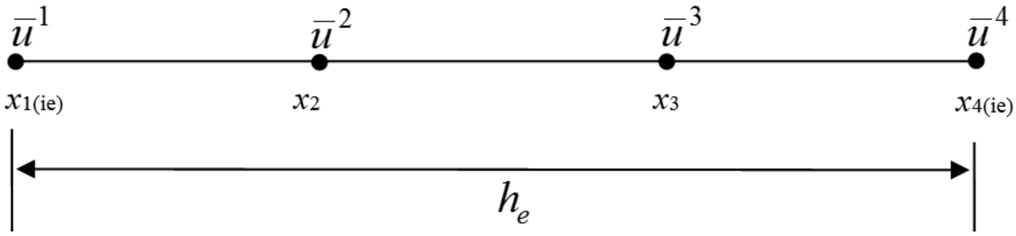


Figure 3.2. Finite element discretication for $N_k = 3$ *(sample).*

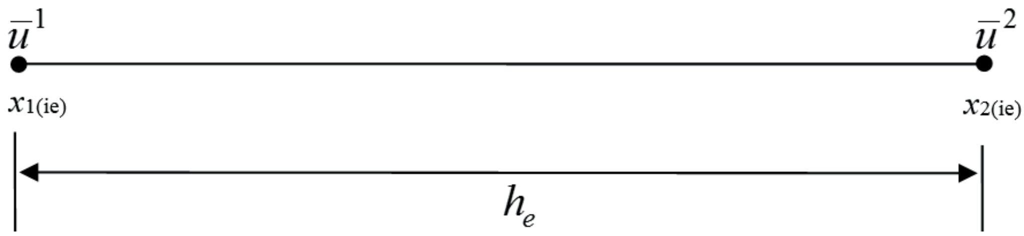


Figure 3.3. Finite element discretication for $N_k = 1$ *(sample).*

3. SOME GENERAL ASPECTS OF FINITE ELEMENT APPROXIMATION

The discrete component of the numerical solution is represented by the direction along the axis corresponding to x_1. The fulfillment within an element (interval) for all components of a vector functions $\bar{u}$ and $\bar{v}$ (see (1.10), (1.13)) is the same. Therefore, let us use the following notation for simplicity:

$$x = x_1,\ \ell = \ell_1,\ y = y(x), \tag{3.1}$$

where $y = y(x)$ is unknown function (component of vector function).

Let us divide the interval $(0, \ell)$ segment into N_e parts (elements). Therefore $h_e = \ell / N_e$ is the length of the element. Besides, let us also divide each element into N_k parts. It should be noted that on the elements of the localization of the solution, parameter N_k is of greater importance than on the other elements. For example, on localization elements, we can set $N_k = 5$, i.e. unknown functions will be represented by polynomials (B-splines) of the 5th degree (Figure 3.1).

Let us use the following notation system: i_e is the element number; $N_p = N_k + 1 = 6$ is the number of nodes within the element; $x_1(i_e)$ is the coordinate of the starting point of the i_e-th element; $x_6(i_e)$ is the coordinate of the end point of the i_e-th element. Thus, the number of unknowns per element with such approximation is equal to

$$N_{ie} = 2N_p = 12\,.$$

For the elements of localization we can take reduced number of N_k. For instance, if we take

$N_k = 3$ (Figure 3.2) we get $N_p = N_k + 1 = 4$ and the number of unknowns per element with such approximation is equal to

$$N_{ie} = 2N_p = 8;$$

$x_1(i_e)$ is the coordinate of the starting point of the i_e-th element; $x_4(i_e)$ is the coordinate of the end point of the i_e-th element.

Besides, let us consider the case with $N_k = 1$ (Figure 3.3). Therefor we have $N_p = N_k + 1 = 2$ and the number of unknowns per element with such approximation is equal to

$$N_{ie} = 2N_p = 4,$$

where $x_1(i_e)$ is the coordinate of the starting point of the i_e-th element; $x_2(i_e)$ is the coordinate of the end point of the i_e-th element.

4. LOCAL CONSTRUCTIONS FOR ARBITRARY FINITE ELEMENT

Let us introduce local coordinates:

$$t = (x - x_{1(ie)})/h_e, \quad x_{1(ie)} \le x \le x_{N_p(ie)}, \ 0 \le t \le 1. \tag{4.1}$$

In this case, we have the following relations:

$$x = x_i \Rightarrow t_i = (x_i - x_{1(ie)})/h_e, \quad i = 1, ..., N_p; \tag{4.2}$$

$$\frac{d^p}{dx^p} = \frac{1}{h_e^p}\frac{d^p}{dt^p}; \quad dx = h_e \cdot dt. \tag{4.3}$$

Since the number of unknowns on the element is equal to $N_{ie} = 12$, we use a B-spline of the fifth degree in order to represent the unknown deflection function.

Let us use the following notation:

$$\varphi(t) = \varphi_{0,6}(t+3);$$

$$\begin{aligned}\varphi(t) &= \frac{1}{5!}\frac{1}{2}(\Delta_2)^3(t^4\,|t|) = \\ &= \frac{1}{5!\cdot 2}[(t+3)^4\,|t+3| - 6(t+2)^4\,|t+2| + \\ &+ 15(t+1)^4\,|t+1| - 20t^4\,|t| + \\ &+ 15(t-1)^4\,|t-1| - 6(t-2)^4\,|t-2| + \\ &+ (t-3)^4\,|t-3|].\end{aligned} \tag{4.4}$$

This function is a B-spline, symmetric with respect to $t = 0$ and its support is defined by an interval $[-3, 3]$ (Figure 4.1).

We take the following eight functions as basis functions on the unit interval (Figures 4.2, 4.3):

$$\begin{aligned}&\varphi_1(t) = \varphi(t+2), \quad \varphi_2(t) = \varphi(t+1), \\ &\varphi_3(t) = \varphi(t), \quad \varphi_4(t) = \varphi(t-1), \\ &\varphi_5(t) = \varphi(t-2), \quad \varphi_6(t) = \varphi(t-3), \\ &0 \le t \le 1.\end{aligned} \tag{4.5}$$

Since the number of unknowns on the element is equal to $N_{ie} = 8$, we use a B-spline of the third degree in order to represent the unknown deflection function.

Let us use the following notation:

$$\varphi(t) = \varphi_{0,4}(t+4);$$

$$\begin{aligned}\varphi(t) &= \frac{1}{3!}\frac{1}{2}(\Delta_2)^2(t^2\,|t|) = \\ &= \frac{1}{3!\cdot 2}[(t+2)^2\,|t+2| - 4(t+1)^2\,|t+1| + \\ &+ 6t^2\,|t| - 4(t-1)^2\,|t-1| + \\ &+ (t-2)^2\,|t-2|].\end{aligned} \tag{4.6}$$

This function is a B-spline, symmetric with respect to $t = 0$ and its support is defined by an interval $[-2, 2]$ (Figure 4.4).

