LOCALIZATION OF SOLUTION OF THE PROBLEM OF ISOTROPIC PLATE ANALYSIS WITH THE USE OF B-SPLINE DISCRETE-CONTINUAL FINITE ELEMENT METHOD

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Abstract: Localization of solution of the problem of isotropic plate analysis with the use of B-spline discrete-continual finite element method (specific version of wavelet-based discrete-continual finite element method) is under consideration in the distinctive paper. The original operational continual and discrete-continual formulations of the problem are given, some actual aspects of construction of normalized basis functions of a B-spline are considered, the corresponding local constructions for an arbitrary discrete-continual finite element are described, some information about the numerical implementation and an example of analysis are presented.

Keywords: localization, wavelet-based discrete-continual finite element method, B-spline discrete-continual finite element method, discrete-continual finite element method, finite element method, B-spline, numerical solution, isotropic plate, plate analysis

ЛОКАЛИЗАЦИЯ РЕШЕНИЯ ЗАДАЧИ О ПОПЕРЕЧНОМ ИЗГИБЕ ИЗОТРОПНОЙ ПЛАСТИНЫ НА ОСНОВЕ ВЕЙВЛЕТ-РЕАЛИЗАЦИИ ДИСКРЕТНО-КОНТИНУАЛЬНОГО МЕТОДА КОНЕЧНЫХ ЭЛЕМЕНТОВ С ИСПОЛЬЗОВАНИЕМ В-СПЛАЙНОВ

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Аннотация: В настоящей статье рассматривается локализация решения задачи о поперечном изгибе изотропной пластины на основе вейвлет-реализации дискретно-континуального метода конечных элементов с использованием В-сплайнов. Приведены исходные операторные континуальная и дискретно-континуальная постановки задачи, рассмотрены некоторые актуальные вопросы построения нормализованных базисных функций В-сплайна, описаны соответствующие локальные построения для произвольного дискретно-континуального конечного элемента, представлены некоторые сведения о численной реализации и пример расчета.

Ключевые слова: локализация, вейвлет-реализация метода конечных элементов, дискретно-континуальный метод конечных элементов, метод конечных элементов, В-сплайны, численное решение, изотропная пластина, изгиб изотропной пластины

INTRODUCTION

As we have already mentioned [1, 2], the B-spline in a given simple knot sequence can be constructed by employing piecewise polynomials between the knots and joining them together at the knots [1–3]. For instance, compared with commonly used Daubechies wavelets [4–8] B-spline wavelet on interval (BSWI) has explicit expressions, facilitating the calculation of coefficient integration and differentiation [1–3]. Besides, the multiresolution and localization properties of BSWI can also supply some superiority for engineering structural analysis [1–3]. The early applications of spline can be found in papers of H. Antes [9], J.G. Han [10, 11, 27], Y. Huang [10, 11], W.X. Ren [10, 11]. The spline wavelet finite element method was further developed in papers of D.P. Chen [28], X.F. Chen [12, 13, 15–18, 23, 24, 26], H.B. Dong [23], J.G. Han [25], Y.M. He [17], Z.H. He [18], Z.J. He [12, 13, 15–17, 23, 24, 26], Y. Huang [25, 27], Z.S. Jiang [22], B. Li [13, 15, 17, 23], M. Liang [19, 21], J.Q. Long [20], G. Ma [20], T. Matsumoto [20, 22], S.T. Mau [30], H.H. Miao [15], Q.M. Mo [18], T.H.H. Pian [28–30], K.Y. Qi [23], W.X. Ren [25, 27], K. Sumihara [29], P. Tong [30], Y.W. Wang [22], J.W. Xiang [12–14, 17–22], Z.B. Yang [15, 16, 24], X.W. Zhang [16, 24, 26], Y.H. Zhang [12], Y.T. Zhong [14].

As is known, generally the structural analysis normally require accurate computer-intensive calculations using numerical (discrete) methods. The field of application of discrete-continual finite element method (DCFFEM), proposed by A.B. Zolotov [33] and P.A. Akimov [31–33] comprises structures with regular (in particular, constant or piecewise constant) physical and geometrical parameters in some dimension (so-called “basic” direction (dimension)). Considering problems remain continual along “basic” direction while along other directions DCFEM presupposes finite element approximation. Solution of corresponding resultant multipoint boundary problems [34] for systems of ordinary differential equations with piecewise constant coefficients and immense number of unknowns is the most time-consuming

stage of the computing, especially if we take into account the limitation in performance of personal computers, contemporary software and necessity to obtain correct semianalytical solution in a reasonable time.

High-accuracy solution at all points of the model is not required normally, it is necessary to find only the most accurate solution in some pre-known domains. Generally the choice of these domains is a priori data with respect to the structure being modelled. Designers usually choose domains with the so-called edge effect (with the risk of significant stresses that could potentially lead to the destruction of structures, etc.) and regions which are subject to specific operational requirements. It is obvious that the stress-strain state in such domains is of paramount importance. Specified factors along with the obvious needs of the designer or researcher to reduce computational costs by application of DCFEM cause considerable urgency of constructing of special algorithms for obtaining local solutions (in some domains known in advance) of boundary problems. Wavelet analysis provides effective and popular tool for such researches. Solution of the considering problem within multilevel wavelet analysis is represented as a composition of local and global components. Wavelet-based DCFEM is presented in papers of P.A. Akimov [35–42], M. Aslami [38–40], T.B. Kaytukov, M.L. Mozgaleva [35–42] and O.A. Negrozov [38–40].

