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ANALYSIS OF COMBINED DISKS WITH PIECEWISE THICKNESS

Elena B. Koreneva

Moscow Higher Combined-Arms Command Academy, Moscow, RUSSIA

Abstract: The combined constructions subjected to an action of expanding loads and consisting of separate sections are examined. Each of the mentioned sections has its own rigidity. These parts may be made from the same or from the various materials. The materials can be anisotropic or isotropic, homogeneous or inhomogeneous. The constructions under study have the round scheme and they are considered as circular disks with piecewise thickness. In the places of the separate parts conjugation the disks' thickness can be discontinuous or continuous. The analytical approach is used. The solutions are obtained in closed form and expressed in terms of Legendre functions, Legendre, Gegenbauer and Laguerre polynomials.

Keywords: combined disks, piecewise thickness, special functions.

РАСЧЕТ КОМБИНИРОВАННЫХ ДИСКОВ КУСОЧНО-ПЕРЕМЕННОЙ ТОЛЩИНЫ

Е.Б. Коренева

Московское высшее общевойсковое командное орденов Жукова, Ленина и Октябрьской Революции Краснознаменное училище, г. Москва, РОССИЯ

Аннотация: Изучаются комбинированные конструкции, работающие преимущественно на растяжение и состоящие из отдельных участков, каждый из которых обладает своим законом изменения жесткости. Эти участки могут быть сделаны из одного и того же или из различных материалов. Эти материалы могут быть анизотропными и изотропными, однородными и неоднородными. В рассматриваемых конструкциях в местах соединения отдельных частей толщина может быть непрерывной или иметь разрывы непрерывности. Изучаемые конструкции имеют в плане круговую форму и рассматриваются как круглые диски кусочно-переменной толщины. В данной работе для расчета подобных конструкций впервые используется аналитическая методика. Решения получены в замкнутом виде и выражены в функциях Лежандра, в полиномах Лежандра, Гегенбауэра, Лагерра.

Ключевые слова: комбинированные диски, кусочно-переменная толщина, специальные функции.

1. INTRODUCTION

In literature the considerable number of works are devoted for computation of plates and shells of various forms. For example the monographies [1] and [2] are to be mentioned. In the monography [3] the orthotropic and isotropic plates of variable thickness, subjected to the action of complicated loads are examined; the analytical methods were applied.

The modern software allows to investigate the similar constructions in detail. The numerical methods, in particular, the finite elements method, are widely used. The work [4] concerns the problem of buckling of orthotropic plates with free and rotationally restrained edges. 3D vibration of cross-ply laminated plates is studied in [5]. The oscillation problems of isotropic and orthotropic rectangular plates of linear thickness are considered in [6]. The work [7] is devoted to the numerical analysis of experimental research on buckling of closed shallow conical shells under external pressure. Free vibration analysis of a rotating varying-thickness-twisted blade with arbitrary boundary conditions is examined in [8]. The article [9] considers the optimization of three-dimensional up to yield bending behaviour using the full layer-wise theory for FGM rectangular plate subjected to thermo-mechanical loads. Comparative assessment of finite element modelling techniques for wind turbine rotors blades is represented in [10]. The work [11] concerns nonlinear primary resonance analysis of nanoshells. Geometrical influence on the vibration of layered plates is discussed in [12].

Some problems of statics, vibration and stability of thin-walled constructions are solved in [13] by the use of the equation decomposition method.

The elements with piecewise variable thickness occur in modern structures and buildings. First the analytical approach for the solution of similar problems was proposed in the works [14], [15]. In the mentioned works the circular plates resting on an elastic basis are examined. The inner part of these plates has the variable thickness and the outer part has the constant thickness. The conditions of the parts conjugation were fulfilled. The solutions were obtained in terms of Bessel functions. The problems of symmetric flexure of orthotropic and isotropic combined plates with piecewise thickness were considered in [16], [17]. The separate parts of these plates have various laws of cylindrical rigidity variation. In [16], [17] the solutions were obtained in terms of Gegenbauer and Laguerre polynomials.