We take the following four functions as basis functions on the unit interval (Figures 4.5):

$$\begin{aligned}&\varphi_1(t) = \varphi(t+1), \quad \varphi_2(t) = \varphi(t), \\ &\varphi_3(t) = \varphi(t-1), \quad \varphi_4(t) = \varphi(t-2), \\ &0 \le t \le 1.\end{aligned} \tag{4.7}$$

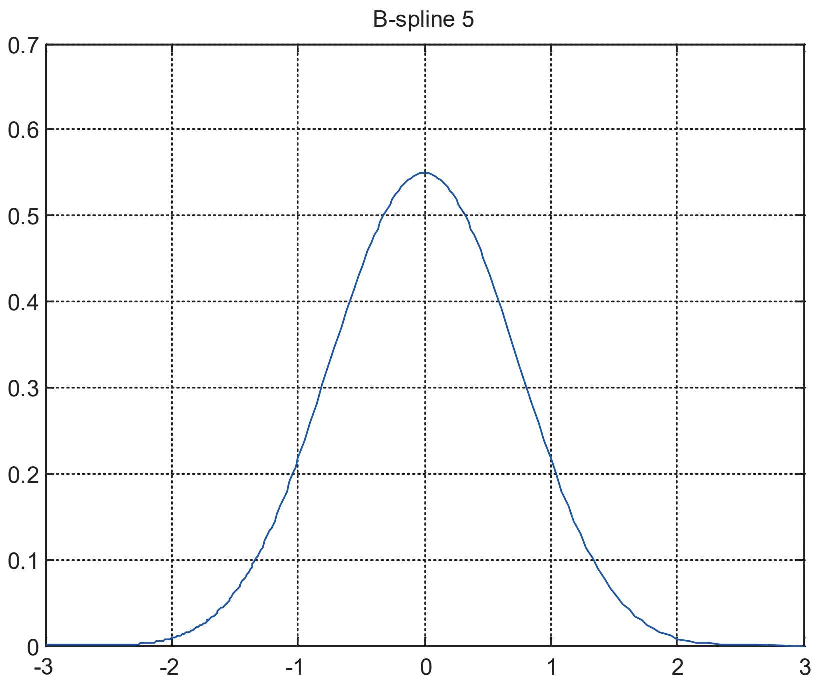


Figure 4.1. B-spline of the fifth order $\varphi(t)=\varphi_{0,6}(t+3)$.

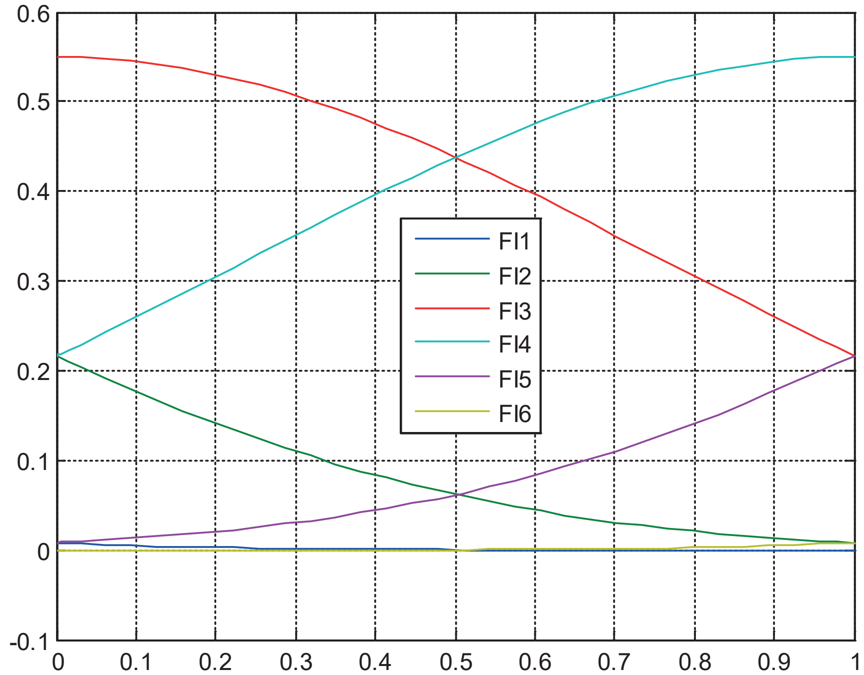


Figure 4.2. Basis functions $\varphi_k(t)$, $k=1,2,...,6$.

Since the number of unknowns on the element is equal to $N_{ie}=4$, we use a B-spline of the first degree in order to represent the unknown deflection function.

Let us use the following notation:

$$\varphi(t)=\varphi_{0,2}(t+1);$$

$$\varphi(t)=\frac{1}{2}\Delta_2\,|\,t\,|=\frac{1}{2}[|\,t+1\,|-2\,|\,t\,|+|\,t-1\,|]. \quad (4.8)$$

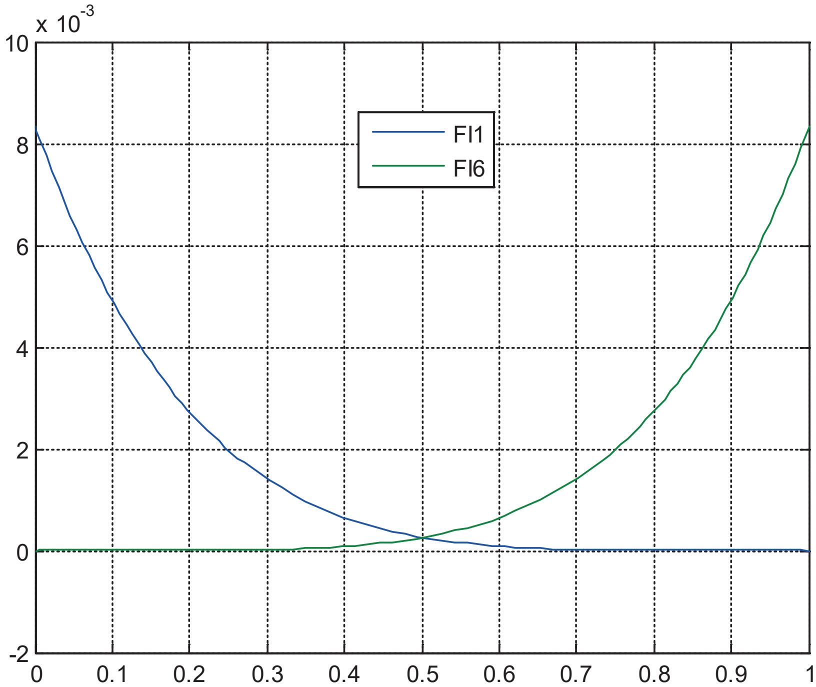


Figure 4.3. Basis functions $\varphi_1(t)$ and $\varphi_6(t)$.

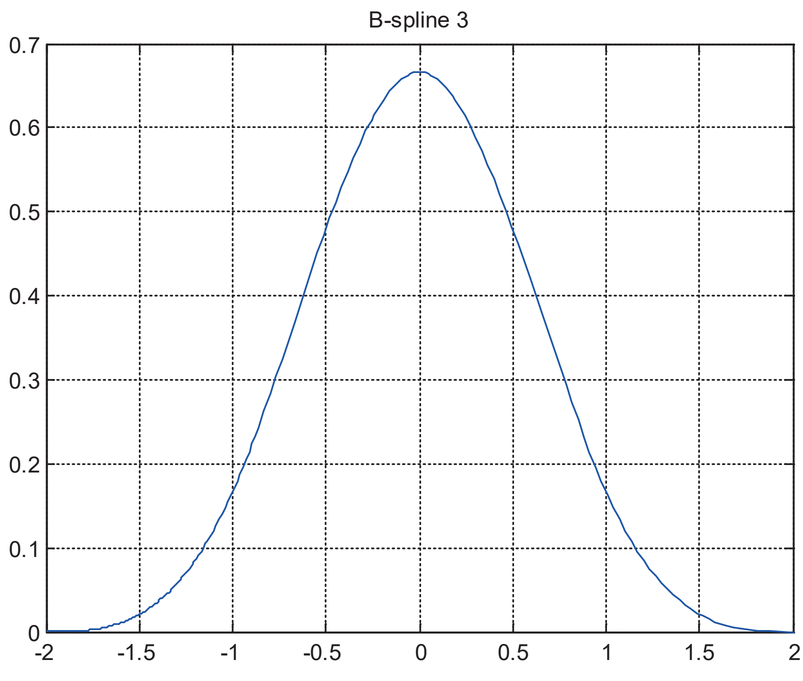


Figure 4.4. B-spline of the third order $\varphi(t) = \varphi_{0,4}(t+2)$.

