LOCALIZATION OF SOLUTION OF THE PROBLEM OF TWO-DIMENSIONAL THEORY OF ELASTICITY WITH THE USE OF B-SPLINE DISCRETE-CONTINUAL FINITE ELEMENT METHOD

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Abstract: Localization of solution of the problem of two-dimensional theory of elasticity with the use of B-spline discrete-continual finite element method (specific version of wavelet-based discrete-continual finite element method) is under consideration in the distinctive paper. The original operational continual and discrete-continual formulations of the problem are given, some actual aspects of construction of normalized basis functions of a B-spline are considered, the corresponding local constructions for an arbitrary discrete-continual finite element are described, some information about the numerical implementation and an example of analysis are presented.

Keywords: localization, wavelet-based discrete-continual finite element method, B-spline discrete-continual finite element method, discrete-continual finite element method, finite element method, B-spline, numerical solution, two-dimensional theory of elasticity, structural analysis.

INTRODUCTION

As we have already mentioned [1, 2], the B-spline in a given simple knot sequence can be constructed by employing piecewise polynomials between the knots and joining them together at the knots [1-3].
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In order to obtain semi-analytical solution in a reasonable time. High accuracy solution at all points of the model is not required normally, it is necessary to find only the most accurate solution in some pre known domains. Generally the choice of these domains is a priori data with respect to the structure being modelled. Designers usually choose domains with the so-called edge effect (with the risk of significant stresses that could potentially lead to the destruction of structures, etc.) and regions which are subject to specific operational requirements. It is obvious that the stress strain state in such domains is of paramount importance. Specified factors along with the obvious needs of the designer or researcher to reduce computational costs by application of DCFEM cause considerable urgency of constructing of special algorithms for obtaining local solutions (in some domains known in advance) of boundary problems. Wavelet analysis provides effective and popular tool for such researches. Solution of the considering problem within multilevel wavelet analysis is represented as a composition of local and global components. Wavelet based DCFEM is presented in papers of P.A. Akimov [35-42], M. Azlam [38-40], T.B. Kaytukov, M.L. Mozgaleva [35-42] and O.A. Negazov [38-40].

The distinctive paper is devoted to numerical solution of the problem of two dimensional theory of elasticity with the use of B-spline DCFEM.

1. FORMULATIONS OF THE PROBLEM

In accordance with [1] let the constancy of the parameters of the problem be in the direction corresponding to \( x_1 \) (main direction). The operational formulation of the problem with the use of so called method of extended domain [43], taking into account the selection of the main direction, is determined by the equation:

\[
L \bar{u} = \bar{F}, \quad \bar{u} < x_1 < \bar{L}_1, \quad \bar{u} < x_2 < \bar{L}_2, \tag{1.1}
\]

where we have

\[
L = -L_\omega \partial_x^2 + L_\omega \partial_x + L_\omega; \tag{1.2}
\]

\[
L_\omega = D_x^2 CD_\omega; \tag{1.3}
\]

\[
L_\omega = \partial_x \partial_x^2 CD_\omega + \partial_x^2 CD \partial_x; \tag{1.4}
\]

\[
L_\omega = \partial_x \partial_x^2 CD \partial_x; \tag{1.5}
\]
\[
\Omega = \{(x_1, x_2) : 0 < x_1 < L_1; \ 0 < x_2 < L_2\};
\]

\(L_1, L_2\) are corresponding dimensions of extended domain (linear dimensions of considering structure); \(x - (x_1, x_2)\); \(x_1, x_2\) are Cartesian coordinates; \(\delta(x_1, x_2)\) is characteristic function of domain \(\Omega\); \(\delta_r - \delta_r(x_1, x_2)\) is the delta function of boundary \(\Gamma - \partial \Omega\); \(n = [n_1, n_2]^{T}\) is boundary normal vector; \(\bar{u}\) is the vector of displacements (unknown vector function),

\[
\bar{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix};
\]

\[
D_l = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}; \\
D_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}; \\
C = \begin{bmatrix} 2 \mu + \lambda & \lambda & 0 \\ \lambda & 2 \mu + \lambda & 0 \\ 0 & 0 & \mu \end{bmatrix};
\]

\(\mu, \lambda\) are Lamé parameters; \(\bar{F}\) is the load vector in domain \(\Omega\); \(\bar{f}\) is the corresponding boundary load vector; \(\bar{\delta}_s - \bar{\partial} / \partial x_s, \ s = 1, 2\).

Let us introduce the following notations

\[
\bar{v} = \bar{\delta}_s \bar{u} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}; \\
\bar{\alpha} = \bar{\partial} / \partial x_1 \\
\bar{\beta} = \bar{\partial} / \partial x_2.
\]

Thus we can rewrite (1.1):

\[
L_w \bar{u} + L_w \bar{v} = L_w \bar{\partial} \bar{v} - \bar{F}.
\]
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\[ k-1: \quad \psi_{\alpha_1}(\xi) = \begin{cases} 1, & 0 < \xi < 1 \\ 0, & \xi < 0 \text{ or } \xi > 1 \end{cases}; \quad (2.3) \]

\[ k > 2: \quad \psi_{\alpha_k}(\xi) = -\frac{1}{k} \left[ \psi_{\alpha_{k-1}}(\xi) + \frac{1}{k} \psi_{\alpha_k}(\xi) \right]. \quad (2.4) \]

The function \( \psi_{\alpha_k}(\xi) \) can be represented by formula

\[ \psi_{\alpha_k}(\xi) = \frac{1}{2} \left[ \text{sign}(\xi) \text{ sign}(\xi - 1) \right]. \quad (2.5) \]