In the present work the analytical method for the first time is applied for the analysis of the combined circular disks with piecewise variable thickness subjected to an action of expanding loads. The solutions of the problems under study are obtained in closed forms and expressed in terms of Legendre functions; Legendre, Gegenbauer and Laguerre polynomials.

2. THE BASIC SOLUTIONS EXPRESSED IN TERMS OF LEGENDRE FUNCTIONS

As it was mentioned above, the circular disks with piecewise variable thickness subjected to an action of expanding loads are analyzed.

We will write the differential equation, describing the symmetric deformation of the circular isotropic disks with the radially variable thickness and loaded by the surface stretching radial forces with the intensity q:

$$\frac{d^2 N_r}{dx^2} + \left(\frac{1+\frac{2}{\alpha_0}}{x} - \frac{1}{D}\frac{dD}{dx} \right)\frac{dN_r}{dx} - \frac{1-\sigma}{\alpha_0 x}\frac{1}{D}\frac{dD}{dx}N_r + \frac{r_0 x^{\frac{1}{\alpha_0}-2} q_r}{\alpha_0^2} \times \times \left(2+\sigma-\alpha_0\frac{x}{D}\frac{dD}{dx} \right) = 0, \quad (1)$$

where $D = \frac{Eh(x)}{1-\sigma^2}$ is the cylindrical rigidity for the tension; N_r - the normal stress; σ - Poisson's ratio; $x = \left(\frac{r}{r_0}\right)^{\alpha_0}$; r_0, α_0 - the parameters.

Further we will determine the laws of cylindrical rigidity variation which allow to receive the solutions in terms of Legendre functions. For this aim we compare the coefficients of the homogeneous differential equation, corresponding to (1), with the coefficients of the Legendre differential equation [3], [18], [19]:

$$\frac{d^2u}{dx^2} - \frac{2x}{1-x^2}\frac{du}{dx}\left[\frac{v(v-1)}{1-x^2} - \frac{\mu^2}{\left(1-x^2\right)^2} \right]u = 0, \quad (2)$$

where μ and v are the parameters of the Legendre functions. Producing the comparision, we have

$$\frac{1+\frac{2}{\alpha_0}}{x} - \frac{1}{D}\frac{dD}{dx} = -\frac{2x}{1-x^2},$$

$$-\frac{1-\sigma}{\alpha_0 x}\frac{1}{D}\frac{dD}{dx} = \frac{v(v+1)}{1-x^2} - \frac{\mu}{\left(1-x^2\right)^2}.$$

We get that for the values of the parameters

$$\alpha_0 = -2\,,\ \mu = 0\,,\ v_{1,2} = \pm\sqrt{\frac{1}{4} + (1-\sigma)} \qquad (3)$$

and for the following law for the rigidity for the tension

$$D = D_0\left(1 - x^2\right)^{-1}, \qquad (4)$$

which corresponds to the thickness

$$\begin{aligned} h &= h_0\left(1 - x^2\right)^{-1}, 0 \le x < 1; \\ h &= h_0\left(x^2 - 1\right)^{-1}, 1 \le x < \infty; \end{aligned} \qquad (5)$$

the solution of the homogeneous differential equation, corresponding to (1), has the form:

$$N_r = AP_v(x) + BQ_v(x)\,. \qquad (6)$$

Then we compare the coefficients of the corresponding to (1) differential equation with the coefficients of another Legendre equation:

$$\frac{d^2v}{dx^2} - \frac{2(\mu+1)x}{1-x^2}\frac{dv}{dx} + \frac{(v-\mu)|v+\mu+1|}{1-x^2}v = 0\,, \qquad (7)$$

which comes out of (2) by means of the following substitution:

$$u = \left(z^2 - 1\right)^{\frac{\mu}{2}} v\,.$$

As a result we get the following expression for the parameters:

$$\alpha_0 = -2\,,\ v_{1,2} = \frac{1}{2} \pm \sqrt{\frac{1}{4} + (1-\sigma-\mu)(\mu+1)}\,, \qquad (8)$$

where μ is the arbitrary value.
We get the rigidity for the tension

$$D = D_0\left(1 - x^2\right)^{-(\mu+1)}, \qquad (9)$$

as a result the thickness of disks is expressed in the form:

$$\begin{aligned} h &= h_0\left(1 - x^2\right)^{-(\mu+1)}, 0 \le x < 1; \\ h &= h_0\left(x^2 - 1\right)^{-(\mu+1)}, 1 \le x < \infty. \end{aligned} \qquad (10)$$