The distinctive paper is devoted to numerical solution of the problem of isotropic plate analysis with the use of B-spline DCFEM.

1. FORMULATIONS OF THE PROBLEM

In accordance with [1] let the constancy of the parameters of the problem be in the direction corresponding to x_2 (main direction). The operational formulation of the problem with the use of so-called method of extended domain [43], taking into account the selection of the main direction, is determined by the equation:

$$L y = \widetilde{F}, 0 \leq x_1 \leq \ell_1, 0 \leq x_2 \leq \ell_2 \quad (1.1) \quad \text{or}$$

where we have

$$L = -L_4 \partial_2^4 + L_2 \partial_2^2 + L_0; \quad (1.2)$$

$$L_4 = \theta D; \quad (1.3)$$

$$L_2 = -[\partial_1^2 \theta D \nu + 2 \partial_1 \theta D (1 - \nu) \partial_1 + \theta D \nu \partial_1^2]; \quad (1.4)$$

$$L_0 = -\partial_1^2 \theta D \partial_1^2; \quad (1.5)$$

$$\widetilde{F} = \theta F + \delta_\Gamma f; \quad (1.6)$$

$$\theta(x_1, x_2) = \begin{cases} 1, & (x_1, x_2) \in \Omega \\ 0, & (x_1, x_2) \notin \Omega; \end{cases} \quad (1.7)$$

$$\delta_\Gamma(x_1, x_2) = \partial \theta / \partial \bar{n}; \quad (1.8)$$

Ω is the domain, occupied by plate;

$$\Omega = \{(x_1, x_2): 0 < x_1 < \ell_1; 0 < x_2 < \ell_2\}; \quad (1.9)$$

ℓ_1, ℓ_2 are corresponding dimensions of extended domain (linear dimensions of plate); $x = (x_1, x_2)$; x_1, x_2 are Cartesian coordinates; $\theta(x_1, x_2)$ is characteristic function of domain Ω ; $\delta_\Gamma = \delta_\Gamma(x_1, x_2)$ is the delta function of boundary $\Gamma = \partial\Omega$; $\bar{n} = [n_1, n_2]^T$ is boundary normal vector; y is deflection of plate; D is flexural rigidity of plate; ν is Poisson's ratio of plate; F is the load in domain Ω ; f is the corresponding boundary load; $\partial_s = \partial / \partial x_s, s = 1, 2$. Let us introduce the following notations

$$y_1 = y, \quad y_2 = \partial_2 y = y'_1, \quad y_3 = \partial_2^2 y = y'_2, \\ y_4 = \partial_2^3 y = y'_3. \quad (1.10)$$

Thus we can rewrite (1.1):

$$-L_4 y'_4 + L_2 y_3 + L_0 y_1 = \widetilde{F}. \quad (1.11)$$

Finally we obtain system of differential equations with operational coefficients:

$$\begin{bmatrix} y'_1 \\ y'_2 \\ y'_3 \\ y'_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ L_4^{-1} L_0 & 0 & L_4^{-1} L_2 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -L_4^{-1} \widetilde{F} \end{bmatrix}, \quad (1.12)$$

where

$$\widetilde{L} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ L_4^{-1} L_0 & 0 & L_4^{-1} L_2 & 0 \end{bmatrix}; \quad (1.14)$$

$$\widetilde{F} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -L_4^{-1} \widetilde{F} \end{bmatrix}; \quad \overline{U} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}. \quad (1.15)$$

The system of equations (1.12) is supplemented by boundary conditions, which are set in sections with coordinates $x_2^1 = 0$ and $x_2^2 = \ell_2$.

2. SOME ASPECTS OF THE CONSTRUCTION OF NORMALIZED BASIS FUNCTIONS OF THE B-SPLINE

The construction of B-spline basic functions is determined by the recursive Cox-de Boer formulas [1]:

$$k = 1: \quad \varphi_{i,1}(t) = \begin{cases} 1, & x_i \leq t < x_{i+1} \\ 0, & t < x_i \vee t \geq x_{i+1} \end{cases}, \quad (2.1)$$

$$k \geq 2: \quad \varphi_{i,k}(t) = \frac{(t - x_i) \varphi_{i,k-1}(t)}{x_{i+k-1} - x_i} + \frac{(x_{i+k} - t) \varphi_{i+1,k-1}(t)}{x_{i+k} - x_{i+1}}. \quad (2.2)$$

We will consider such a construction for the case $x_i = i$ are integers. Let us note that,

$$\varphi_{i,k}(t) = \varphi_{0,k}(t - i)$$

and therefore, recursive formulas (2.1)–(2.2) can be represented in the form

$$k=1: \quad \varphi_{0,1}(t) = \begin{cases} 1, & 0 \leq t < 1 \\ 0, & t < 0 \vee t \geq 1; \end{cases} \quad (2.3)$$

$$k \geq 2: \quad \varphi_{0,k}(t) = \frac{1}{k-1} [t \cdot \varphi_{0,k-1}(t) + (k-t)\varphi_{0,k-1}(t-1)]. \quad (2.4)$$

