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ASYMPTOTICS OF THE FILTRATION PROBLEM WITH ALMOST CONSTANT COEFFICIENTS

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Abstract: During the construction of hydraulic and underground structures, a grout solution is pumped into the ground to create waterproof partitions. The liquid grout is filtered in the porous rock and clogs the pores when hardened. The mathematical model of deep bed filtration describes the transfer of suspension particles and colloids by a fluid flow through the pores of a rock. For a one-dimensional filtration problem in a homogeneous porous medium with almost constant coefficients, an asymptotic solution is constructed. The asymptotics is compared with the numerical solution.

Keywords: deep bed filtration, suspensions and colloids, porous medium, suspended and retained particles, asymptotic solution.

АСИМПТОТИКА ЗАДАЧИ ФИЛЬТРАЦИИ С ПОЧТИ ПОСТОЯННЫМИ КОЭФФИЦИЕНТАМИ

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Аннотация: При строительстве гидротехнических и подземных сооружений для создания водонепроницаемых перегородок в грунт закачивается раствор укрепителя. Жидкий укрепитель фильтруется в пористой породе и при застывании закупоривает поры. Математическая модель фильтрации описывает перенос жидкостью частиц суспензий и коллоидов через поры горной породы. Для одномерной задачи фильтрации в однородной пористой среде с мало меняющимися коэффициентами построено асимптотическое решение. Асимптотика сравнивается с численным решением.

Ключевые слова: фильтрация, суспензии и коллоиды, пористая среда, взвешенные и осажденные частицы, асимптотическое решение.

1. INTRODUCTION

Filtration of suspensions and colloids in porous media occurs in many natural and technological processes: the spread of microorganisms in the aquatic environment, filtration of water in rocks, treatment of industrial and domestic wastewater, a decrease in oil production due to the deposition of small particles entrained in water near the well, and much more [1–3]. During the construction of tunnels and underground storage facilities for hazardous toxic and radioactive waste, a liquid grout is pumped into the rock under pressure to create watertight walls. The grout filters in the porous soil and clogs the pores after solidification [4].

The transport of micro- and nanoparticles in a porous medium is accompanied by the retention of particles and the formation of a deposit. Various retention mechanisms of particles carried by a fluid flow in a porous medium of complex structure are determined by electric, gravitational and hydrodynamic forces [5–7]. Filtration models take into account either a single prevailing retention mechanism, or several mechanisms acting simultaneously [8, 9].

The mathematical model of deep bed filtration includes the equation for the balance of the masses of suspended and retained particles and the kinetic equation of deposit growth, which form a quasilinear hyperbolic system of the first order partial differential equations [10]. To solve filtration problems, both numerical and analytical methods are used [11–16]. Analytical methods allow to obtain exact and asymptotic solutions and their dependence on parameters. This makes it possible to fine-tune experiments and to solve inverse filtration problems [17–19].

The classical filtration model assumes that the properties of the porous medium do not change with the formation of deposit. More sophisticated models take into account the dependence of porosity and permissible flow on the concentration of deposit [20]. In these models, it is assumed that a suspension or colloidal solution of constant volume concentration is injected at the inlet of a porous medium.

We consider a one-dimensional model for deep bed filtration of particles carried by a fluid flow in a homogeneous porous medium. It is assumed that the carrier fluid is incompressible; at the porous medium inlet the suspended particles concentration is variable. Experiments show that the coefficients of the filtration equations depending on the retained concentration do not change much. This allows us to construct an asymptotic solution to the filtration problem. The asymptotics is compared with the numerical solution.

2. MATHEMATICAL MODEL

In the domain $\Omega = \{0 \le x \le 1, t \ge 0\}$, consider the system of first-order differential equations

$$\frac{\partial}{\partial t}\big(g(S)C\big)+\frac{\partial}{\partial x}\big(f(S)C\big)+\frac{\partial S}{\partial t}=0\,, \qquad (1)$$

$$\frac{\partial S}{\partial t}=\Lambda(S)C\,. \qquad (2)$$

Here the blocking filtration function $\Lambda(S)$ is smooth and positive at $0 \le S < S_m$, $S_m > 0$; $\Lambda(S) = 0$ at $S \ge S_m$; the functions $g(S)$ and $f(S)$ are positive at $0 \le S \le S_m$; $C(x,t)$, $S(x,t)$ – the unknown volumetric concentrations of suspended and retained particles [21].