This function is a B-spline, symmetric with respect to $t = 0$ and its support is defined by an interval $[-1,1]$ (Figure 4.6).

We take the following four functions as basis functions on the unit interval (Figures 4.7):

$$\varphi_1(t) = \varphi(t), \quad \varphi_2(t) = \varphi(t-1), \quad 0 \le t \le 1. \quad (4.9)$$

We represent the unknown function $y(x)$ within the element number i_e in the form

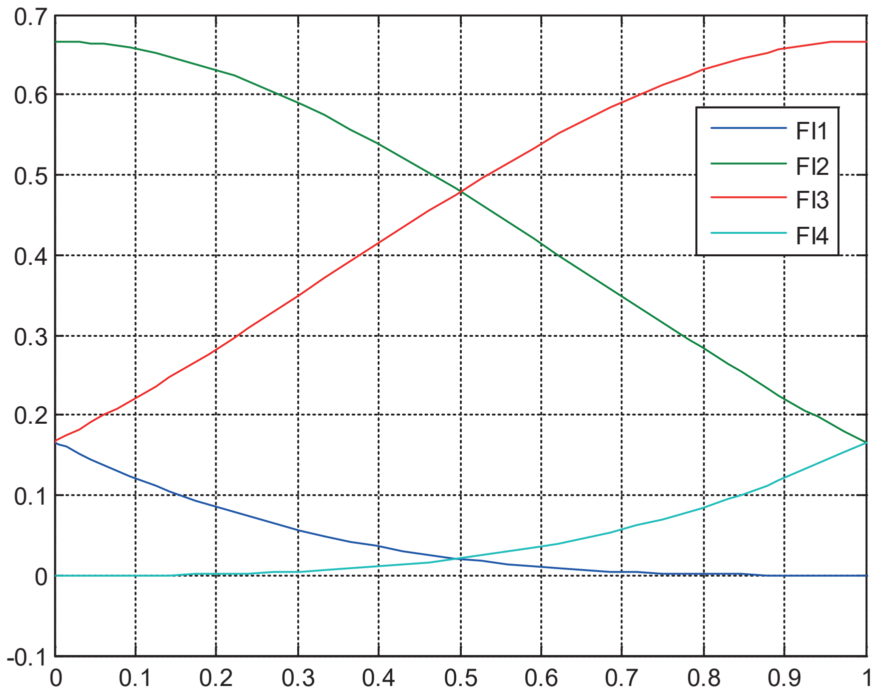


Figure 4.5. Basis functions $\varphi_k(t)$, $k = 1, 2, 3, 4$.

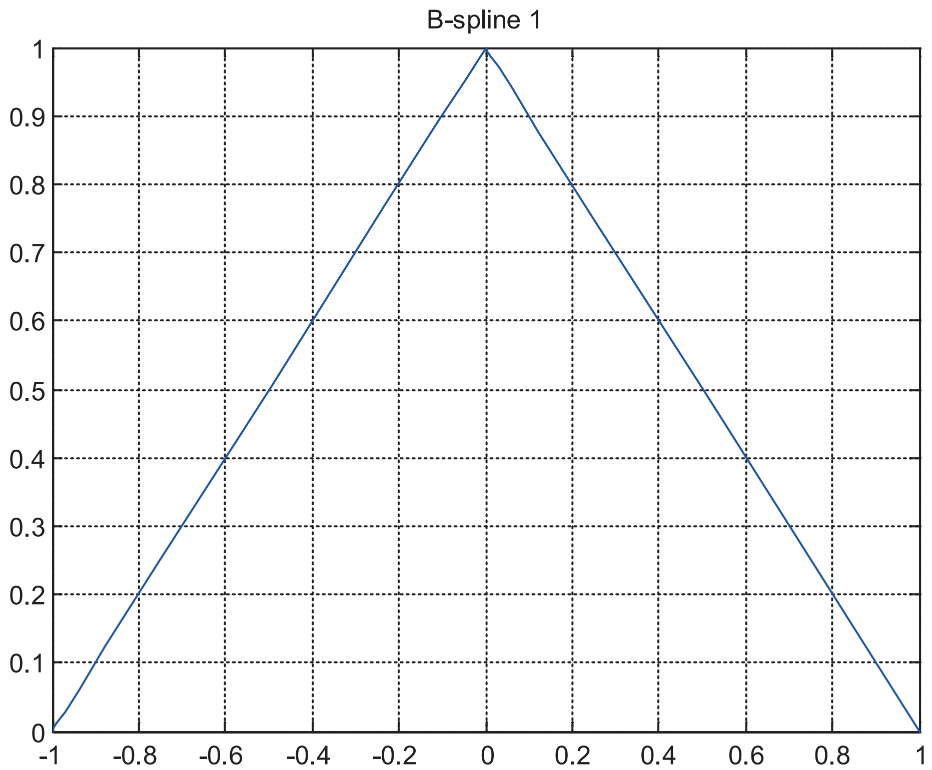


Figure 4.6. B-spline of the first order $\varphi(t) = \varphi_{0,4}(t+2)$.

$$y(x) = w(t) = \sum_{k=1}^{N_p} \alpha_k \varphi_k(t), \quad x_{1(i_e)} \le x \le x_{N_p(i_e)}, \quad 0 \le t \le 1. \quad (4.10)$$

We can define parameters α_k with the use of nodal unknowns of the element:

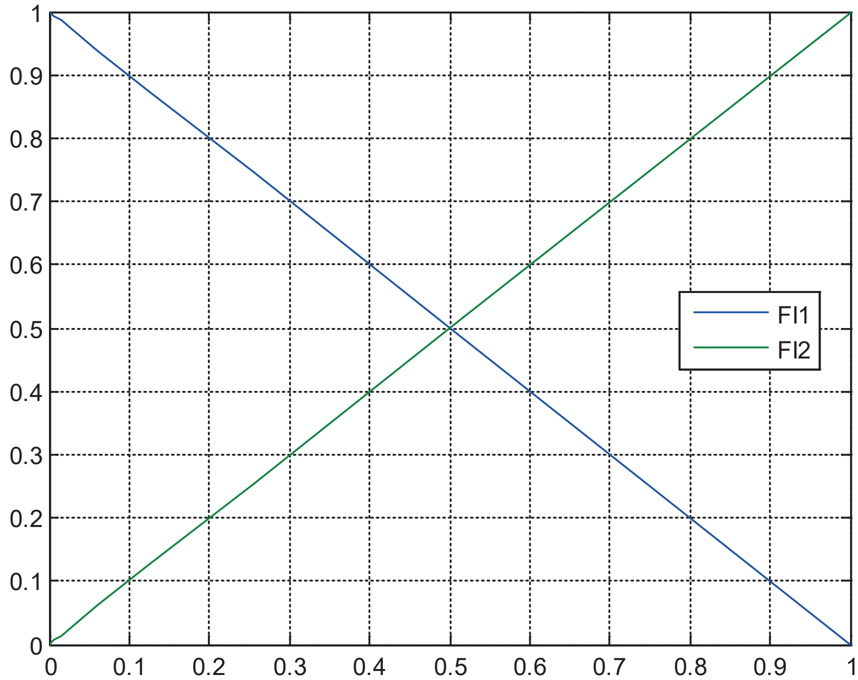


Figure 4.7. Basis functions $\varphi_k(t)$, $k = 1,2$.