Let us denote by \( \Delta_1 \) the operator of the first difference. Then we have

\[ \psi_{\alpha_k}(\xi) = \frac{1}{2} \Delta_1 \text{sign}(\xi). \quad (2.6) \]

We can substitute formula (2.5) into (2.4) in order to determine \( \psi_{\alpha_2}(\xi) \):

\[ \psi_{\alpha_2}(\xi) = \psi_{\alpha_1}(\xi) + (2 - \xi) \psi_{\alpha_1}(\xi - 1) - \frac{1}{2} \left[ \psi_{\alpha_1}(\xi) \text{ sign}(\xi) \text{ sign}(\xi - 1) \right] + \frac{1}{2} \left[ 2 \text{sign}(\xi) \text{ sign}(\xi - 1) \right] - \frac{1}{2} \left[ \text{sign}(\xi) \text{ sign}(\xi - 1) \right]. \quad (2.7) \]

Let us denote by \( \Delta_2 \) the operator of the second difference. Then we have

\[ \psi_{\alpha_2}(\xi) = \frac{1}{2} \left[ \psi_{\alpha_1}(\xi) + 2 - \xi \right] \psi_{\alpha_1}(\xi - 1) - \frac{1}{2} \left[ \psi_{\alpha_1}(\xi) \text{ sign}(\xi) \text{ sign}(\xi - 1) \right] + \frac{1}{2} \left[ 2 \text{sign}(\xi) \text{ sign}(\xi - 1) \right] - \frac{1}{2} \left[ \text{sign}(\xi) \text{ sign}(\xi - 1) \right]. \quad (2.8) \]

Omitting intermediate calculations, as a result we get

\[ \psi_{\alpha_1}(\xi) = \psi_{\alpha_k}(\xi) - \frac{1}{2} \left[ \psi_{\alpha_k}(\xi) + \psi_{\alpha_k}(\xi - 1) \right]. \quad (2.9) \]

It can be proved that for even \( k - 2m \) we have

\[ \psi_{\alpha_k}(\xi) = \psi_{\alpha_{k+1}}(\xi) - \frac{1}{(2m - 1)!} \left( \Delta_1 \psi_{\alpha_k}(\xi) \right)^{m - 1} \psi_{\alpha_k}(\xi). \quad (2.10) \]

and for odd (uneven) \( k - 2m + 1 \) we have

\[ \psi_{\alpha_k}(\xi) = \frac{1}{(2m - 1)!} \left( \Delta_1 \psi_{\alpha_k}(\xi) \right)^{m - 1} \psi_{\alpha_k}(\xi). \quad (2.11) \]

Note that \( \psi_{\alpha_k}(\xi) \) is a polynomial of degree \( k-1 \) with bounded support and, as follows from the difference operator, this support is equal to the interval \([0, k]\).

In addition, we should note the following property of B-spline basis functions:

\[ \sum \psi_{\alpha_k}(\xi - j) = 1 \text{ for arbitrary } \xi. \quad (2.12) \]
3. SOME GENERAL ASPECTS OF FINITE ELEMENT APPROXIMATION

The discrete component of the numerical solution is represented by the direction along the axis corresponding to $x_1$. The fulfillment within an element (interval) for all components of a vector functions $\mathbf{u}$ and $\mathbf{v}$ (see (1.10), (1.13)) is the same. Therefore, let us use the following notation for simplicity:

$$ x - x_i, \ell - \ell_i, y - y(x), $$

(3.1)

where $y - y(x)$ is unknown function (component of vector function).

Let us divide the interval $(0, \ell)$ segment into $N_s$ parts (elements). Therefore $h_s = \ell / N_s$ is the length of the element. Besides, let us also divide each element into $N_k$ parts. It should be noted that on the elements of the localization of the solution, parameter $N_k$ is of greater importance than on the other elements. For example, on localization elements, we can set $N_k = S$, i.e. unknown functions will be represented by polynomials ($B$ splines) of the 5th degree (Figure 3.1).

Let us use the following notation system: $i_*$ is the element number; $N_{s*} = N_k + 1 - \delta$ is the number of nodes within the element; $x_i(i_*)$ is the coordinate of the starting point of the $i_*$ th element; $x_i(i_*)$ is the coordinate of the end point of the $i_*$ th element. Thus, the number of unknowns per element with such approximation is equal to

$$ N_{s*} = 2N_{s*} - 12. $$

For the elements of localization we can take reduced number of $N_k$. For instance, if we take
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\( N_e - 3 \) (Figure 3.2) we get \( N_p - N_e + 1 - 4 \) and the number of unknowns per element with such approximation is equal to

\[
N_a = 2N_p - 8;
\]

\( x_i(i_\ast) \) is the coordinate of the starting point of the \( i \) \( \ast \) th element; \( x_i(i_\ast) \) is the coordinate of the end point of the \( i \) \( \ast \) th element.

Besides, let us consider the case with \( N_e - 1 \) (Figure 3.3). Therefore we have \( N_p - N_e + 1 - 2 \) and the number of unknowns per element with such approximation is equal to

\[
N_a = 2N_p - 4,
\]

where \( x_i(i_\ast) \) is the coordinate of the starting point of the \( i \) \( \ast \) th element; \( x_i(i_\ast) \) is the coordinate of the end point of the \( i \) \( \ast \) th element.