The function $\varphi_{0,1}(t)$ can be represented by formula

$$\varphi_{0,1}(t) = \frac{1}{2} [\text{sign}(t) - \text{sign}(t-1)]. \quad (2.5)$$

Let us denote by Δ_1 the operator of the first difference. Then we have

$$\varphi_{0,1}(t) = -\frac{1}{2} \Delta_1 \text{sign}(t). \quad (2.6)$$

We can substitute formula (2.5) into (2.4) in order to determine $\varphi_{0,2}(t)$:

$$\begin{aligned} \varphi_{0,2}(t) &= 1 \cdot [t \cdot \varphi_{0,1}(t) + (2-t)\varphi_{0,1}(t-1)] = \\ &= \frac{1}{2} \{t \cdot [\text{sign}(t) - \text{sign}(t-1)] + \\ &\quad (2-t)[\text{sign}(t-1) - \text{sign}(t-2)]\} = \\ &= \frac{1}{2} [t \text{sign}(t) - 2(t-1) \text{sign}(t-1) + \\ &\quad (t-2) \text{sign}(t-2)] = \frac{1}{2} [|t| - 2|t-1| + |t-2|]. \end{aligned}$$

Let us denote by Δ_2 the operator of the second difference. Then we have

$$\varphi_{0,2}(t) = \frac{1}{2} [|t| - 2|t-1| + |t-2|] = \frac{1}{2} \Delta_2 |t-1|. \quad (2.7)$$

We can define function $\varphi_{0,3}(t)$:

$$\varphi_{0,3}(t) = \frac{1}{2} [t \cdot \varphi_{0,2}(t) + (3-t)\varphi_{0,2}(t-1)].$$

Omitting intermediate calculations, we get

$$\begin{aligned} \varphi_{0,3}(t) &= \frac{1}{4} [t \cdot |t| - 3(t-1)|t-1| + \\ &\quad + 3(t-2)|t-2| - (t-3)|t-3|] = \\ &= -\frac{1}{2!} \frac{1}{2} \Delta_1 \Delta_2 ((t-1)|t-1|). \end{aligned} \quad (2.8)$$

Based on formulas (2.8) and (2.4), we can define the function

$$\varphi_{0,4}(t) = \frac{1}{3} [t \cdot \varphi_{0,3}(t) + (4-t)\varphi_{0,3}(t-1)].$$

Omitting intermediate calculations, as a result we get

$$\begin{aligned} \varphi_{0,4}(t) &= \\ &= \frac{1}{2 \cdot 3} \cdot \frac{1}{2} [t^2 \cdot |t| - 4(t-1)^2 |t-1| + \\ &\quad + 6(t-2)^2 |t-2| - 4(t-3)^2 |t-3| + \\ &\quad + (t-4)^2 |t-4|] = \\ &= \frac{1}{3!} \frac{1}{2} (\Delta_2)^2 ((t-2)^2 |t-2|). \end{aligned} \quad (2.9)$$

It can be proved that for even $k = 2m$ we have

$$\varphi_{0,k}(t) = \frac{1}{(2m-1)!} \frac{1}{2} (\Delta_2)^m ((t-m)^{2m-2} |t-m|) \quad (2.10)$$

and for odd (uneven) $k = 2m + 1$ we have

$$\varphi_{0,k}(t) = -\frac{1}{(2m)!} \frac{1}{2} \Delta_1 (\Delta_2)^m ((t-m)^{2m-1} |t-1|). \quad (2.11)$$

Note that $\varphi_{0,k}(t)$ is a polynomial of degree $k-1$ with bounded support and, as follows from the difference operator, this support is equal to the interval $[0, k]$.

In addition, we should note the following property of B-spline basis functions:

$$\sum_i \varphi_{0,k}(t-i) \equiv 1 \text{ for arbitrary } t. \quad (2.12)$$

3. SOME GENERAL ASPECTS OF FINITE ELEMENT APPROXIMATION

The discrete component of the numerical solution is represented by the direction along the axis corresponding to x_1 . The fulfillment within an element (interval) for all components of a vector function $\bar{U}$ (see (1.15)) is the same. Therefore, let us use the following notation for simplicity:

$$x = x_1, \ell = \ell_1, y = y_1, j = 1, 2, 3, 4. \quad (3.1)$$

Let us divide the interval $(0, \ell)$ segment into N_e parts (elements). Therefore $h_e = \ell/N_e$ is the length of the element. Besides, let us also divide each element into N_k parts (see Figures 3.1-3.3), for example, $N_k = 5$ (see Figure 3.1).

Let us use the following notation system: i_e is the element number; $N_p = N_k + 1 = 6$ is the number of nodes within the element; $x_1(i_e)$ is the coordinate of the starting point of the i_e -th element; $x_6(i_e)$ is the coordinate of the end point of the i_e -th element. We take y_i and $y'_i = \partial_1 y(x_i)$, $i = 1, 6$ as unknowns at the boundary points. Besides, we take y_i , $i = 2, 3, 4, 5$ as unknowns at the inner points. Thus, the number of unknowns per element with such approximation is equal to

$$N_{ie} = N_k - 1 + 2 \cdot 2 = N_k + 3 = 8.$$

Figure 3.1. Finite element discretization for $N_k = 5$ (sample).

Figure 3.2. Finite element discretization for $N_k = 3$ (sample).

Figure 3.3. Finite element discretization for $N_k = 1$ (sample).