$$T_6 = \begin{bmatrix} \varphi_1(0) & \varphi_2(0) & \varphi_3(0) & \varphi_4(0) & \varphi_5(0) & \varphi_6(0) \\ \varphi_1(0.2) & \varphi_2(0.2) & \varphi_3(0.2) & \varphi_4(0.2) & \varphi_5(0.2) & \varphi_6(0.2) \\ \varphi_1(0.4) & \varphi_2(0.4) & \varphi_3(0.4) & \varphi_4(0.4) & \varphi_5(0.4) & \varphi_6(0.4) \\ \varphi_1(0.6) & \varphi_2(0.6) & \varphi_3(0.6) & \varphi_4(0.6) & \varphi_5(0.6) & \varphi_6(0.6) \\ \varphi_1(0.8) & \varphi_2(0.8) & \varphi_3(0.8) & \varphi_4(0.8) & \varphi_5(0.8) & \varphi_6(0.8) \\ \varphi_1(1) & \varphi_2(1) & \varphi_3(1) & \varphi_4(1) & \varphi_5(1) & \varphi_6(1) \end{bmatrix}$$

Figure 4.8. Matrix T_6.

$$y_i = w(t_i) = \sum_{k=1}^{N_p} \alpha_k \varphi_k(t_i), \quad x_{1(i_e)} \le x \le x_{N_p(i_e)}, \quad 0 \le t \le 1. \quad (4.11)$$

In case $N_p = 6$ we have (Figure 4.8)

$$\bar{y}^{i_e} = T_6 \bar{\alpha}, \quad (4.12)$$

$$\bar{y}^{i_e} = [y_1 \;\; y_2 \;\; y_3 \;\; y_4 \;\; y_5 \;\; y_6]^T; \quad (4.13)$$

$$\bar{\alpha} = [\alpha_1 \;\; \alpha_2 \;\; \alpha_3 \;\; \alpha_4 \;\; \alpha_5 \;\; \alpha_6]^T. \quad (4.14)$$

In case $N_p = 4$ we have

$$\bar{y}^{i_e} = T_4 \bar{\alpha}, \quad (4.15)$$

$$\bar{y}^{i_e} = [y_1 \;\; y_2 \;\; y_3 \;\; y_4]^T; \quad (4.16)$$

$$\bar{\alpha} = [\alpha_1 \;\; \alpha_2 \;\; \alpha_3 \;\; \alpha_4]^T; \quad (4.17)$$

$$T_4 = \begin{bmatrix} \varphi_1(0) & \varphi_2(0) & \varphi_3(0) & \varphi_4(0) \\ \varphi_1(1/3) & \varphi_2(1/3) & \varphi_3(1/3) & \varphi_4(1/3) \\ \varphi_1(2/3) & \varphi_2(2/3) & \varphi_3(2/3) & \varphi_4(2/3) \\ \varphi_1(1) & \varphi_2(1) & \varphi_3(1) & \varphi_4(1) \end{bmatrix}. \quad (4.18)$$

In case $N_p = 2$ we have

$$\bar{y}^{i_e} = T_2 \bar{\alpha}, \quad (4.19)$$

$$\bar{y}^{i_e} = [y_1 \;\; y_2]^T; \quad \bar{\alpha} = [\alpha_1 \;\; \alpha_2]^T; \quad (4.20)$$

$$T_2 = \begin{bmatrix} \varphi_1(0) & \varphi_2(0) \\ \varphi_1(1) & \varphi_2(1) \end{bmatrix}. \quad (4.21)$$

Using (4.12)-(4.21), we get

$$\bar{\alpha} = T_{N_p}^{-1} \bar{y}^{i_e}, \quad (4.22)$$

where

$$T_{N_p} = \{T_{ij}\}_{i,j=1,..,N_p}; \quad T_{ij} = \varphi_j(t_i). \quad (4.23)$$

Let us introduce the following notation system:

$$\overline{u}^{ie} = \begin{bmatrix} \overline{u}^1 \\ \vdots \\ \overline{u}^{N_p} \end{bmatrix} \quad (4.24)$$

is nodal vector-function of element number i_e;

$$\overline{u}^i = \begin{bmatrix} u_1^i \\ u_2^i \end{bmatrix} \quad (4.25)$$

is vector-unction in node number i with the element number i_e;

$$\overline{u}_k^{ie} = \begin{bmatrix} u_k^1 \\ \vdots \\ u_k^{N_p} \end{bmatrix}, \quad k = 1,2 \quad (4.26)$$

is nodal component number k of vector-function of element number i_e.

Let P be permutation matrix,

$$\overline{u}^{ie} = P\begin{bmatrix} \overline{u}_1^{ie} \\ \overline{u}_2^{ie} \end{bmatrix}; \quad (4.27)$$

$$P = \underbrace{\left[\begin{array}{cccc|cccc} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & 0 & \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & 1 & \vdots & \vdots & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & & 1 \end{array}\right]}_{2\cdot N_p}; \quad (4.28)$$

Due to $P^{-1} = P^T$ we have

$$\begin{bmatrix} \overline{u}_1^{ie} \\ \overline{u}_2^{ie} \end{bmatrix} = P^T \overline{u}^{ie}. \quad (4.29)$$

We have to consider bilinear forms with allowance for relations (4.2)-(4.3) in order to construct local stiffness matrices corresponding to the operators L_{uu}, L_{uv}, L_{vv} (see (1.3)-(1.5)):

$$(L_{uu}\overline{u}, \overline{z}) = (\partial^* D_1^T C D_1 \partial \overline{u}, \overline{z}) = (D_1^T C D_1 \partial \overline{u}, \partial \overline{z}) =$$
$$= \int_{x_{1(ie)}}^{x_{Np(ie)}} (\begin{bmatrix} 2\mu+\lambda & \\ & \mu \end{bmatrix} \partial \overline{u}, \partial \overline{z}) dx; \quad (4.30)$$

$$(L_{uv}\overline{u}, \overline{z}) = (\partial^* D_1^T C D_2 \overline{u} - D_2^T C D_1 \partial \overline{u}, \overline{z}) =$$
$$= (D_1^T C D_2 \overline{u}, \partial \overline{z}) - (D_2^T C D_1 \partial \overline{u}, \overline{z}) =$$
$$= \int_{x_{1(ie)}}^{x_{Np(ie)}} (\begin{bmatrix} & \lambda \\ \mu & \end{bmatrix} \overline{u}, \partial \overline{z}) dx - \int_{x_{1(ie)}}^{x_{Np(ie)}} (\begin{bmatrix} & \mu \\ \lambda & \end{bmatrix} \partial \overline{u}, \overline{z}) dx; \quad (4.31)$$