For the uniqueness of the solution to the system (1), (2), the initial and boundary conditions are set

$$C\big|_{x=0}=p(t),\;\; p(t)>0\,, \qquad (3)$$

$$C\big|_{t=0}=0,\quad S\big|_{t=0}=0\,. \qquad (4)$$

Condition (3) means that a suspension of variable concentration is injected at the inlet of the porous medium; by condition (4), at the initial moment of time, the porous medium does not contain any suspended and retained particles. The concentrations front of the suspended and retained particles given by the formula $x = vt$, $v = f(0)/g(0)$ moves with a speed v from the inlet to the outlet of the porous medium. Ahead of the front in the domain $\Omega_0 = \{0 \le x \le 1, t < x\}$, the solution is zero; behind the front in the domain $\Omega_1 = \{0 \le x \le 1, t > x\}$, the decision is positive. Since conditions (3) and (4) do not matched at the origin, the solution C is discontinuous at the concentration front; the solution S is continuous in the whole domain Ω.

Consider the condition on the concentrations front

$$S\big|_{x=vt}=0\,. \qquad (5)$$

In the domain $\overline{\Omega}_1$, the solution to problem (1)–(4) coincides with the solution to the Goursat problem (1)–(3), (5).

In characteristic variables $\tau = t - x/v$, $y = x$, the Goursat problem takes the form

$$\begin{aligned}&\frac{\partial}{\partial \tau}\big(g(S)C\big)-\frac{1}{v}\frac{\partial}{\partial \tau}\big(f(S)C\big)+\\&+\frac{\partial}{\partial y}\big(f(S)C\big)+\frac{\partial S}{\partial \tau}=0\end{aligned}\,, \qquad (6)$$

$$\frac{\partial S}{\partial \tau}=\Lambda(S)C\,. \qquad (7)$$

Conditions (3) and (5) take the form

$$C\big|_{y=0} = p(\tau)\,, \tag{8}$$

$$S\big|_{\tau=0} = 0\,. \tag{9}$$

3. ASYMPTOTICS FOR ALMOST CONSTANT COEFFICIENTS

Assume that the coefficients of equations (1), (2) admit expansions

$$\begin{aligned} g(S) &= g_0 + \varepsilon g_1 S + \ldots, \\ f(S) &= f_0 + \varepsilon f_1 S + \ldots, \\ \Lambda(S) &= \Lambda_0 + \varepsilon \Lambda_1 S + \ldots\,. \end{aligned} \tag{10}$$

Here ε is a small positive parameter.
The solution to the system (6), (7) is obtained in the form [22, 23]

$$\begin{aligned} S(y,\tau) &= s_0(y,\tau) + \varepsilon s_1(y,\tau) + \ldots, \\ C(y,\tau) &= c_0(y,\tau) + \varepsilon c_1(y,\tau) + \ldots\,. \end{aligned} \tag{11}$$

Substitute the expansions (10), (11) into the equations (6), (7) and equate the terms at the same powers of ε. We obtain a recurrent system of differential equations

$$f_0 \frac{\partial c_0}{\partial y} + \Lambda_0 c_0 = 0\,, \tag{12}$$

$$\frac{\partial s_0}{\partial \tau} = \Lambda_0 c_0\,, \tag{13}$$

$$\begin{aligned} & f_0 \frac{\partial c_1}{\partial y} + \Lambda_0 c_1 + f_1 \frac{\partial s_0}{\partial y} c_0 + \\ & + f_0 \frac{\partial c_1}{\partial y} + \Lambda_1 s_0 c_0 = 0, \end{aligned} \tag{14}$$

$$\frac{\partial s_0}{\partial \tau} = \Lambda_0 c_1 + \Lambda_1 s_0 c_0\,. \tag{15}$$

Conditions for the equations (12)–(15) follow from (8) and (9)

$$c_0\big|_{y=0} = p(\tau),\; c_1\big|_{y=0} = 0\,. \tag{16}$$

$$s_0\big|_{\tau=0} = 0,\; s_1\big|_{\tau=0} = 0\,. \tag{17}$$

Solution to the system (12)–(15) with the conditions (16), (17)

$$c_0 = p(\tau) e^{-Ay},\;\; s_0 = \Lambda_0 P(\tau) e^{-Ay}, \tag{18}$$

$$c_1 = p(\tau)\big(B + DP(\tau)\big) e^{-Ay} \left(e^{-Ay} - 1\right), \tag{19}$$

$$\begin{aligned} s_1 = {} & \Lambda_0 \left(Bp_1(\tau) + 0.5DP^2(\tau)\right) e^{-Ay} \cdot \\ & \cdot \left(e^{-Ay} - 1\right) + 0.5\Lambda_0 \Lambda_1 P^2(\tau) e^{-2Ay}. \end{aligned} \tag{20}$$