In this case the solutions are represented in terms of adjoined Legendre functions:

$$N_r = \left[AP_v^{\mu}(x) + BQ_v^{\mu}(x)\right]\left(x^2 - 1\right)^{-\frac{\mu}{2}}. \qquad (11)$$

We mark that the use of another Legendre equations for the consideration of disks of variable thickness symmetric deformation will not give any results.
Some disks' profiles for the cases when the solutions are given in terms of the adjoined Legendre functions are shown on the fig.1.

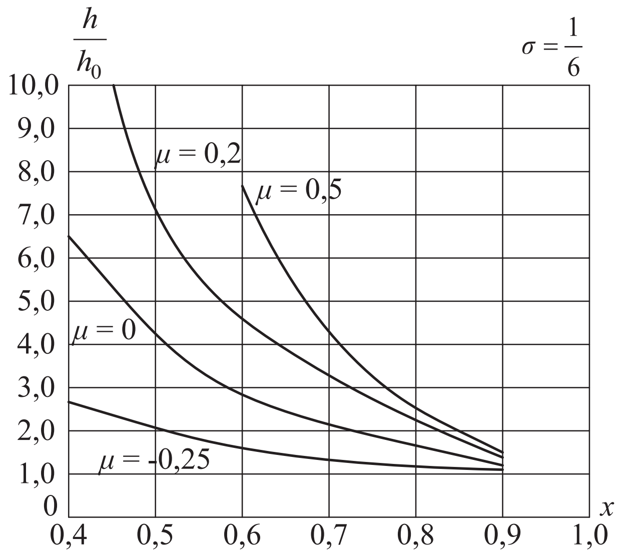

Figure 1. Some disks' profiles for the cases when the solutions are obtained in terms of the adjoined Legendre functions

We note the special case when the solutions are given in terms of the cone functions. We set $v = -\frac{1}{2} + ip$. Then the solution can be written in the following form:

$$N_r = AP^{\mu}_{-\frac{1}{2}+ip} + BQ^{\mu}_{-\frac{1}{2}+ip}\, x\,. \qquad (12)$$

The case of specific interest is when one of the adjoined Legendre functions parameters $\mu = \pm\left(\frac{1}{2}+n\right)$, when n is the positive integer; then the solutions of the equation (1) for the rigidities (4) and (9) are expressed in terms of Legendre polynomials. For example, the adjoined Legendre functions $P_\nu^{\frac{1}{2}}(x)$, $P_\nu^{-\frac{1}{2}}(x)$, $Q_\nu^{\frac{1}{2}}(x)$, $Q_\nu^{-\frac{1}{2}}(x)$ can be represented by the following formulae:

$$P_\nu^{\frac{1}{2}}(x) = \frac{1}{\sqrt{2\pi}}\left(x^2-1\right)^{-\frac{1}{4}} \times$$
$$\times\left[\left(x+\sqrt{x^2-1}\right)^{\nu+\frac{1}{2}} + \left(x+\sqrt{x^2-1}\right)^{-\nu-\frac{1}{2}}\right];$$
$$Q_\nu^{\frac{1}{2}}(x) = i\sqrt{\frac{\pi}{2}}\left(x^2-1\right)^{-\frac{1}{4}}\left(x+\sqrt{x^2-1}\right)^{-\nu-\frac{1}{2}};$$
$$P_\nu^{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi}}\,\frac{\left(x^2-1\right)^{-\frac{1}{4}}}{2\nu+1} \times \qquad (13)$$
$$\times\left[\left(x+\sqrt{x^2-1}\right)^{\nu+\frac{1}{2}} - \left(x+\sqrt{x^2-1}\right)^{-\nu-\frac{1}{2}}\right];$$
$$Q_\nu^{-\frac{1}{2}}(x) = i\sqrt{2\pi}\,\frac{\left(x^2-1\right)^{-\frac{1}{4}}}{2\nu+1}\left(x+\sqrt{x^2-1}\right)^{-\nu-\frac{1}{2}},$$