For the elements of localization we can take reduced number of N_k . For instance, if we take $N_k = 3$ (Figure 3.2) we get $N_p = N_k + 1 = 4$ and the number of unknowns per element with such approximation is equal to

$$N_{ie} = N_k - 1 + 2 \cdot 2 = N_k + 3 = 6;$$

$x_1(i_e)$ is the coordinate of the starting point of the i_e -th element; $x_4(i_e)$ is the coordinate of the end point of the i_e -th element; y_i and y'_i , $i = 1, 4$ are unknowns at the boundary points; y_i , $i = 2, 3$ are unknowns at the inner points.

Besides, let us consider the case with (Figure 3.3). Therefore we have and the number of unknowns per element with such approximation is equal to

$$N_{ie} = N_k - 1 + 2 \cdot 2 = N_k + 3 = 4;$$

$x_1(i_e)$ is the coordinate of the starting point of the i_e -th element; $x_2(i_e)$ is the coordinate of the end point of the i_e -th element; y_i and y'_i , $i = 1, 2$ are unknowns at the boundary points.

4. LOCAL CONSTRUCTIONS FOR ARBITRARY FINITE ELEMENT

Let us introduce local coordinates:

$$t = (x - x_{1(ie)}) / h_e, \quad x_{1(ie)} \leq x \leq x_{N_p(ie)}, \quad 0 \leq t \leq 1. \quad (4.1)$$

In this case, we have the following relations:

$$x = x_i \Rightarrow t_i = (x_i - x_{1(ie)}) / h_e, \quad i = 1, \dots, N_p; \quad (4.2)$$

$$\frac{d^p}{dx^p} = \frac{1}{h_e^p} \frac{d^p}{dt^p}; \quad dx = h_e \cdot dt. \quad (4.3)$$

Since the number of unknowns on the element is equal to $N = 8$, we use a B-spline of the seventh degree in order to represent the unknown deflection function.

Let us use the following notation:

$$\begin{aligned} \varphi(t) &= \varphi_{0,8}(t+4); \\ \varphi(t) &= \frac{1}{7!} \frac{1}{2} (\Delta_2)^4 (t^6 | t |) = \frac{1}{2 \cdot 7!} [(t+4)^6 | t+4 | - \\ &\quad - 8(t+3)^6 | t+3 | + 28(t+2)^6 | t+2 | - \\ &\quad - 56(t+1)^6 | t+1 | + 70t^6 | t | - \\ &\quad - 56(t-1)^6 | t-1 | + 28(t-2)^6 | t-2 | - \\ &\quad - 8(t-3)^6 | t-3 | + (t-4)^6 | t-4 |]. \end{aligned} \quad (4.4)$$

This function is a B-spline, symmetric with respect to $t = 0$ and its support is defined by an interval $[-4, 4]$ (Figure 4.1).

We take the following eight functions as basis functions on the unit interval (Figures 4.2, 4.3):

$$\begin{aligned} \varphi_1(t) &= \varphi(t+3), \quad \varphi_2(t) = \varphi(t+2), \\ \varphi_3(t) &= \varphi(t+1), \quad \varphi_4(t) = \varphi(t), \\ \varphi_5(t) &= \varphi(t-1), \quad \varphi_6(t) = \varphi(t-2), \\ \varphi_7(t) &= \varphi(t-3), \quad \varphi_8(t) = \varphi(t-4), \\ &0 \leq t \leq 1. \end{aligned} \quad (4.5)$$

Since the number of unknowns on the element is equal to $N = 6$, we use a B-spline of the fifth degree in order to represent the unknown deflection function.

Let us use the following notation:

$$\begin{aligned} \varphi(t) &= \varphi_{0,6}(t+4); \\ \varphi(t) &= \frac{1}{5!} \frac{1}{2} (\Delta_2)^3 (t^4 | t |) = \frac{1}{2 \cdot 5!} [(t+3)^4 | t+3 | - \\ &\quad - 6(t+2)^4 | t+2 | + 15(t+1)^4 | t+1 | - \\ &\quad - 20t^4 | t | + 15(t-1)^4 | t-1 | - \\ &\quad - 6(t-2)^4 | t-2 | + (t-3)^4 | t-3 |]. \end{aligned} \quad (4.6)$$

Figure 4.1. B-spline of the seventh order $\varphi(t) = \varphi_{0,8}(t + 4)$.

Figure 4.2. Basis functions $\varphi_k(t)$, $k = 1, 2, \dots, 8$.

Figure 4.3. Basis functions $\varphi_1(t)$ and $\varphi_8(t)$.

This function is a B-spline, symmetric with respect to $t = 0$ and its support is defined by an interval $[-3, 3]$ (Figure 4.4).

We take the following six functions as basis functions on the unit interval (Figures 4.5, 4.6):

$$\begin{aligned} \varphi_1(t) &= \varphi(t+2), & \varphi_2(t) &= \varphi(t+1), \\ \varphi_3(t) &= \varphi(t), & \varphi_4(t) &= \varphi(t-1), \\ \varphi_5(t) &= \varphi(t-2), & \varphi_6(t) &= \varphi(t-3), \end{aligned} \quad 0 \leq t \leq 1. \quad (4.7)$$

Figure 4.4. B-spline of the fifth order $\varphi(t) = \varphi_{0,6}(t + 3)$.