4. LOCAL CONSTRUCTIONS FOR ARBITRARY FINITE ELEMENT

Let us introduce local coordinates:

\[
i(x - x_i(i_\ast)) / h, \quad x_i(i_\ast) \leq x < x_{i+1(i_\ast)}, \quad 0 < i < 1.
\]

(4.1)

In this case, we have the following relations:

\[
x - x_\ast \Rightarrow \xi = (x - x_i(i_\ast)) / h, \quad i = 1, \ldots, N_p; \quad \xi = 0, \ldots, 3.
\]

(4.2)

\[
\frac{d^\xi}{dx} = \frac{1}{h} \frac{d\xi}{dx}, \quad \xi = h \cdot \xi.
\]

(4.3)

Since the number of unknowns on the element is equal to \( N_a = 8 \), we use a B-spline of the third degree in order to represent the unknown deflection function.

Let us use the following notation:

\[
\psi(\xi) - \psi_{0,i}(\xi + 1);
\]

(4.4)

This function is a B-spline, symmetric with respect to \( \xi = 0 \) and its support is defined by an interval \( [0, 3] \) (Figure 4.1).

We take the following eight functions as basis functions on the unit interval (Figures 4.2, 4.3):

\[
\begin{align*}
\psi_1(\xi) & - \psi_1(\xi + 1), \\
\psi_2(\xi) & - \psi_2(\xi + 1), \\
\psi_3(\xi) & - \psi_3(\xi + 1), \\
\psi_4(\xi) & - \psi_4(\xi + 1), \\
\psi_5(\xi) & - \psi_5(\xi + 1), \\
\psi_6(\xi) & - \psi_6(\xi + 1), \\
\psi_7(\xi) & - \psi_7(\xi + 1), \\
\psi_8(\xi) & - \psi_8(\xi + 1).
\end{align*}
\]

(4.5)

Since the number of unknowns on the element is equal to \( N_a = 8 \), we use a B-spline of the third degree in order to represent the unknown deflection function.

Let us use the following notation:

\[
\psi(\xi) - \psi_{0,i}(\xi + 1);
\]

(4.6)

This function is a B-spline, symmetric with respect to \( \xi = 0 \) and its support is defined by an interval \( [0, 2] \) (Figure 4.4).

We take the following four functions as basis functions on the unit interval (Figures 4.5):

\[
\begin{align*}
\psi_1(\xi) & - \psi_1(\xi + 1), \\
\psi_2(\xi) & - \psi_2(\xi + 1), \\
\psi_3(\xi) & - \psi_3(\xi + 1), \\
\psi_4(\xi) & - \psi_4(\xi + 1).
\end{align*}
\]

(4.7)
Since the number of unknowns on the element is equal to $N_u - 4$, we use a B spline of the first degree in order to represent the unknown deflection function.

Let us use the following notation:

$$
\psi(s) - \psi_{02}(s + 1);
\psi(s) = \frac{1}{2} \left[ \frac{1}{2} s + 1 \right]^{0} \left[ 2 | s | + 1 \right]^{0} \left[ s \right]^{0} \left[ s + 1 \right]^{0} \left[ 1 \right]^{0} . (9.8)
$$
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![Graph showing basis functions and B-spline](image)

This function is a B-spline, symmetric with respect to \( s = 0 \) and its support is defined by an interval \([1,1]\) (Figure 4.6).

We take the following four functions as basis functions on the unit interval (Figures 4.7):

\[
\psi_1(s) - \psi(s), \quad \psi_2(s) - \psi(s-1), \quad 0 < s < 1. \tag{4.9}
\]

We represent the unknown function \( y(x) \) with in the element number \( i \), in the form...
We can define parameters \( \alpha_k \) with the use of nodal unknowns of the element:

\[
y(x) = w(x) + \sum_{k=1}^{n} \alpha_k \psi_k(x), \quad x_{k-1} < x < x_{k+1}.
\]

\( 0 < s < 1. \quad (4.10) \)
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\[
T_a = \begin{bmatrix}
\phi_1(0) & \phi_2(0) & \phi_3(0) & \phi_4(0) & \phi_5(0) & \phi_6(0) \\
\phi_1(0.2) & \phi_2(0.2) & \phi_3(0.2) & \phi_4(0.2) & \phi_5(0.2) & \phi_6(0.2) \\
\phi_1(0.4) & \phi_2(0.4) & \phi_3(0.4) & \phi_4(0.4) & \phi_5(0.4) & \phi_6(0.4) \\
\phi_1(0.6) & \phi_2(0.6) & \phi_3(0.6) & \phi_4(0.6) & \phi_5(0.6) & \phi_6(0.6) \\
\phi_1(0.8) & \phi_2(0.8) & \phi_3(0.8) & \phi_4(0.8) & \phi_5(0.8) & \phi_6(0.8) \\
\phi_1(1) & \phi_2(1) & \phi_3(1) & \phi_4(1) & \phi_5(1) & \phi_6(1)
\end{bmatrix}
\]

In case \( N_p = 6 \) we have (Figure 4.8)

\[
\tilde{y}^v - T_6 \tilde{\sigma}, \quad (4.12)
\]

\[
\tilde{y}^v = [y_1 \ y_2 \ y_3 \ y_4 \ y_5 \ y_6]^T; \quad (4.13)
\]

\[
\tilde{\sigma} = [\alpha_1 \ \alpha_2 \ \alpha_3 \ \alpha_4 \ \alpha_5 \ \alpha_6]^T. \quad (4.14)
\]