where $x \geq 1$.
Using the recurrence relations

$$\left(\nu-\mu+1\right)P_{\nu+1}^{\mu}(x) - \left(\nu+\mu+1\right)xP_\nu^{\mu}(x) =$$
$$= \sqrt{x^2-1}P_\nu^{\mu}(x);$$
$$P_{\nu-1}^{\mu}(x) - xP_\nu^{\mu}(x) = -\left(\nu-\mu+1\right)\sqrt{x^2-1}P_\nu^{\mu-1}(x);$$
$$xP_\nu^{\mu}(x) - P_{\nu+1}^{\mu}(x) = -\left(\nu+\mu\right)\sqrt{x^2-1}P_\nu^{\mu-1}(x),$$

from these formulae the expressions for the functions $P_\nu^{\frac{3}{2}}(x)$, $P_\nu^{\frac{5}{2}}(x)$ etc. can be consequently received. For instance, we can write:

$$Q_\nu^{1,5}(x) = \frac{i\sqrt{\pi}}{\sqrt{2}}\left(x^2-1\right)^{-\frac{3}{4}}\left(x+\sqrt{x^2-1}\right)^{-\nu-\frac{1}{2}} \times$$
$$\times\left[x+\left(\nu+\frac{1}{2}\right)\sqrt{x^2-1}\right];$$
$$P_\nu^{1,5}(x) = \frac{1}{\sqrt{2}}\left(x^2-1\right)^{-\frac{3}{4}}\left(x+\sqrt{x^2-1}\right)^{-\nu-\frac{1}{2}} \times$$
$$\times\left[\left(\nu-\frac{1}{2}\right)x\left(x+\sqrt{x^2-1}\right)-\left(\nu+\frac{1}{2}\right)-x-\right.$$
$$\left.-\left(\nu+\frac{1}{2}\right)\sqrt{x^2-1}\right].$$

Next the particular solution of inhomogeneous equation (1) is to be considered. For this purpose the Cauchy functions for the solution received above are to be obtained.
The Wronskian for the solutions of the Legendre equation is used for this aim:

$$W(x) = \frac{K(\mu,\nu)}{1-x^2};$$
$$K(\mu,\nu) = \qquad (14)$$
$$= \frac{e^{i\mu\pi}2^{2\mu}\Gamma\left(1+\frac{\mu}{2}+\frac{\nu}{2}\right)\Gamma\left(\frac{1}{2}+\frac{\mu}{2}+\frac{\nu}{2}\right)}{\Gamma\left(1+\frac{\nu}{2}-\frac{\mu}{2}\right)\Gamma\left(\frac{1}{2}+\frac{\nu}{2}-\frac{\mu}{2}\right)},$$

where $\Gamma(x)$ - gamma-function.
The Cauchy functions for the solution (6) and the rigidity (4) are:

$$Y_1(x_1;x) = \nu\{-\left[x_1Q_\nu(x_1) - Q_{\nu-1}(x_1)\right]P_\nu(x) +$$
$$+\left[x_1P_\nu(x_1) - P_{\nu-1}(x_1)\right]Q_\nu(x)\};$$
$$Y_2(x_1;x) = \left(x_1^2-1\right)\left[Q_\nu(x_1)P_\nu(x) - \right. \qquad (15)$$
$$\left.-P_\nu(x_1)Q_\nu(x)\right],$$

in this case $\mu = 0$ and, hence, $K(\mu,\nu) = 1$.