Figure 4.5. Basis functions $\varphi_k(t)$, $k = 1, 2, \dots, 3$.

Figure 4.6. Basis functions $\varphi_1(t)$ and $\varphi_6(t)$.

Since the number of unknowns on the element is equal to $N = 4$, we use a B-spline of the third degree in order to represent the unknown deflection function.

Let us use the following notation:

$$\varphi(t) = \varphi_{0,4}(t + 4) ;$$

$$\begin{aligned} \varphi(t) = \frac{1}{3!} \frac{1}{2} (\Delta_2)^2 (t^2 |t|) = \frac{1}{2 \cdot 3!} [& (t+2)^4 |t+2| - \\ & - 4(t+1)^4 |t+1| + 6t^2 |t| - \\ & - 4(t-1)^2 |t-1| + (t-2)^2 |t-2|]. \end{aligned} \quad (4.8)$$

This function is a B-spline, symmetric with respect to $t = 0$ and its support is defined by an interval $[-2, 2]$ (Figure 4.7).

Figure 4.7. B-spline of the fifth order $\varphi(t) = \varphi_{0,6}(t + 3)$.

Figure 4.8. Basis functions $\varphi_k(t)$, $k = 1, 2, \dots, 4$.

We take the following four functions as basis functions on the unit interval (Figures 4.8):

$$\begin{aligned} \varphi_1(t) &= \varphi(t+1), \quad \varphi_2(t) = \varphi(t), \\ \varphi_3(t) &= \varphi(t-1), \quad \varphi_4(t) = \varphi(t-2), \\ 0 \leq t \leq 1. \end{aligned} \quad (4.7)$$

We represent the unknown deflection function $y(x)$ within the element number i_e in the form

$$y(x) = w(t) = \sum_{k=1}^N \alpha_k \varphi_k(t), \quad x_{1(i_e)} \leq x \leq x_{5(i_e)}, \quad 0 \leq t \leq 1. \quad (4.8)$$

We have to consider bilinear forms with allowance for relations (4.2) in order to construct local stiffness matrices corresponding to the operators L_0 , L_2 and L_4 (see (1.3)–(1.5)):

$$\begin{aligned} B_0(y, z) &= \langle L_0 y, z \rangle = - \int_{x_{1(i_e)}}^{x_{5(i_e)}} \frac{d^2}{dx^2} \theta D \frac{d^2 y}{dx^2} \cdot z \, dx = \\ &= - \theta_{i_e} D_{i_e} \int_{x_{1(i_e)}}^{x_{5(i_e)}} \frac{d^2 y}{dx^2} \cdot \frac{d^2 z}{dx^2} \, dx = \\ &= - \frac{1}{h_e^3} \theta_{i_e} D_{i_e} \int_0^1 \frac{d^2 w}{dt^2} \cdot \frac{d^2 v}{dt^2} \, dt = B_0(w, v); \end{aligned} \quad (4.9)$$

$$\begin{aligned} B_4(y, z) &= \langle L_4 y, z \rangle = \theta_{i_e} D_{i_e} \int_{x_{1(i_e)}}^{x_{5(i_e)}} y \cdot z \, dx = \\ &= h_e \theta_{i_e} D_{i_e} \int_0^1 w \cdot v \, dt = B_4(w, v); \end{aligned} \quad (4.10)$$

$$\begin{aligned} B_2(y, z) &= \langle L_2 y, z \rangle = \langle L_{21} y, z \rangle + \\ &+ \langle L_{22} y, z \rangle + \langle L_{23} y, z \rangle, \end{aligned} \quad (4.11)$$

where

$$\begin{aligned} \langle L_{21} y, z \rangle &= - \theta_{i_e} D_{i_e} v_{i_e} \int_{x_{1(i_e)}}^{x_{5(i_e)}} \frac{d^2 y}{dx^2} z \, dx = \\ &= - \frac{1}{h_e} \theta_{i_e} D_{i_e} v_{i_e} \int_0^1 \frac{d^2 w}{dt^2} \cdot v \, dt = B_{21}(w, v); \end{aligned} \quad (4.12)$$

$$\begin{aligned} \langle L_{23} y, z \rangle &= - \theta_{i_e} D_{i_e} v_{i_e} \int_{x_{1(i_e)}}^{x_{5(i_e)}} y \cdot \frac{d^2 z}{dx^2} \, dx = \\ &= - \frac{1}{h_e} \theta_{i_e} D_{i_e} v_{i_e} \int_0^1 w \cdot \frac{d^2 v}{dt^2} \, dt = B_{23}(w, v); \end{aligned} \quad (4.13)$$

$$\begin{aligned} \langle L_{22} y, z \rangle &= - 2 \int_{x_{1(i_e)}}^{x_{5(i_e)}} \frac{d}{dx} \theta D (1 - v) \frac{dy}{dx} \cdot z \, dx = \\ &= 2 \theta_{i_e} D_{i_e} (1 - v_{i_e}) \int_{x_{1(i_e)}}^{x_{5(i_e)}} \frac{dy}{dx} \cdot \frac{dz}{dx} \, dx = \\ &= \frac{1}{h_e} 2 \theta_{i_e} D_{i_e} (1 - v_{i_e}) \int_0^1 \frac{dw}{dt} \cdot \frac{dv}{dt} \, dt = B_{22}(w, v). \end{aligned} \quad (4.14)$$

for the following type of functions

$$\begin{aligned} y(x) &= w(t) = \sum_{k=1}^{N_{ie}} \alpha_k \varphi_k(t), \\ z(x) &= v(t) = \sum_{k=1}^{N_{ie}} \beta_k \varphi_k(t), \\ x_{1(ie)} &\leq x \leq x_{N_p(ie)}, \quad 0 \leq t \leq 1 \end{aligned} \quad (4.15)$$