In case \( N_p = 2 \) we have

\[
\tilde{y}^v - T_2 \tilde{\sigma}, \quad (4.19)
\]

\[
\tilde{y}^v = [y_1 \ y_2]^T; \quad \tilde{\sigma} = [\alpha_1 \ \alpha_2]^T; \quad (4.20)
\]

\[
T_2 = \begin{bmatrix}
\phi_1(0) & \phi_2(0) \\
\phi_1(1/3) & \phi_2(1/3) \\
\phi_3(1/3) & \phi_4(1/3) \\
\phi_2(2/3) & \phi_3(2/3) \\
\phi_4(2/3) & \phi_5(2/3) \\
\phi_5(1) & \phi_6(1)
\end{bmatrix}. \quad (4.21)
\]

In case \( N_p = 4 \) we have

\[
\tilde{y}^v - T_4 \tilde{\sigma}, \quad (4.15)
\]

\[
\tilde{y}^v = [y_1 \ y_2 \ y_3 \ y_4]^T; \quad (4.16)
\]

\[
\tilde{\sigma} = [\alpha_1 \ \alpha_2 \ \alpha_3 \ \alpha_4]^T. \quad (4.17)
\]

Using (4.12) (4.21), we get

\[
\tilde{\sigma} - T_{n,2} \tilde{y}^v, \quad (4.22)
\]
where
\[
T_{\psi} = \{T_\psi\}_{\psi} \quad \text{and} \quad T_\psi - \psi_\psi \tag{4.23}
\]

Let us introduce the following notation system:
\[
\begin{bmatrix}
\mathbf{u}_1 \\
\vdots \\
\mathbf{u}_n
\end{bmatrix}
\]
\[
\mathbf{u}_n = \begin{bmatrix}
\mathbf{u}_1^T \\
\vdots \\
\mathbf{u}_n^T
\end{bmatrix}
\tag{4.24}
\]
is nodal vector function of element number \( i \);
\[
\mathbf{u}_n = \begin{bmatrix}
\mathbf{u}_1 \\
\vdots \\
\mathbf{u}_n
\end{bmatrix}
\tag{4.25}
\]
is vector function in node number \( j \) with the element number \( i \);
\[
\mathbf{u}_k = \begin{bmatrix}
\mathbf{u}_k^1 \\
\vdots \\
\mathbf{u}_k^{n_k}
\end{bmatrix}, \quad k = 1, 2
\tag{4.26}
\]
is nodal component number \( k \) of vector function of element number \( i \).
Let \( P \) be permutation matrix,
\[
\begin{bmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 1
\end{bmatrix}
\]
\[
P^{-1} = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 1
\end{bmatrix}
\tag{4.27}
\]

We have to consider bilinear forms with allowance for relations (4.2) (4.3) in order to construct local stiffness matrices corresponding to the operators \( L_{\omega}, L_{\omega} \), \( L_{\omega} \) (see (1.3) (1.5)):
\[
\begin{align*}
\langle L_{\omega} \mathbf{u}, \mathbf{v} \rangle &= \langle \delta \mathbf{D}_1 \mathbf{C}_1 \mathbf{C}_1 \mathbf{D}_1 \mathbf{u}, \mathbf{v} \rangle - \langle \delta \mathbf{D}_1 \mathbf{C}_1 \mathbf{C}_1 \mathbf{D}_1 \mathbf{u}, \mathbf{v} \rangle - \\
& \quad - \int_{\omega} \left( \begin{array}{c}
2\mu + \lambda \\
\mu
\end{array} \right) \mathbf{u}, \mathbf{v} \mathbf{x} dx \tag{4.30}
\end{align*}
\]
\[
\begin{align*}
\langle L_{\omega} \mathbf{u}, \mathbf{v} \rangle &= \langle \delta \mathbf{D}_1 \mathbf{C}_1 \mathbf{C}_1 \mathbf{D}_1 \mathbf{u}, \mathbf{v} \rangle - \langle \delta \mathbf{D}_1 \mathbf{C}_1 \mathbf{C}_1 \mathbf{D}_1 \mathbf{u}, \mathbf{v} \rangle - \\
& \quad - \int_{\omega} \left( \begin{array}{c}
\lambda \\
2\mu + \lambda
\end{array} \right) \mathbf{u}, \mathbf{v} \mathbf{x} dx \tag{4.32}
\end{align*}
\]

for the following type of functions
\[
\mathbf{z}(x) - \mathbf{w}(x) - \sum_{j=1}^{n_k} \psi_j \mathbf{f}^j
\]
\[
\mathbf{z}(x) - \mathbf{w}(x) - \sum_{j=1}^{n_k} \psi_j \mathbf{f}^j
\tag{4.33}
\]

where we have
\[
x_{i_{\omega}} < x < x_{i_{\omega}+1}, \quad 0 < j < 1
\tag{4.34}
\]

Let us substitute (4.33) sequentially into (4.30) (4.32), changing the variable of integration (see (4.2) (4.3)). Let us consider (4.30):

Due to \( P^{-1} - P^T \) we have
\[
\begin{align*}
\begin{bmatrix}
\mathbf{u}_1 \\
\vdots \\
\mathbf{u}_n
\end{bmatrix} - \begin{bmatrix}
\mathbf{u}_1 \\
\vdots \\
\mathbf{u}_n
\end{bmatrix}
\end{align*}
\tag{4.29}
\]
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\[ \int_{\Omega} \left[ \frac{2\mu + \lambda}{\rho} \right] \bar{\sigma} : \bar{\varepsilon} \, dx \]

\[ - \frac{h_x}{h_x^2} \int_{\Omega} \left[ \frac{2\mu + \lambda}{\rho} \right] \sum_{j=1}^{N_x} \psi_j(x) \left[ \frac{2\mu + \lambda}{\rho} \right] \bar{\sigma}_j \, dx - \sum_{i=1}^{N_\nu} \sum_{j=1}^{N_\nu} b_{ij} \psi_i(x) \psi_j(x) \left[ \frac{2\mu + \lambda}{\rho} \right] \bar{\sigma}_i \bar{\sigma}_j, \tag{4.35} \]

where we have

\[ b_{ij} = \frac{1}{h_x} \int_{\Omega} \psi_i(x) \psi_j(x) \, dx. \tag{4.36} \]