Here

$$A = \frac{\Lambda_0}{f_0},\;\; P(\tau) = \int_0^\tau p(z)dz,$$
$$B = g_1 - f_1 / v,\;\; D = \Lambda_1 - \frac{f_1}{f_0}\Lambda_0.$$

Substituting the solutions (18)–(20) into the expansions (11) and passing to the Cartesian coordinates, we obtain an asymptotic solution to the problem (1)–(4) in the domain $\overline{\Omega}_1$

$$\begin{aligned} S(x,t) &= \Lambda_0 P(t - x/v) e^{-Ax} \left(1 + \varepsilon\left(\left(B + 0.5P(t - x/v)\right)\left(D(e^{-Ax} - 1) + \Lambda_1 e^{-Ax}\right)\right)\right), \\ C(x,t) &= p(t - x/v) e^{-Ax} \left(1 + \varepsilon\left(B + DP(t - x/v)\right)\left(e^{-Ax} - 1\right)\right). \end{aligned} \tag{21}$$

4. RESULTS OF NUMERICAL MODELLING

The numerical calculation was carried out for the coefficients of equations (1), (2) obtained from the results of experiments with particles of a suspension with a radius of 2.179 microns in the laboratory of the University of Adelaide, Australia [24]

$$g(S) = 0.9743 - 8.8818 \cdot 10^{-14} S,$$
$$f(S) = 0.9947 + 6.2733 \cdot 10^{-5} S,$$
$$\Lambda(S) = 0.5106 - 0.0060 S.$$

The calculation was made for a linearly increasing suspended concentration at the porous medium inlet $p(t) = 1 + 0.01t$. Figures 1–4 show the asymptotics at $\varepsilon = 0.01$ (yellow line) and the numerical solution (blue line).

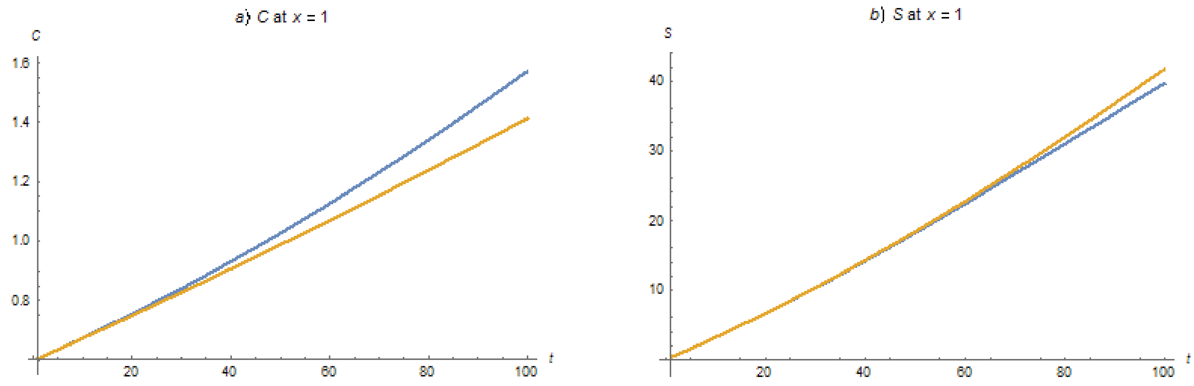

Figure 1. Concentrations at the porous medium outlet x=1 a) suspended C(1,t); b) retained S(1,t).

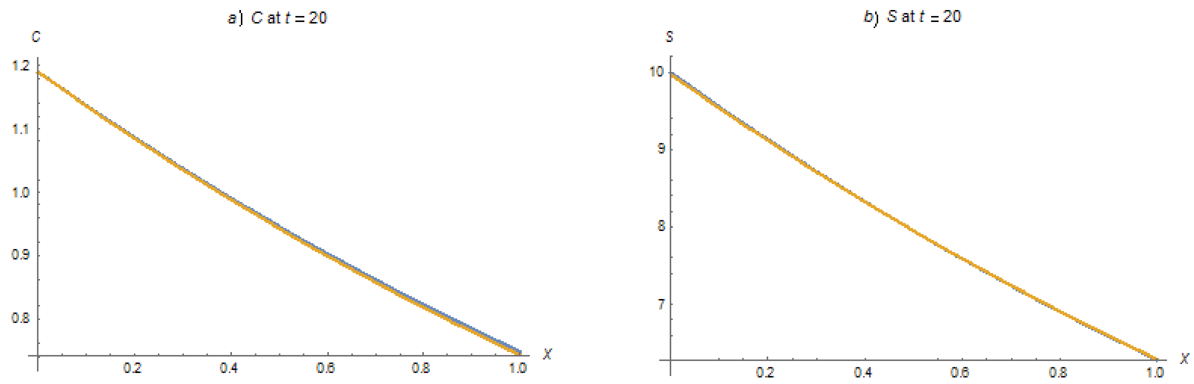

Figure 2. Concentrations at fixed time t=20 a) suspended C(x,20); b) retained S(x,20).