Let us substitute (4.15) into (4.9)–(4.14):

$$\begin{aligned} B_0(w, v) &= -\frac{1}{h_e^3} \theta_{i_e} D_{i_e} \int_0^1 \frac{d^2 w}{dt^2} \cdot \frac{d^2 v}{dt^2} dt = \\ &= -\frac{\theta_{i_e} D_{i_e}}{h_e^3} \sum_{i=1}^N \sum_{j=1}^N \alpha_i \beta_j \int_0^1 \varphi_i''(t) \varphi_j''(t) dt = \\ &= -\frac{\theta_{i_e} D_{i_e}}{h_e^3} (K_{\alpha\beta}^0 \bar{\alpha}, \bar{\beta}), \end{aligned} \quad (4.16)$$

where

$$K_{\alpha\beta}^0(i, j) = \int_0^1 \varphi_i''(t) \varphi_j''(t) dt; \quad \varphi'' = \frac{d^2 \varphi}{dt^2}; \quad (4.17)$$

$$\begin{aligned} B_4(w, v) &= h_e \theta_{i_e} D_{i_e} \int_0^1 w \cdot v dt = \\ &= \theta_{i_e} D_{i_e} h_e \sum_{i=1}^N \sum_{j=1}^N \alpha_i \beta_j \int_0^1 \varphi_i(t) \varphi_j(t) dt = \\ &= h_e \theta_{i_e} D_{i_e} (K_{\alpha\beta}^4 \bar{\alpha}, \bar{\beta}), \end{aligned} \quad (4.18)$$

where

$$\begin{aligned} K_{\alpha\beta}^4(i, j) &= \int_0^1 \varphi_i(t) \varphi_j(t) dt; \quad (4.19) \\ B_{21}(w, v) &= -\frac{1}{h_e} \theta_{i_e} D_{i_e} \nu_{i_e} \int_0^1 \frac{d^2 w}{dt^2} \cdot v dt = \\ &= -\frac{\theta_{i_e} D_{i_e} \nu_{i_e}}{h_e} \sum_{i=1}^N \sum_{j=1}^N \alpha_i \beta_j \int_0^1 \varphi_i''(t) \varphi_j(t) dt = \\ &= -\frac{\theta_{i_e} D_{i_e} \nu_{i_e}}{h_e} (K_{\alpha\beta}^{21} \bar{\alpha}, \bar{\beta}), \end{aligned} \quad (4.20)$$

where

$$K_{\alpha\beta}^{21}(i, j) = \int_0^1 \varphi_i''(t) \varphi_j(t) dt; \quad (4.21)$$

$$\begin{aligned} B_{23}(w, v) &= -\frac{1}{h_e} \theta_{i_e} D_{i_e} \nu_{i_e} \int_0^1 w \cdot \frac{d^2 v}{dt^2} dt = \\ &= -\frac{\theta_{i_e} D_{i_e} \nu_{i_e}}{h_e} \sum_{i=1}^N \sum_{j=1}^N \alpha_i \beta_j \int_0^1 \varphi_i(t) \varphi_j''(t) dt = \\ &= -\frac{\theta_{i_e} D_{i_e} \nu_{i_e}}{h_e} (K_{\alpha\beta}^{23} \bar{\alpha}, \bar{\beta}), \end{aligned} \quad (4.22)$$

where

$$K_{\alpha\beta}^{23}(i, j) = \int_0^1 \varphi_i(t) \varphi_j''(t) dt = K_{\alpha\beta}^{21}(j, i); \quad (4.23)$$

$$\begin{aligned} B_{22}(w, v) &= \frac{1}{h_e} 2 \theta_{i_e} D_{i_e} (1 - \nu_{i_e}) \int_0^1 \frac{dw}{dt} \cdot \frac{dv}{dt} dt = \\ &= 2 \frac{\theta_{i_e} D_{i_e} (1 - \nu_{i_e})}{h_e} \sum_{i=1}^N \sum_{j=1}^N \alpha_i \beta_j \int_0^1 \varphi_i'(t) \varphi_j'(t) dt = \\ &= 2 \frac{\theta_{i_e} D_{i_e} (1 - \nu_{i_e})}{h_e} (K_{\alpha\beta}^{22} \bar{\alpha}, \bar{\beta}), \end{aligned} \quad (4.24)$$

where

$$K_{\alpha\beta}^{22}(i, j) = \int_0^1 \varphi_i'(t) \varphi_j'(t) dt, \quad \varphi' = \frac{d\varphi}{dt}. \quad (4.25)$$