The Cauchy functions for the solution (11) and the disk's rigidity (9) are:

$$Y_1(x_1;x)=\frac{\left(x_1^2-1\right)^{-\frac{\mu}{2}}\left(x^2-1\right)^{-\frac{\mu}{2}}}{K(\mu,\nu)}\times$$
$$\times\left\{-\left[(\nu-\mu)x_1Q_\nu^\mu(x_1)-(\nu+\mu)Q_{\nu-1}^\mu(x_1)\right]P_\nu^\mu(x)+\right.$$
$$\left.+\left[(\nu-\mu)x_1P_\nu^\mu(x_1)-(\nu+\mu)P_{\nu-1}^\mu(x_1)\right]Q_\nu^\mu(x)\right\};\quad(16)$$
$$Y_2(x_1;x)=\frac{\left(x_1^2-1\right)^{-\frac{\mu}{2}+1}\left(x^2-1\right)^{-\frac{\mu}{2}}}{K(\mu,\nu)}\times$$
$$\times\left\{Q_\nu^\mu(x_1)P_\nu^\mu(x)-P_\nu^\mu(x_1)Q_\nu^\mu(x)\right\}.$$

For the real x, μ, ν the Wronskian $W(x)$ is complex-valued, if $\mu\neq\frac{n+1}{2}$, where n is integer. In this case $W(x)$ contains the factor $e^{i\mu\pi}$. However the Cauchy functions will be real, because the adjoined Legendre function contains the similar factor.

Thus, the particular solution of the inhomogeneous equation (1) is:

$$N_C=\int_{x_1}^{x}f(z)Y_2(z;x)dz,\qquad(17)$$

where $f(z)$ is the right part of (1).

It is recommended to determine numerically the values of (17) for the actual parameters.

Next we go to the cases when the solutions are expressed in terms of orthogonal polynomials.

3. THE BASIC SOLUTIONS IN TERMS OF GEGENBAUER POLYNOMIALS

For receiving of the solutions we compare the coefficients of the homogeneous equation, corresponding to (1), with the coefficients of Jacobi equation [18], [20]. The analysis shows that the solution is possible, when the parameters $\alpha=\beta$; it corresponds to the case of the ultraspherical Gegenbauer polynomials $C_m^\lambda(x)$.

The differential equation for the Gegenbauer polynomials is

$$y''-\frac{2\lambda+1}{1-x^2}xy'+\frac{m(m+2\lambda)}{1-x^2}y=0.\qquad(18)$$

Fulfilling the above-mentioned comparision, we get the following parameters

$$\alpha_0=-2,\qquad m_{1,2}=\lambda\pm\sqrt{\lambda^2+\frac{(1-\sigma)}{2}(2\lambda+1)}\qquad(19)$$

and the rigidity for the tension is

$$D=D_0\left(1-x^2\right)^{-\left(\lambda+\frac{1}{2}\right)},0\le x<1;\qquad D=D_0\left(x^2-1\right)^{-\left(\lambda+\frac{1}{2}\right)},0\le x<\infty.\qquad(20)$$

The solution of the homogeneous equation has the following form:

$$N_1=AC_m^\lambda(x)+B\left(\frac{1-x}{2}\right)^{\frac{1}{2}-\lambda}\times F\left(-n-\lambda+\frac{1}{2},n+\lambda+\frac{1}{2};\frac{3}{2}-\lambda;\frac{1-x}{2}\right),\qquad(21)$$

where $F(\ )$ is the hypergeometric function.

For the disks with the rigidity (20) and the solution (21) the Cauchy functions are to be determined. The expression for Wronskian is

$$W(x)=\frac{\Gamma(m+2\lambda)}{m!\,\Gamma(2\lambda)}\left(\frac{1}{2}-\lambda\right)2^{2\lambda+1}\times\left(1-x^2\right)^{-\lambda-\frac{1}{2}}.\qquad(22)$$