$$(L_{vv}\overline{u}, \overline{z}) = (D_2^T C D_2 \overline{u}, \overline{z}) =$$
$$= \int_{x_{1(ie)}}^{x_{Np(ie)}} (\begin{bmatrix} \mu & \\ & 2\mu+\lambda \end{bmatrix} \overline{u}, \overline{z}) dx \quad (4.32)$$

for the following type of functions

$$y(x) = w(t) = \sum_{j=1}^{N_p} \varphi_j(t) \overline{\alpha}^j,$$
$$z(x) = \overline{q}(t) = \sum_{j=1}^{N_p} \varphi_j(t) \overline{\beta}^j, \quad (4.33)$$

where we have

$$x_{1(ie)} \le x \le x_{N_p(ie)}, \quad 0 \le t \le 1;$$

$$\overline{\alpha}^j = \begin{bmatrix} \alpha_1^j \\ \alpha_2^j \end{bmatrix}, \quad \overline{\beta}^j = \begin{bmatrix} \beta_1^j \\ \beta_2^j \end{bmatrix}, \quad j = 1, ..., N_p; \quad (4.34)$$

Let us substitute (4.33) sequentially into (4.30)-(4.32), changing the variable of integration (see (4.2)-(4.3)). Let us consider (4.30):

$$\int_{x_{1(ie)}}^{x_{Np(ie)}} (\begin{bmatrix} 2\mu+\lambda & \\ & \mu \end{bmatrix} \partial\overline{u}, \partial\overline{z}) dx$$

$$= \frac{h_e}{h_e^2} \int_0^1 (\begin{bmatrix} 2\mu+\lambda & \\ & \mu \end{bmatrix} \sum_{j=1}^{N_p} \varphi_j'(t)\overline{\alpha}^j, \sum_{j=1}^{N_p} \varphi_j'(t)\overline{\beta}^j) dt =$$

$$= \sum_{i=1}^{N_p}\sum_{j=1}^{N_p} (\frac{1}{h_e}\int_0^1 \varphi_i'(t)\varphi_j'(t)dt \begin{bmatrix} 2\mu+\lambda & \\ & \mu \end{bmatrix} \overline{\alpha}^j, \overline{\beta}^i) =$$

$$= \sum_{i=1}^{N_p}\sum_{j=1}^{N_p} b_{ij} (\begin{bmatrix} 2\mu+\lambda & \\ & \mu \end{bmatrix} \overline{\alpha}^j, \overline{\beta}^i), \quad (4.35)$$

where we have

$$b_{ij} = \frac{1}{h_e}\int_0^1 \varphi_i'(t)\varphi_j'(t)dt. \quad (4.36)$$

It should be noted, in particular, that, $b_{ji} = b_{ij}$ i.e. if $B = \{b_{ij}\}_{i,j=1,..,N_p}$, then $B^T = B$.

For further transformations, we use the representation

$$\overline{\alpha}^j = \begin{bmatrix} \alpha_1^j \\ \alpha_2^j \end{bmatrix} = \alpha_1^j \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \alpha_2^j \begin{bmatrix} 0 \\ 1 \end{bmatrix};$$

$$\overline{\beta}^j = \begin{bmatrix} \beta_1^j \\ \beta_2^j \end{bmatrix} = \beta_1^j \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \beta_2^j \begin{bmatrix} 0 \\ 1 \end{bmatrix};$$

$$\begin{bmatrix} 2\mu+\lambda & \\ & \mu \end{bmatrix} \overline{\alpha}^j = \alpha_1^j (2\mu+\lambda) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \alpha_2^j \mu \begin{bmatrix} 0 \\ 1 \end{bmatrix};$$

$$(\begin{bmatrix} 2\mu+\lambda & \\ & \mu \end{bmatrix} \overline{\alpha}^j, \overline{\beta}^i) = (2\mu+\lambda)\alpha_1^j\beta_1^i + \mu\alpha_2^j\beta_2^i. \quad (4.37)$$

We can substitute (4.37) in (4.35). Taking into account (4.22)-(4.29) and the adopted notation we get

$$\sum_{i=1}^{N_p}\sum_{j=1}^{N_p} b_{ij}((2\mu+\lambda)\alpha_1^j\beta_1^i + \mu\alpha_2^j\beta_2^i) =$$

$$= (2\mu+\lambda)\sum_{i=1}^{N_p}\sum_{j=1}^{N_p} b_{ij}\alpha_1^j\beta_1^i + \mu\sum_{i=1}^{N_p}\sum_{j=1}^{N_p} b_{ij}\alpha_2^j\beta_2^i =$$

$$= (2\mu+\lambda)(B\overline{\alpha}_1, \overline{\beta}_1) + \mu(B\overline{\alpha}_2, \overline{\beta}_2) =$$

$$= (2\mu+\lambda)(BT_{N_p}^{-1}\overline{u}_1^{ie}, T_{N_p}^{-1}\overline{z}_1^{ie}) +$$

$$+ \mu(BT_{N_p}^{-1}\overline{u}_2^{ie}, T_{N_p}^{-1}\overline{z}_2^{ie}) =$$

$$= (2\mu+\lambda)((T_{N_p}^{-1})^T BT_{N_p}^{-1}\overline{u}_1^{ie}, \overline{z}_1^{ie}) +$$

$$+ \mu((T_{N_p}^{-1})^T BT_{N_p}^{-1}\overline{u}_2^{ie}, \overline{z}_2^{ie}) =$$

$$= (2\mu+\lambda)(A_{11}\overline{u}_1^{ie}, \overline{z}_1^{ie}) + \mu(A_{11}\overline{u}_2^{ie}, \overline{z}_2^{ie}) =$$

$$= \left(\left[\begin{array}{c|c} (2\mu+\lambda)A_{11} & 0 \\ \hline 0 & 0 \end{array} \right] \begin{bmatrix} \overline{u}_1^{ie} \\ \overline{u}_2^{ie} \end{bmatrix}, \begin{bmatrix} \overline{z}_1^{ie} \\ \overline{z}_2^{ie} \end{bmatrix} \right) +$$

$$+ \left(\left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & \mu A_{11} \end{array} \right] \begin{bmatrix} \overline{u}_1^{ie} \\ \overline{u}_2^{ie} \end{bmatrix}, \begin{bmatrix} \overline{z}_1^{ie} \\ \overline{z}_2^{ie} \end{bmatrix} \right) =$$

$$= \left(\left[\begin{array}{c|c} (2\mu+\lambda)A_{11} & 0 \\ \hline 0 & \mu A_{11} \end{array} \right] \begin{bmatrix} \overline{u}_1^{ie} \\ \overline{u}_2^{ie} \end{bmatrix}, \begin{bmatrix} \overline{z}_1^{ie} \\ \overline{z}_2^{ie} \end{bmatrix} \right) =$$

$$= \left(\left[\begin{array}{c|c} (2\mu+\lambda)A_{11} & 0 \\ \hline 0 & \mu A_{11} \end{array} \right] P^T\overline{u}^{ie}, P^T\overline{z}^{ie} \right) =$$

$$= \left(P \left[\begin{array}{c|c} (2\mu+\lambda)A_{11} & 0 \\ \hline 0 & \mu A_{11} \end{array} \right] P^T\overline{u}^{ie}, \overline{z}^{ie} \right) =$$