It should be noted, in particular, that \( b_{ij} \neq b_{ji} \), i.e. if \( \Omega = \{ b_{ij} \}_{i=1}^{N_\nu}, \nu \), then \( B^T = B \).

For further transformations, we use the representation

\[ \bar{\sigma}_i = \begin{bmatrix} \alpha_i^1 \\ \alpha_i^2 \\ \alpha_i^3 \\ \alpha_i^4 \end{bmatrix} \]

\[ \bar{\sigma}_i = \begin{bmatrix} \beta_i^1 \\ \beta_i^2 \\ \beta_i^3 \\ \beta_i^4 \end{bmatrix} \]

\[ \begin{bmatrix} 2\mu + \lambda \\ \mu \end{bmatrix} \bar{\sigma}_i - \alpha_i^1 \left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) + \alpha_i^4 \left( \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) \]

\[ \begin{bmatrix} 2\mu + \lambda \\ \mu \end{bmatrix} \bar{\sigma}_i - \beta_i^1 \left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) + \beta_i^4 \left( \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) \]

\[ \begin{bmatrix} 2\mu + \lambda \\ \mu \end{bmatrix} \bar{\sigma}_i - \alpha_i^1 \left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) + \alpha_i^4 \left( \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) \]

\[ \begin{bmatrix} 2\mu + \lambda \\ \mu \end{bmatrix} \bar{\sigma}_i - \beta_i^1 \left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) + \beta_i^4 \left( \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) \]

Thus, an expression is obtained for the local stiffness matrix corresponding to the operator \( L_{ij} \) within the element number \( i \):

\[ K_{ij} = F \left[ \frac{\left( 2\mu + \lambda \right) \mu A_{ij}}{\rho} \right] F^T, \tag{4.39} \]

where we have

\[ A_{ij} = \begin{bmatrix} \mu A_{ij} \\ \mu A_{ij} \end{bmatrix} \tag{4.40} \]

Then we can consider (4.31) in a similar way:

\[ \int_{\Omega} \left[ \begin{bmatrix} \mu \\ \mu \end{bmatrix} \right] \bar{\sigma}_i \bar{\varepsilon}_j \, dx - \int_{\Omega} \left[ \begin{bmatrix} \mu \\ \mu \end{bmatrix} \right] \bar{\sigma}_i \bar{\varepsilon}_j \, dx - \sum_{i=1}^{N_x} \sum_{j=1}^{N_x} b_{ij} \mu \alpha_i \beta_j - \sum_{i=1}^{N_\nu} \sum_{j=1}^{N_\nu} b_{ij} \mu \alpha_i \beta_j, \tag{4.41} \]

where we have
\[ r_s = \frac{1}{2} \int_{x}^{} \phi_i(x)\phi_j(x)dx; \quad s_j = \frac{1}{2} \int_{x}^{} \phi_i(x)\psi_j(x)dx. \tag{4.42} \]

We should note that if
\[ R = \{r_s\}_{1 \leq i \leq n_s}, \quad S = \{s_j\}_{1 \leq i \leq n_s}, \]
we get
\[ R^s - S. \]

Let us define the elements of the sums (4.41):
\[
\begin{align*}
\begin{bmatrix} \mu \\
\lambda 
\end{bmatrix} & \begin{bmatrix} \bar{\alpha} \\
\gamma 
\end{bmatrix} = \alpha_{1} \lambda \begin{bmatrix} 1 \\
0 
\end{bmatrix} + \alpha_{1} \mu \begin{bmatrix} 0 \\
1 
\end{bmatrix}; \\
\begin{bmatrix} \mu \\
\lambda 
\end{bmatrix} & \begin{bmatrix} \bar{\alpha} \\
\gamma 
\end{bmatrix} = \mu_{0} \beta_{1} + \lambda \gamma_{1} \\
\begin{bmatrix} \lambda \\
\mu 
\end{bmatrix} & \begin{bmatrix} \bar{\alpha} \\
\gamma 
\end{bmatrix} = \mu_{0} \gamma_{1} + \lambda \beta_{1}. \tag{4.43} \\
\end{align*}
\]