Let us define the parameters α_k and β_k through the nodal unknowns on the element:

$$\begin{aligned} y_i &= w(t_i) = \sum_{k=1}^{N_{ie}} \alpha_k \varphi_k(t_i), \\ t_i &= (x_i - x_{1(ie)}) / h_e, \quad i = 1, \dots, N_p; \end{aligned} \quad (4.26)$$

$$\begin{aligned} \frac{dy_i}{dx} &= \frac{1}{h_e} w'(t_i) = \frac{1}{h_e} \sum_{k=1}^{N_{ie}} \alpha_k \varphi_k'(t_i), \\ t_i &= (x_i - x_{1(ie)}) h_e, \quad i = 1, N_p. \end{aligned} \quad (4.27)$$

For the case $N_{ie} = 8$ we have

$$\bar{y}^{ie} = T_8 \bar{\alpha}, \quad (4.28)$$

where

$$\bar{y}^{ie} = [y_1 \quad \frac{dy_1}{dx} \quad y_2 \quad y_3 \quad y_4 \quad y_5 \quad y_6 \quad \frac{dy_6}{dx}]^T; \quad (4.29)$$

$$\bar{\alpha} = [\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4 \quad \alpha_5 \quad \alpha_6 \quad \alpha_7 \quad \alpha_8]^T; \quad (4.30)$$

$$D = \text{diag}(1 \quad 1/h_e \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1/h_e). \quad (4.31)$$

$$T = D \begin{bmatrix} \varphi_1(0) & \varphi_2(0) & \varphi_3(0) & \varphi_4(0) & \varphi_5(0) & \varphi_6(0) & \varphi_7(0) & \varphi_8(0) \\ \varphi'_1(0) & \varphi'_2(0) & \varphi'_3(0) & \varphi'_4(0) & \varphi'_5(0) & \varphi'_6(0) & \varphi'_7(0) & \varphi'_8(0) \\ \varphi_1(0.2) & \varphi_2(0.2) & \varphi_3(0.2) & \varphi_4(0.2) & \varphi_5(0.2) & \varphi_6(0.2) & \varphi_7(0.2) & \varphi_8(0.2) \\ \varphi_1(0.4) & \varphi_2(0.4) & \varphi_3(0.4) & \varphi_4(0.4) & \varphi_5(0.4) & \varphi_6(0.4) & \varphi_7(0.4) & \varphi_8(0.4) \\ \varphi_1(0.6) & \varphi_2(0.6) & \varphi_3(0.6) & \varphi_4(0.6) & \varphi_5(0.6) & \varphi_6(0.6) & \varphi_7(0.6) & \varphi_8(0.6) \\ \varphi_1(0.8) & \varphi_2(0.8) & \varphi_3(0.8) & \varphi_4(0.8) & \varphi_5(0.8) & \varphi_6(0.8) & \varphi_7(0.8) & \varphi_8(0.8) \\ \varphi_1(1) & \varphi_2(1) & \varphi_3(1) & \varphi_4(1) & \varphi_5(1) & \varphi_6(1) & \varphi_7(1) & \varphi_8(1) \\ \varphi'_1(1) & \varphi'_2(1) & \varphi'_3(1) & \varphi'_4(1) & \varphi'_5(1) & \varphi'_6(1) & \varphi'_7(1) & \varphi'_8(1) \end{bmatrix}, \quad (4.32)$$

For the case $N_{ie} = 6$ we have

$$\bar{y}^{ie} = T_6 \bar{\alpha}, \quad (4.33)$$

$$\bar{y}^{ie} = [y_1 \quad \frac{dy_1}{dx} \quad y_2 \quad y_3 \quad y_4 \quad \frac{dy_4}{dx}]^T; \quad (4.34)$$

$$\bar{\alpha} = [\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4 \quad \alpha_5 \quad \alpha_6]^T; \quad (4.35)$$

where

$$D = \text{diag}(1 \quad 1/h_e \quad 1 \quad 1 \quad 1 \quad 1/h_e). \quad (4.36)$$

$$T_6 = D_6 \begin{bmatrix} \varphi_1(0) & \varphi_2(0) & \varphi_3(0) & \varphi_4(0) & \varphi_5(0) & \varphi_6(0) \\ \varphi'_1(0) & \varphi'_2(0) & \varphi'_3(0) & \varphi'_4(0) & \varphi'_5(0) & \varphi'_6(0) \\ \varphi_1(1/3) & \varphi_2(1/3) & \varphi_3(1/3) & \varphi_4(1/3) & \varphi_5(1/3) & \varphi_6(1/3) \\ \varphi_1(2/3) & \varphi_2(2/3) & \varphi_3(2/3) & \varphi_4(2/3) & \varphi_5(2/3) & \varphi_6(2/3) \\ \varphi_1(1) & \varphi_2(1) & \varphi_3(1) & \varphi_4(1) & \varphi_5(1) & \varphi_6(1) \\ \varphi'_1(0) & \varphi'_2(0) & \varphi'_3(0) & \varphi'_4(0) & \varphi'_5(0) & \varphi'_6(0) \end{bmatrix}; \quad (4.37)$$

For the case $N_{ie} = 4$ we have

$$\bar{y}^{ie} = T_4 \bar{\alpha}, \quad (4.38)$$

$$T_4 = D_4 \begin{bmatrix} \varphi_1(0) & \varphi_2(0) & \varphi_3(0) & \varphi_4(0) \\ \varphi'_1(0) & \varphi'_2(0) & \varphi'_3(0) & \varphi'_4(0) \\ \varphi_1(1) & \varphi_2(1) & \varphi_3(1) & \varphi_4(1) \\ \varphi'_1(1) & \varphi'_2(1) & \varphi'_3(1) & \varphi'_4(1) \end{bmatrix}. \quad (4.41)$$

where

$$\bar{y}^{ie} = [y_1 \quad \frac{dy_1}{dx} \quad y_2 \quad \frac{dy_2}{dx}]^T; \quad (4.39)$$

Similarly, we get

$$\bar{\alpha} = [\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4]^T; \quad (4.35)$$

$$\bar{z}_{ie} = T_{N_{ie}} \bar{\beta} \quad (4.42)$$

$$D = \text{diag}(1 \quad 1/h_e \quad 1 \quad 1/h_e); \quad (4.40)$$

for $N_{ie} = 8, N_{ie} = 6, N_{ie} = 4$.