$$= (K_{uu}^{ie}\overline{u}^{ie}, \overline{z}^{ie}). \quad (4.38)$$

Thus, an expression is obtained for the local stiffness matrix corresponding to the operator L_{uu} within the element number i_e:

$$K_{uu}^{ie} = P \left[\begin{array}{c|c} (2\mu+\lambda)A_{11} & 0 \\ \hline 0 & \mu A_{11} \end{array} \right] P^T, \quad (4.39)$$

where we have

$$A_{11} = (T_{N_p}^{-1})^T BT_{N_p}^{-1}. \quad (4.40)$$

Then we can consider (4.31) in a similar way:

$$\int_{x_{1(ie)}}^{x_{Np(ie)}} (\begin{bmatrix} & \lambda \\ \mu & \end{bmatrix} \overline{u}, \partial\overline{z}) dx - \int_{x_{1(ie)}}^{x_{Np(ie)}} (\begin{bmatrix} & \mu \\ \lambda & \end{bmatrix} \partial\overline{u}, \overline{z}) dx =$$

$$= \sum_{i=1}^{N_p}\sum_{j=1}^{N_p} r_{ij} (\begin{bmatrix} & \lambda \\ \mu & \end{bmatrix} \overline{\alpha}^j, \overline{\beta}^i) - \sum_{i=1}^{N_p}\sum_{j=1}^{N_p} s_{ij} (\begin{bmatrix} & \mu \\ \lambda & \end{bmatrix} \overline{\alpha}^j, \overline{\beta}^i), \quad (4.41)$$

where we have

$$r_{ij}=\int_0^1\varphi_i'(t)\varphi_j(t)dt\,;\quad s_{ij}=\int_0^1\varphi_i(t)\varphi_j'(t)dt\,. \quad (4.42)$$

We should note that if

$$R=\{r_{ij}\}_{i,j=1,..,N_p}\,,\quad S=\{s_{ij}\}_{i,j=1,..,N_p}\,,$$

we get

$$R^T=S\,.$$

Let us define the elements of the sums (4.41):

$$\begin{bmatrix} & \lambda\\ \mu & \end{bmatrix}\overline{\alpha}^j=\alpha_2^j\lambda\begin{bmatrix}1\\0\end{bmatrix}+\alpha_1^j\mu\begin{bmatrix}0\\1\end{bmatrix};$$

$$(\begin{bmatrix} & \lambda\\ \mu & \end{bmatrix}\overline{\alpha}^j,\overline{\beta}^i)=\mu\alpha_1^j\beta_2^i+\lambda\alpha_2^j\beta_1^i\,; \quad (4.43)$$

$$\begin{bmatrix} & \mu\\ \lambda & \end{bmatrix}\overline{\alpha}^j=\alpha_2^j\mu\begin{bmatrix}1\\0\end{bmatrix}+\alpha_1^j\lambda\begin{bmatrix}0\\1\end{bmatrix};$$

$$(\begin{bmatrix} & \mu\\ \lambda & \end{bmatrix}\overline{\alpha}^j,\overline{\beta}^i)=\lambda\alpha_1^j\beta_2^i+\mu\alpha_2^j\beta_1^i\,. \quad (4.44)$$

Substituting (4.43) into (4.41) and, taking into account (4.22)-(4.29) and the accepted notation, we obtain

$$\begin{aligned}&\sum_{i=1}^{N_p}\sum_{j=1}^{N_p}s_{ij}(\lambda\alpha_1^j\beta_2^i+\mu\alpha_2^j\beta_1^i)=\\
&=\lambda\sum_{i=1}^{N_p}\sum_{j=1}^{N_p}s_{ij}\alpha_1^j\beta_2^i+\mu\sum_{i=1}^{N_p}\sum_{j=1}^{N_p}s_{ij}\alpha_2^j\beta_1^i=\\
&=\lambda(S\overline{\alpha}_1,\overline{\beta}_2)+\mu(S\overline{\alpha}_2,\overline{\beta}_1)=\\
&=\lambda(ST_{N_p}^{-1}\overline{u}_1^{ie},T_{N_p}^{-1}\overline{z}_2^{ie})+\mu(ST_{N_p}^{-1}\overline{u}_2^{ie},T_{N_p}^{-1}\overline{z}_1^{ie})=\\
&=\lambda((T_{N_p}^{-1})^T ST_{N_p}^{-1}\overline{u}_1^{ie},\overline{z}_2^{ie})\\
&+\mu((T_{N_p}^{-1})^T ST_{N_p}^{-1}\overline{u}_2^{ie},\overline{z}_1^{ie})=\lambda(A_{01}\overline{u}_1^{ie},\overline{z}_2^{ie})+\\
&+\mu(A_{01}\overline{u}_2^{ie},\overline{z}_1^{ie})=\\
&=\left(\left[\begin{array}{c|c}0 & \mu A_{01}\\ \hline \lambda A_{01} & 0\end{array}\right]\begin{bmatrix}\overline{u}_1^{ie}\\ \overline{u}_2^{ie}\end{bmatrix},\begin{bmatrix}\overline{z}_1^{ie}\\ \overline{z}_2^{ie}\end{bmatrix}\right)=\\
&=\left(P\left[\begin{array}{c|c}0 & \mu A_{01}\\ \hline \lambda A_{01} & 0\end{array}\right]P^T\overline{u}^{ie},\overline{z}^{ie}\right)=\\
&=(K_{2,uv}^{ie}\overline{u}^{ie},\overline{z}^{ie})\,. \quad (4.45)\end{aligned}$$

Thus, an expression is obtained for the local stiffness matrix corresponding to the operator L_{uv} within the element number i_e:

$$K_{uv}^{ie}=K_{1,uv}^{ie}-K_{2,uv}^{ie}\,, \quad (4.46)$$

where we have

$$K_{1,uv}^{ie}=P\left[\begin{array}{c|c}0 & \lambda A_{10}\\ \hline \mu A_{10} & 0\end{array}\right]P^T;\quad A_{10}=(T_{N_p}^{-1})^T RT_{N_p}^{-1}; \quad (4.47)$$

$$K_{2,uv}^{ie}=P\left[\begin{array}{c|c}0 & \mu A_{01}\\ \hline \lambda A_{01} & 0\end{array}\right]P^T;\quad A_{01}=(T_{N_p}^{-1})^T ST_{N_p}^{-1}. \quad (4.48)$$