Substituting (4.43) into (4.41) and, taking into account (4.22) (4.29) and the accepted notation, we obtain
\[
\sum_{i=1}^{n_s} \sum_{j=1}^{n_s} \left( \lambda \alpha_{1} \beta_{1} + \mu \gamma_{1} \right) - \lambda \sum_{i=1}^{n_s} \sum_{j=1}^{n_s} \alpha_{1} \beta_{1} + \mu \sum_{i=1}^{n_s} \gamma_{1} = - \lambda (\alpha_{1} \overline{\beta_{1}}) + \mu (\overline{\gamma_{1}}) - \lambda (\alpha_{1} \overline{\beta_{1}}) + \mu (\overline{\gamma_{1}}) - \lambda (\alpha_{1} \overline{\beta_{1}}) + \mu (\overline{\gamma_{1}}) - \lambda (\alpha_{1} \overline{\beta_{1}}) + \mu (\overline{\gamma_{1}}) + \mu (\overline{\alpha_{1} \overline{\gamma_{1}}} \overline{\beta_{1}}) + \lambda (\overline{\alpha_{1} \gamma_{1}} \overline{\beta_{1}}) - \lambda (\overline{\alpha_{1} \gamma_{1}} \overline{\beta_{1}}) + \mu (\overline{\alpha_{1} \gamma_{1}} \overline{\beta_{1}}) - \lambda (\overline{\alpha_{1} \gamma_{1}} \overline{\beta_{1}}) + \mu (\overline{\alpha_{1} \gamma_{1}} \overline{\beta_{1}}) - \lambda (\overline{\alpha_{1} \gamma_{1}} \overline{\beta_{1}}) + \mu (\overline{\alpha_{1} \gamma_{1}} \overline{\beta_{1}}) - \lambda (\overline{\alpha_{1} \gamma_{1}} \overline{\beta_{1}}) + \mu (\overline{\alpha_{1} \gamma_{1}} \overline{\beta_{1}}) = - \left( \begin{bmatrix} 0 \\
\lambda \alpha_{1} 
\end{bmatrix} \begin{bmatrix} \overline{\alpha_{1}} \\
\overline{\alpha_{1}} 
\end{bmatrix} \right) - \left( \begin{bmatrix} 0 \\
\mu \alpha_{1} 
\end{bmatrix} \begin{bmatrix} \overline{\alpha_{1}} \\
\overline{\alpha_{1}} 
\end{bmatrix} \right) - \left( \begin{bmatrix} 0 \\
\mu \alpha_{1} 
\end{bmatrix} \begin{bmatrix} \overline{\alpha_{1}} \\
\overline{\alpha_{1}} 
\end{bmatrix} \right) - \left( \begin{bmatrix} 0 \\
\mu \alpha_{1} 
\end{bmatrix} \begin{bmatrix} \overline{\alpha_{1}} \\
\overline{\alpha_{1}} 
\end{bmatrix} \right) - \left( \begin{bmatrix} 0 \\
\mu \alpha_{1} 
\end{bmatrix} \begin{bmatrix} \overline{\alpha_{1}} \\
\overline{\alpha_{1}} 
\end{bmatrix} \right) - \left( \begin{bmatrix} 0 \\
\mu \alpha_{1} 
\end{bmatrix} \begin{bmatrix} \overline{\alpha_{1}} \\
\overline{\alpha_{1}} 
\end{bmatrix} \right) - \left( \begin{bmatrix} 0 \\
\mu \alpha_{1} 
\end{bmatrix} \begin{bmatrix} \overline{\alpha_{1}} \\
\overline{\alpha_{1}} 
\end{bmatrix} \right) - \left( \begin{bmatrix} 0 \\
\mu \alpha_{1} 
\end{bmatrix} \begin{bmatrix} \overline{\alpha_{1}} \\
\overline{\alpha_{1}} 
\end{bmatrix} \right). \tag{4.45} \]

Thus, an expression is obtained for the local stiffness matrix corresponding to the operator \( L_{we} \) within the element number \( i_e \):
\[
K_{we}^{*} = K_{we}^{*} - K_{we}^{*}, \tag{4.46} \]
where we have
\[
K_{we}^{*} - P \begin{bmatrix} 0 \\
\lambda \alpha_{1} 
\end{bmatrix} = A_{01} - (T_{we}^{*})^{T} R T_{we}^{*}; \tag{4.47} \]
\[
K_{we}^{*} - P \begin{bmatrix} 0 \\
\mu \alpha_{1} 
\end{bmatrix} = A_{01} - (T_{we}^{*})^{T} S T_{we}^{*}; \tag{4.48} \]
\[
K_{we}^{*} - P \begin{bmatrix} 0 \\
\mu \alpha_{1} 
\end{bmatrix} = A_{01} - (T_{we}^{*})^{T} S T_{we}^{*}. \tag{4.49} \]

Let us note that
\[
K_{we}^{*} - \frac{1}{2} \left( \begin{bmatrix} \mu \\
\lambda 
\end{bmatrix} \begin{bmatrix} \bar{\alpha} \\
\gamma 
\end{bmatrix} = \frac{1}{2} \left( \begin{bmatrix} \mu \\
\lambda 
\end{bmatrix} \begin{bmatrix} \bar{\alpha} \\
\gamma 
\end{bmatrix} \right. \}
\end{align*}
\]

Let us further consider (4.32) in a similar way:
\[
\int_{a_{i}}^{b_{i}} \left( \begin{bmatrix} \mu \\
\lambda 
\end{bmatrix} \begin{bmatrix} \bar{\alpha} \\
\gamma 
\end{bmatrix} = \frac{1}{2} \left( \begin{bmatrix} \mu \\
\lambda 
\end{bmatrix} \begin{bmatrix} \bar{\alpha} \\
\gamma 
\end{bmatrix} \right. \}
\end{align*}
\]

where we have
\[
M_{we}^{*} - \frac{1}{2} \left( \begin{bmatrix} \mu \\
\lambda 
\end{bmatrix} \begin{bmatrix} \bar{\alpha} \\
\gamma 
\end{bmatrix} = \frac{1}{2} \left( \begin{bmatrix} \mu \\
\lambda 
\end{bmatrix} \begin{bmatrix} \bar{\alpha} \\
\gamma 
\end{bmatrix} \right. \}
\end{align*}
\]