Further we obtain the following formulae for the Cauchy functions:

$$Y_1(x_1;x)=\frac{m!\,\Gamma(2\lambda)\left(1-x_1^2\right)^{\lambda+\frac{1}{2}}}{2^{2\lambda+1}\left(\frac{1}{2}-\lambda\right)\Gamma(m+2\lambda)}\left\{\left[\left(\frac{\lambda}{2}-\frac{1}{4}\right)\left(\frac{1-x_1}{2}\right)^{-\lambda-\frac{1}{2}}F\left(-m-\lambda+\frac{1}{2},m+\lambda+\frac{1}{2};\frac{3}{2}-\lambda;\frac{1-x_1}{2}\right)-\right.\right.$$

$$\left.-\left(\frac{1-x_1}{2}\right)^{-\frac{1}{2}-\lambda}\frac{\frac{1}{4}-(m+\lambda)^2}{\frac{3}{2}-\lambda}F\left(-m-\lambda+\frac{3}{2},m+\lambda+\frac{3}{2};\frac{5}{2}-\lambda;\frac{1-x_1}{2}\right)\right]C_m^{\lambda}(x)+$$

$$\left.+\left[\frac{mx_1}{1-x_1^2}C_m^{\lambda}(x_1)-\frac{m+2\lambda-1}{1-x_1^2}C_{m-1}^{\lambda}(x_1)\right]\left(\frac{1-x}{2}\right)^{\frac{1}{2}-\lambda}F\left(-m-\lambda+\frac{1}{2},m+\lambda+\frac{1}{2};\frac{3}{2}-\lambda;\frac{1-x}{2}\right)\right\}; \quad (23)$$

$$Y_2(x_1;x)=\frac{n!\,\Gamma(2\lambda)\left(1-x_1^2\right)^{\lambda+\frac{1}{2}}}{2^{2\lambda+1}\left(\frac{1}{2}-\lambda\right)\Gamma(m+2\lambda)}\left\{\left(\frac{1-x_1}{2}\right)^{\frac{1}{2}-\lambda}F\left(-m-\lambda+\frac{1}{2},m+\lambda+\frac{1}{2};\frac{3}{2}-\lambda;\frac{1-x_1}{2}\right)C_m^{\lambda}(x)+\right.$$

$$\left.+C_m^{\lambda}(x_1)\left(\frac{1-x}{2}\right)^{\frac{1}{2}-\lambda}F\left(-m-\lambda+\frac{1}{2},m+\lambda+\frac{1}{2};\frac{3}{2}-\lambda;\frac{1-x}{2}\right)\right\}.$$

The particular solution of the inhomogeneous equation is determined by means of the expression (17).

4. THE BASIC SOLUTIONS IN TERMS OF LAGUERRE POLYNOMIALS

Let us determine the possibility to obtain the solutions in terms of Laguerre polynomials $L_m^{(\alpha)}(x)$ [18], [20]. The differential equation for these polynomials is

$$y''+\frac{\alpha+1-x}{x}y'+\frac{m}{x}y=0. \quad (24)$$

After the described above transformations we get the following parameters:

$$\alpha_0=-\frac{2}{\alpha},\ \alpha=\frac{2m}{1-\sigma}. \quad (25)$$

The rigidity for the tension is

$$D=D_0\mathrm{e}^x. \quad (26)$$

The general solution of the homogeneous equation for the case under study is

$$N_r=AL_m^{\alpha}(x)+Bx^{-\alpha}{}_1F_1(-n-\alpha;1-\alpha;x), \quad (27)$$

where ${}_1F_1(x)$ is the confluent hypergeometric function.

The Wronskian for the solution (27) is determined by the following formula:

$$W(x)=\binom{m+\alpha}{m}\alpha x^{-\alpha-1}e^x,$$

where

$$\binom{m+\alpha}{m}=\frac{(m+\alpha-m+1)m}{m!}=\frac{(\alpha+1)m}{m!};$$
$$(\alpha+1)_m=(\alpha+1)(\alpha+2)...(\alpha+m). \quad (28)$$