Let us note that

$$K_{2,uv}^{ie}=(K_{1,uv}^{ie})^T\,. \quad (4.49)$$

Let us further consider (4.32) in a similar way:

$$\begin{aligned}&\int_{x_{1(ie)}}^{x_{Np(ie)}}(\begin{bmatrix}\mu & \\ & 2\mu+\lambda\end{bmatrix}\overline{u},\overline{z})dx=\\
&=h_e\int_0^1(\begin{bmatrix}\mu & \\ & 2\mu+\lambda\end{bmatrix}\sum_{j=1}^{N_p}\varphi_j(t)\overline{\alpha}^j,\sum_{j=1}^{N_p}\varphi_j(t)\overline{\beta}^j)dt=\\
&=\sum_{i=1}^{N_p}\sum_{j=1}^{N_p}(h_e\int_0^1\varphi_i(t)\varphi_j(t)dt\begin{bmatrix}\mu & \\ & 2\mu+\lambda\end{bmatrix}\overline{\alpha}^j,\overline{\beta}^i)=\\
&=\sum_{i=1}^{N_p}\sum_{j=1}^{N_p}m_{ij}(\begin{bmatrix}\mu & \\ & 2\mu+\lambda\end{bmatrix}\overline{\alpha}^j,\overline{\beta}^i)\,, \quad (4.50)\end{aligned}$$

where we have

$$m_{ij}=h_e\int_0^1\varphi_i(t)\varphi_j(t)dt\,. \quad (4.51)$$

We should note that, in particular $m_{ji}=m_{ij}$, i.e. if $M=\{m_{ij}\}_{i,j=1,..,N_p}$ we get

$$M^T=M\,.$$

For further transformations, we use the representation

$$\left(\begin{bmatrix} \mu & \\ & 2\mu+\lambda \end{bmatrix}\overline{\alpha}^j, \overline{\beta}^i\right) = \mu\alpha_1^j\beta_1^i + (2\mu+\lambda)\alpha_2^j\beta_2^i . \qquad (4.52)$$

We can substitute (4.52) in (4.50) and taking into account (4.22)-(4.29) and the adopted notation we get:

$$\sum_{i=1}^{N_p}\sum_{j=1}^{N_p} m_{ij}(\mu\alpha_1^j\beta_1^i + (2\mu+\lambda)\alpha_2^j\beta_2^i) =$$
$$= \mu\sum_{i=1}^{N_p}\sum_{j=1}^{N_p} m_{ij}\alpha_1^j\beta_1^i + (2\mu+\lambda)\sum_{i=1}^{N_p}\sum_{j=1}^{N_p} m_{ij}\alpha_2^j\beta_2^i =$$
$$= \mu(M\overline{\alpha}_1, \overline{\beta}_1) + (2\mu+\lambda)(M\overline{\alpha}_2, \overline{\beta}_2) =$$
$$= \mu((T_{N_p}^{-1})^T M T_{N_p}^{-1}\overline{u}_1^{ie}, \overline{z}_1^{ie}) +$$
$$+ (2\mu+\lambda)((T_{N_p}^{-1})^T B T_{N_p}^{-1}\overline{u}_2^{ie}, \overline{z}_2^{ie}) =$$
$$= \mu(A_{00}\overline{u}_1^{ie}, \overline{z}_1^{ie}) + (2\mu+\lambda)(A_{00}\overline{u}_2^{ie}, \overline{z}_2^{ie}) =$$
$$= \left(P\left[\begin{array}{c|c} \mu A_{00} & 0 \\ \hline 0 & (2\mu+\lambda)A_{00} \end{array}\right]P^T\overline{u}^{ie}, \overline{z}^{ie}\right) =$$
$$= (K_{vv}^{ie}\overline{u}^{ie}, \overline{z}^{ie}) . \qquad (4.53)$$

Thus, an expression is obtained for the local stiffness matrix corresponding to the operator L_{vv} within the element number i_e:

$$K_{uu}^{ie} = P\left[\begin{array}{c|c} \mu A_{00} & 0 \\ \hline 0 & (2\mu+\lambda)A_{00} \end{array}\right]P^T , \qquad (4.54)$$

where we have

$$A_{00} = (T_{N_p}^{-1})^T M T_{N_p}^{-1} . \qquad (4.55)$$

5. SEVERAL ASPECTS OF NUMERICAL IMPLEMENTATION

The presented algorithm can be implemented using MATLAB tools. The MATLAB system has convenient functions for working with polynomials. Moreover, the main parameter of these functions is the vector of coefficients of the polynomial. To determine the coefficients of basic polynomials φ_k on an interval $[0\ \ 1]$, we can firstly determine their values at N_p points of the interval $t = [t_1, t_2, ..., t_{N_p}]$, $t_i \in [0\ \ 1]$, $i = 1, 2, ..., N_p$:

$$F_k(i) = \varphi_k(t_i) , \quad i = 1, 2, ..., N_p , \quad k = 1, 2, ..., N_p .$$

Then, using the `polyfit` function, we define their coefficient vector p_k:

```
pk=polyfit(t,Fk,Nk)
```

This function is used to determine the coefficients of the optimal polynomial using the least squares method. In the considering case, we construct polynomial of the $(N_p - 1)$th degree (i.e. we have to define N_p coefficients of polynomial, according to its N_p values), therefore, we get a polynomial passing through the given values.

In order to calculate the derivatives we can sequentially use the `polyder` function:

```
dpk=polyder(pk)
```

is the vector of coefficients φ_k'.

In order to calculate the product of polynomials we can use the `conv` function:

```
pij=conv(pi,pj)
```

is the vector of coefficients $\varphi_i\varphi_j$;

```
d10pij=conv(dpi,pj)
```

is the vector of coefficients $\varphi_i'\varphi_j$;

```
d01pij=conv(pi,dpj)
```

is the vector of coefficients $\varphi_i\varphi_j'$;

```
dpij=conv(dpi, dpj)
```

is the vector of coefficients $\varphi_i'\varphi_j'$.

In order to calculate the antiderivative of a polynomial we can use the `polyint` function:

```
Pi=polyint(pi)
```

is the vector of coefficients $\int \varphi_i dt$;

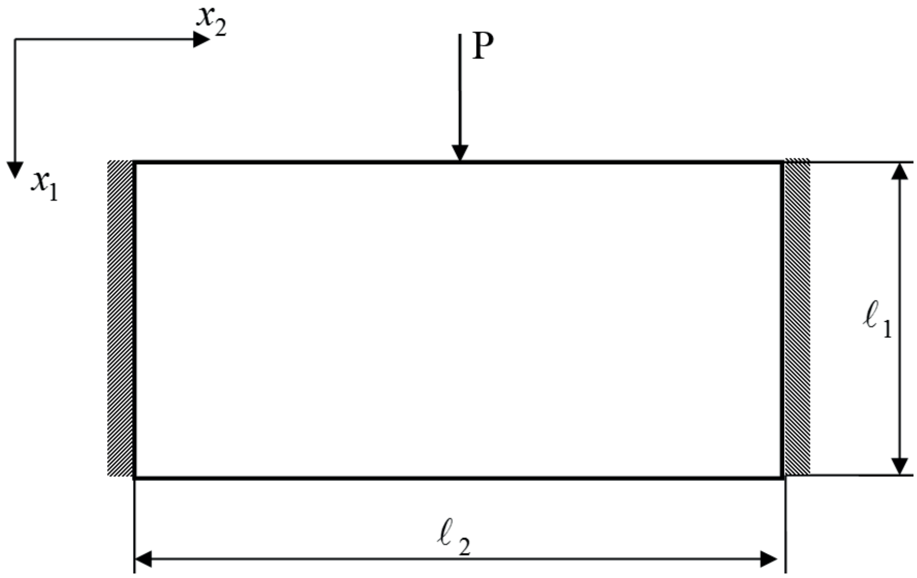


Figure 5.1. Formulation of the problem (Sample).