We should note that, in particular \( m_{we} - m_{we} \), i.e.
if \( M - \{m_{we} \} \), we get
\[
M^{\prime} - M. \tag{4.51} \]

For further transformations, we use the representation
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\[
\begin{align*}
\left( \begin{array}{c} 
\mu \\
2\mu + \lambda
\end{array} \right) \left( \begin{array}{c}
\frac{\partial^2 \sigma}{\partial x^2} \\\n\frac{\partial^2 \sigma}{\partial y^2}
\end{array} \right) - \mu \alpha \left( \begin{array}{c}
\frac{\partial \beta}{\partial x} \\
\frac{\partial \beta}{\partial y}
\end{array} \right) + (2\mu + \lambda) \alpha \left( \begin{array}{c}
\frac{\partial \beta}{\partial x} \\
\frac{\partial \beta}{\partial y}
\end{array} \right) =
\end{align*}
\]  

\( (4.52) \)

We can substitute \((4.52)\) in \((4.51)\) and taking into account \((4.22)\), \((4.29)\) and the adopted notation we get:

\[
\sum_{i=1}^{N_x} \sum_{j=1}^{N_y} m_{ij} \left( \begin{array}{c}
\frac{\partial \alpha}{\partial x} \\
\frac{\partial \alpha}{\partial y}
\end{array} \right) -
\sum_{i=1}^{N_x} \sum_{j=1}^{N_y} m_{ij} \left( \begin{array}{c}
\frac{\partial \beta}{\partial x} \\
\frac{\partial \beta}{\partial y}
\end{array} \right) -
\mu \left( \begin{array}{c}
\frac{\partial \delta}{\partial x} \\
\frac{\partial \delta}{\partial y}
\end{array} \right) - \mu \left( \begin{array}{c}
\frac{\partial \epsilon}{\partial x} \\
\frac{\partial \epsilon}{\partial y}
\end{array} \right) + (2\mu + \lambda) \left( \begin{array}{c}
\frac{\partial \epsilon}{\partial x} \\
\frac{\partial \epsilon}{\partial y}
\end{array} \right) -
\mu \left( \begin{array}{c}
\frac{\partial \gamma}{\partial x} \\
\frac{\partial \gamma}{\partial y}
\end{array} \right) + (2\mu + \lambda) \left( \begin{array}{c}
\frac{\partial \gamma}{\partial x} \\
\frac{\partial \gamma}{\partial y}
\end{array} \right) -
\rho \left( \begin{array}{c}
A_{\alpha\alpha} \delta \\
A_{\beta\beta}
\end{array} \right) + (2\mu + \lambda) \left( \begin{array}{c}
A_{\alpha\alpha} \delta \\
A_{\beta\beta}
\end{array} \right) -
\left( \begin{array}{c}
\mu A_{\alpha\alpha} \\
\mu A_{\beta\beta}
\end{array} \right) \frac{p}{(2\mu + \lambda) A_{\alpha\alpha}} -
\left( \begin{array}{c}
\mu A_{\alpha\alpha} \\
\mu A_{\beta\beta}
\end{array} \right) \frac{p}{(2\mu + \lambda) A_{\alpha\alpha}}
\end{align*}
\]  

\( (4.53) \)

Thus, an expression is obtained for the local stiffness matrix corresponding to the operator \(L_{\alpha}\) within the element number \(i\):

\[
K_{\alpha\alpha} = \frac{\mu A_{\alpha\alpha}}{(2\mu + \lambda) A_{\alpha\alpha}} \frac{p}{(2\mu + \lambda) A_{\alpha\alpha}},
\]

\( (4.54) \)

where we have

\[
A_{\alpha\alpha} = (C_{\alpha\alpha})^T M T_{\alpha\alpha}.
\]  

\( (4.55) \)

5. SEVERAL ASPECTS OF NUMERICAL IMPLEMENTATION

The presented algorithm can be implemented using MATLAB tool. The MATLAB system has convenient functions for working with polynomials. Moreover, the main parameter of these functions is the vector of coefficients of the polynomial. To determine the coefficients of basic polynomials \(\psi_k\) on an interval \([0, 1]\), we can firstly determine their values at \(N_p\) points of the interval \(i = [i_1, i_2, ..., i_{N_p}]\), \(i \in [0, 1]\), \(i = 1, 2, ..., N_p\):

\[
F_k(i) - \psi_k(i), \quad i = 1, 2, ..., N_p, \quad k = 1, 2, ..., N_p.
\]

Then, using the \texttt{polyfit} function, we define their coefficient vector \(p_k\):

\[
p_k \texttt{polyfit} \{k, F_k, N_p\}
\]

This function is used to determine the coefficients of the optimal polynomial using the least squares method. In the considered case, we construct polynomial of the \((N_p + 1)\) th degree \((i.e. we \ have \ to \ define \ N_p \ coefficients \ of \ polynomial, \ according \ to \ its \ N_p \ values)\), therefore, we get a polynomial passing through the given values.

In order to calculate the derivatives we can sequentially use the \texttt{polyder} function:

\[
dp_k \texttt{polyder} \{p_k\}
\]

is the vector of coefficients \(\psi'_k\).

In order to calculate the product of polynomials we can use the \texttt{conv} function:

\[
p_{ij} \texttt{conv} \{p_i, p_j\}
\]

is the vector of coefficients \(\psi_i \psi_j\);

\[
dl_{dp_{ij}} \texttt{conv} \{dp_i, dp_j\}
\]

is the vector of coefficients \(\psi_i' \psi_j\);

\[
dl_{dp_{ij}} \texttt{conv} \{dp_i, dp_j\}
\]

is the vector of coefficients \(\psi_i' \psi_j\);

\[
dp_{ij} \texttt{conv} \{dp_i, dp_j\}
\]

is the vector of coefficients \(\psi_i' \psi_j\).