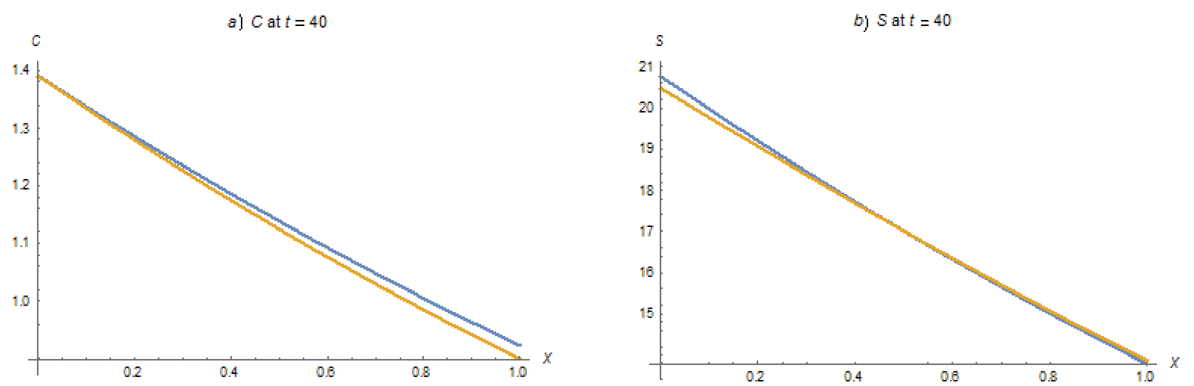

Figure 3. Concentrations at fixed time t=40 a) suspended C(x,40); b) retained S(x,40).

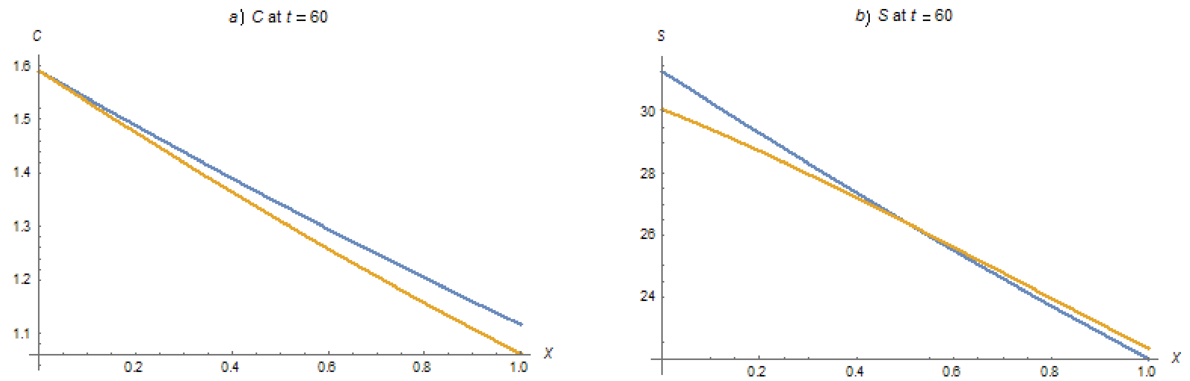

Figure 4. Concentrations at fixed time t=60 a) suspended C(x,60); b) retained S(x,60).

At the porous medium outlet $x = 1$, the relative error of the asymptotics increases with time. For the suspended and retained particles concentrations, the error reaches 2.5% and 0.1% at $t = 40$, 1% and 5% at $t = 60$, 5% and 10% at $t = 100$, respectively. For a fixed time, the relative error of the asymptotics throughout the whole porous medium does not exceed 1% at $t = 20$, 2% at $t = 40$, and 4% at $t = 60$ for both types of particles concentrations.

5. CONCLUSIONS

The study of the mathematical model of deep bed filtration of suspensions and colloids in a porous medium allows us to draw the following conclusions.

– An asymptotic solution to the filtration problem is constructed.

– The main term of the asymptotics coincides with the exact solution of the problem with constant coefficients.

– The asymptotics is close to the numerical solution.

– The asymptotic solution depending on the model parameters can be used to fine-tune laboratory experiments and to solve the inverse filtration problem.

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