`Pij=polyint(pij)`

is the vector of coefficients $\int \varphi_i \varphi_j dt$;

`d10Pij=polyint(d10pij)`

is the vector of coefficients $\int \varphi_i' \varphi_j dt$;

`d01Pij=polyint(d01pij)`

is the vector of coefficients $\int \varphi_i \varphi_j' dt$;

`dPij=polyint(dpij)`

is the vector of coefficients $\int \varphi_i' \varphi_j' dt$;

Then the calculation of

$$B(i,j),\quad R(i,j),\quad S(i,j),\quad M(i,j)$$

can be done in accordance with formulas

$M(i,j)=$ `he[polyval(Pij,1)-polyval(Pij,0)];`

$R(i,j)=$ `polyval(d10Pij,1)-polyval(d10Pij,0);`

$S(i,j)=$ `polyval(d01Pij,1)-polyval(d01Pij,0);`

$B(i,j)=$ `[polyval(dPij,1)-polyval(dPij,0)]/he,`

where the function `polyval (p, t)` allows researcher to calculate the values of a polynomial with a vector of coefficients p at a given point t.

5. EXAMPLE OF ANALYSIS

5.1. Formulation of the problem.

As a model example, let us consider the determination of the displacements of a beam-wall, fixed along the side faces in both directions, under the influence of a load concentrated in the center (Figure 5.1).

Let us consider the following geometric parameters: $\ell_1 = 6$ m, $\ell_2 = 12$ m.

Let us consider the following design parameters of material of plate: coefficient of elasticity $E = 26500 \cdot 10^4$ kN/m², Poisson's ratio $\nu = 0.15$.

Let external load parameter be equal to $P = 100$kN.

5.2. Structural analysis with allowance for localization.

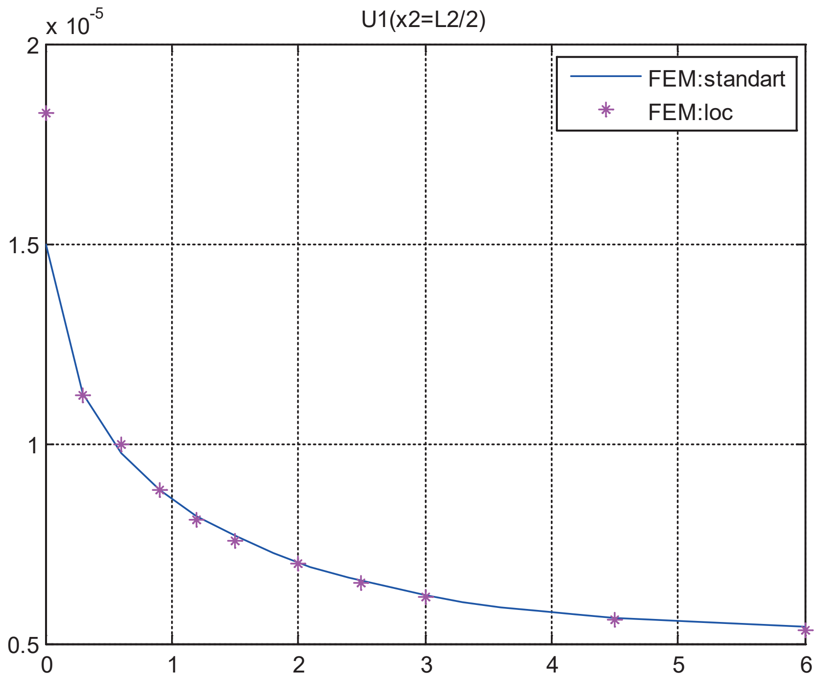
Let the number of elements be equal to $N_e = 6$.

Then we have the following element length:

$$h_e = \ell_1 / N_e = 6/4 = 1.5 .$$

Let's define localization in the load area.

For the first element we have $N_k = 5$ and fifth-order spline; distance between the coordinates of the nodes of the first element is equal to $h_1 = 1.5/5 = 0.3$.



Figures 5.2. Comparison of the results of analysis in the middle sections along x_1 direction (discrete direction).

For the second element we have $N_k = 3$ and third-order spline; distance between the coordinates of the nodes of the second element is equal to $h_2 = 1.5/3 = 0.5$.

For the third element and for the fourth element we have $N_k = 1$ and first-order spline; distance between the coordinates of the nodes of the third element and of the fourth element is equal to $h_3 = h_4 = 1.5/5 = 0.3$.

With such approximation the total number of nodes for all elements is equal to

$$N_x = 5 + 3 + 2 \cdot 1 + 1 = 11 .$$

Then the total number of unknown nodal values for vectors $\bar{u}$ and $\bar{v} = \bar{u}'$ is equal to

$$N_g = 4 \cdot N_x = 4 \cdot 11 = 44 .$$

5.3. Structural analysis without localization.
In this case, we will consider only the standard linear fulfilment. In this case, the length of the element is taken equal to the minimum distance between the nodes, i.e. $h_e = 0.3$. Then the number of elements is equal to

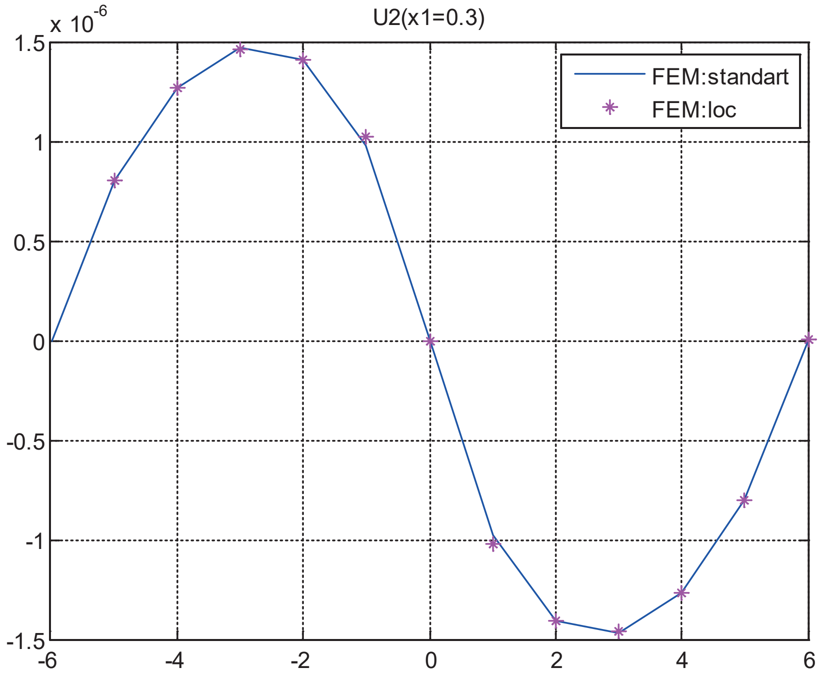
$$N_e = 6/0.3 = 20$$

and the total number of nodes is equal to $N_x = 21$. In this case, the number of nodal unknowns for each component of the vectors $\bar{u}$ and $\bar{v} = \bar{u}'$ is equal to

$$N_g = 4 \cdot N_x = 4 \cdot 21 = 84 .$$

Graphical comparison of corresponding results of analysis is presented at Figure 5.2 and Figure 5.3 (`FEM:loc-spline` are nodal values computed with allowance for localization; `FEM:standart` are nodal values computed without localization).

As researcher can see, the results obtained are almost completely identical. Besides, the use of localization based on application of B-splines of various degrees leads to a significant decrease in the number of unknowns. The difference for this example is equal to $\Delta = 84 - 44 = 40$.



Figures 5.3. Comparison of the results of analysis in the middle sections along x_2 direction (continual direction).

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