The Cauchy functions for the solutions (27) are

$$Y_1(x_1;x)=\begin{pmatrix} m+\alpha \\ m \end{pmatrix}^{-1}\alpha^{-1}e^{-x_1}x^{\alpha+1}\times$$
$$\times\Big\{\Big[\alpha x_1^{-\alpha-1}{}_1F_1(-m-\alpha;1-\alpha;x_1)+$$
$$+x_1^{-\alpha}\frac{m+\alpha}{1-\alpha}{}_1F_1(-m-\alpha+1;2-\alpha;x_1)\Big]\times$$
$$\times L_m^{\alpha}(x)-\left[\frac{m}{x_1}L_m^{\alpha}(x_1)-\frac{m+\alpha}{x_1}L_{m-1}^{\alpha}(x_1)\right]\times \quad (29)$$
$$\times x^{-\alpha}F(-m-\alpha;1-\alpha;x)\Big\};$$
$$Y_2(x_1;x)=\begin{pmatrix} m+\alpha \\ m \end{pmatrix}^{-1}\alpha^{-1}e^{-x_1}x^{\alpha+1}\times$$
$$\times\Big\{-L_m^{(\alpha)}(x_1)x^{\alpha}{}_1F_1(-m-\alpha;1-\alpha;x)+$$
$$+x_1^{\alpha}{}_1F_1(-m-\alpha;1-\alpha;x_1)L_{-m}^{\alpha}(x)\Big\}.$$

Here, the particular solution is also determined with the use of the formula (17), where the function $Y_2(x_1;x)$ is determined by means the expression (29).

5. THE STATEMENT OF THE COMPUTATION PROBLEM OF THE CIRCULAR DISK WITH PIECEWISE THICKNESS

The disks, subjected to an action of the stretching loads and consisting of two sections with different laws of thickness variation, are under study. The disks' profile has a gap in the place of these parts conjugation (fig.2).

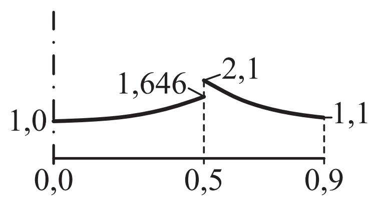

Figure 2. The profile of the disk with piecewise thickness

The rigidity for the tension in the first section is approximated by the formula (26). The normal stress when $0{,}0 \le x \le 0{,}5$ is determined in the following way:

$$N_r^{(1)}=A_1L_m^{\alpha}(x)+B_1x^{-\alpha}{}_1F_1(-n-\alpha;1-\alpha;x)+ \\ +N_C^{(1)}(x). \quad (30)$$

The rigidity for the tension in the second section is approximated by the formula (9). We have when $0{,}5 \le x \le 0{,}9$:

$$N_r^{(2)}=N_r^{(1)}+\left[A_2P_\nu^{\mu}(x)+B_2Q_\nu^{\mu}(x)\right]\times \\ \times\left(x^2-1\right)^{-\frac{\mu}{2}}+N_C^{(2)}(x), \quad (31)$$

where $N_C^{(1)}$, $N_C^{(2)}$ are the particular solutions determined by means the expression (17).

6. THE CONCLUSION

In the present work the exact analytical solutions of computational problems of circular disks with piecewise variable thickness, subjected to an action of expanding loads, are obtained. The constructions under study consist of two or several sections. Each of the mentioned parts has its own law of thickness variation. These sections can be made from the same or from the different materials, which can be homogeneous or inhomogeneous, isotropic and anisotropic. In the places of conjugation the disk's thickness can be continuous or discontinuous. The received solutions are obtained in terms of Legendre functions and Legendre, Gegenbauer and Laguerre polynomials.

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Elena B. Koreneva, Dr.Sc., Professor, Moscow Higher Combined-Arms Command Academy; 2, ul. Golovacheva, Moscow, 109380, Russia; tel.: +7(499)175-82-45, e-mail: elena.koreneva2010@yandex.ru.

Коренева Елена Борисовна, доктор технических наук, профессор, Московское высшее общевойсковое командное орденов Жукова, Ленина и Октябрьской Революции Краснознаменное училище; 109380, Россия, г. Москва, ул. Головачева, д.2; тел.: +7(499)175-82-45, e-mail: elena.koreneva2010@yandex.ru.