From (4.28), (4.33), (4.38) and (4.42) it follows

$$\bar{\alpha} = T_{N_{ie}}^{-1} \bar{y}^{ie}; \quad \bar{\beta} = T_{N_{ie}}^{-1} \bar{z}^{ie}. \quad (4.43)$$

We have the following chain of equalities

$$(K_{\alpha\beta} \bar{\alpha}, \bar{\beta}) = (K_{\alpha\beta} T_{N_{ie}}^{-1} \bar{y}^{ie}, T_{N_{ie}}^{-1} \bar{z}^{ie}) = ((T_{N_{ie}}^{-1})^T K_{\alpha\beta} T_{N_{ie}}^{-1} \bar{y}^{ie}, \bar{z}^{ie}). \quad (4.44)$$

Therefore, substituting (4.43) sequentially in (4.16), (4.18), (4.20), (4.22), (4.24), we obtain local stiffness matrices K_0^{ie} , K_4^{ie} , K_{21}^{ie} , K_{23}^{ie} , K_{22}^{ie} , K_2^{ie} , corresponding to the operators L_0 , L_4 , L_{21} , L_{23} , L_{22} , L_2 .

5. EXAMPLE OF ANALYSIS

5.1. Formulation of the problem.

Let us consider the problem of the bending of a thin plate rigidly fixed along the side faces under the influence of a load concentrated in the center as an example (Figure 5.1).

Let us consider the following geometric parameters: $L_1 = 0.9$ m, $L_2 = 1.0$ m, $h = 0.05$ m is the thickness. Let us consider the following design parameters of material of plate: coefficient of elasticity $E = 3000 \cdot 10^4$ kN/m², Poisson's ratio $\nu = 0.16$. Let external load parameter be equal to $P = 1$ kN.

5.2. Structural analysis with allowance for localization.

Let the number of elements be equal to $N_e = 6$. Then we have the following element length:

$$h_e = L_1 / N_e = 0.9 / 6 = 0.15.$$

Let's define localization in the load area.

For the first element and for the sixth element we have $N_k = 1$ and third-order spline; distance between the coordinates of the nodes of the first element and the sixth element is equal to

$$h_1 = h_6 = 0.15 / 1 = 0.15.$$

For the second element and for the fifth element we have $N_k = 3$ and fifth-order spline; distance between the coordinates of the nodes of the second element and the fifth element is equal to

$$h_2 = h_5 = 0.15 / 3 = 0.05.$$

For the third element and for the fourth element we have $N_k = 5$ and seventh-order spline; distance between the coordinates of the nodes of the third element and the fourth element is equal to

$$h_3 = h_4 = 0.15 / 5 = 0.03.$$

The total number of inner nodes is equal to

$$N_p = 2(4 + 2 + 0) = 12.$$

The total number of boundary nodes is equal to

$$N_b = N_e + 1 = 7.$$

The total number of nodes for all elements is equal to

$$N_x = N_p + N_b = 12 + 7 = 19.$$

Figure 5.1. Example of analysis.

The number of nodal unknowns for each component of the vector function $y_j, j = 1, 2, 3, 4$ is equal to

$$N_g = N_p + 2N_b = 12 + 2 \cdot 7 = 26.$$

The total number of unknowns is equal to

$$N_U = 4N_g = 4 \cdot 26 = 104.$$

Figures 5.2. Comparison of the results of analysis in the middle sections along x_1 direction (discrete direction).

Figures 5.3. Comparison of the results of analysis in the middle sections along x_2 direction (continual direction).

5.3. Structural analysis without localization.

In this case, we will consider only the standard cubic fulfilment. In this case, the length of the element is taken equal to the minimum distance between the nodes, i.e. $h_e = 0.03$. Then the number of elements is equal to

$$N_e = 0.9/0.03 = 30$$

and the total number of nodes is equal to $N_x = 31$. In this case, the number of nodal unknowns

for each component of the vector function $y_j, j = 1, 2, 3, 4$ is equal to

$$N_g = 2 \cdot N_x = 2 \cdot 31 = 62$$

and the total number of nodal unknowns is equal to

$$N_U = 4N_g = 4 \cdot 62 = 248.$$

Graphical comparison of corresponding results of analysis is presented at Figures 5.2 and 5.3

(loc-spline are nodal values computed with allowance for localization; cub are nodal values computed without localization).

As researcher can see, the results obtained are almost completely identical. Besides, the use of localization based on application of B-splines of various degrees leads to a significant decrease in the number of unknowns. The difference for this example is equal to

$$N_{Ucub} - N_{Uloc} = 248 - 104 = 144.$$

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