In order to calculate the antiderivative of a polynomial we can use the \texttt{polyint} function:

\[
F \texttt{polyint} \{p\}
\]

is the vector of coefficients \(\int \psi_i \, dx\).
Marina L. Mozgaleva, Pavel A. Akimov, Taymuraz B. Kaytukov

5. EXAMPLE OF ANALYSIS

5.1. Formulation of the problem.
As a model example, let us consider the determination of the displacements of a beam wall, fixed along the side faces in both directions, under the influence of a load concentrated in the center (Figure 5.1).

Let us consider the following geometric parameters: $l = 6$ m, $l_2 = 12$ m.

Let us consider the following design parameters of material of plate: coefficient of elasticity

\[ E = 26500 \times 10^6 \text{ KPa}, \]

Poisson's ratio $\nu = 0.15$.

Let external load parameter be equal to $P = 100$ kN.

5.2. Structural analysis with allowance for localization.

Let the number of elements be equal to $N_e - 6$.

Then we have the following element length:

\[ h_e = \frac{P}{N_e} - \frac{8}{4} - 1.5. \]

Let's define localization in the load area.

For the first element we have $N_e = 5$ and fifth order spline; distance between the coordinates of the nodes of the first element is equal to $h_1 = 15/5 = 3$. 

\[ \text{Figure 5.1. Formulation of the problem (Sample).} \]

\[ \text{Pij polyint} \{ p_{ij} \} \]

is the vector of coefficients $\int \phi_i \phi_j ds$;
\[ \text{d10Pij polyint} \{ d10p_{ij} \} \]

is the vector of coefficients $\int \phi_i' \phi_j ds$;
\[ \text{d01Pij polyint} \{ d01p_{ij} \} \]

is the vector of coefficients $\int \phi_i \phi_j' ds$;
\[ \text{dPij polyint} \{ dp_{ij} \} \]

is the vector of coefficients $\int \phi_i' \phi_j ds$;

Then the calculation of

\[ E(i,j), R(i,j), S(i,j), M(i,j) \]

can be done in accordance with formulas:

\[ M(i,j) - h_e \{ \text{polyval} \{ \text{Pij}, 1 \} - \text{polyval} \{ \text{Pij}, 0 \} \}; \]
\[ R(i,j) - \text{polyval} \{ \text{d10Pij}, 1 \} - \text{polyval} \{ \text{d10Pij}, 0 \}; \]
\[ S(i,j) - \text{polyval} \{ \text{d01Pij}, 1 \} - \text{polyval} \{ \text{d01Pij}, 0 \}; \]
\[ E(i,j) - \{ \text{polyval} \{ \text{dpij}, 1 \} - \text{polyval} \{ \text{dpij}, 0 \} \} / h_e, \]

where the function polyval \{ p, t \} allows researchers to calculate the values of a polynomial with a vector of coefficients $p$ at a given point $t$. 

\[ \phi_i \]

\[ \theta_i \]

\[ \psi_i \]

\[ \phi_j \]

\[ \theta_j \]

\[ \psi_j \]
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For the second element we have $N_3 = 3$ and third order spline; distance between the coordinates of the nodes of the second element is equal to $h_2 = 1.5/3 = 0.5$.

For the third element and for the fourth element we have $N_4 = 1$ and first order spline; distance between the coordinates of the nodes of the third element and of the fourth element is equal to $h_3 = h_4 = 1.5/5 = 0.3$.

With such approximation the total number of nodes for all elements is equal to

$$N_s = 5 + 3 + 2 \cdot 1 + 1 = 11.$$  

Then the total number of unknown nodal values for vectors $\mathbf{u}$ and $\mathbf{u} - \mathbf{u}'$ is equal to

$$N_k = 4 \cdot N_s = 4 \cdot 11 = 44.$$  

5.3. Structural analysis without localization.

In this case, we will consider only the standard linear fulfilment. In this case, the length of the element is taken equal to the minimum distance between the nodes, i.e. $h_s = 0.3$. Then the number of elements is equal to

$$N_s = 6 / 0.3 = 20$$

and the total number of nodes is equal to $N_s = 21$. In this case, the number of nodal unknowns for each component of the vectors $\mathbf{u}$ and $\mathbf{u} - \mathbf{u}'$ is equal to

$$N_k = 4 \cdot N_s = 4 \cdot 21 = 84.$$  

Graphical comparison of corresponding results of analysis is presented at Figure 5.2 and Figure 5.3 (FEI: local-B-spline are nodal values computed with allowance for localization; FEII: standard are nodal values computed without localization).

As researcher can see, the results obtained are almost completely identical. Besides, the use of localization based on application of B-splines of various degrees leads to a significant decrease in the number of unknowns. The difference for this example is equal to $\Delta = 84 - 44 = 40$. 

Figures 5.2. Comparison of the results of analysis in the middle sections along $x_1$ direction (discrete direction)
REFERENCES


9. Mozgaleva M.L., Akimov P.A., Kaytukov T.B. Wavelet-based discrete-continual finite element plate analysis with the use of


СПИСОК ЛИТЕРАТУРЫ

Localization of Solution of the Problem of Two-Dimensional Theory of Elasticity with the Use of B-Spline Discrete-Continual Finite Element Method


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