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UNSYMMETRIC OSCILLATIONS OF ANISOTROPIC PLATE HAVING AN ADDITIONAL MASS

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Abstract: The problem of unsymmetric oscillations of circular plate made from anisotropic material is examined. The plate under consideration has an additional point mass attached off the center or a system of additional masses. Also the oscillations of anisotropic circular plate with a point support placed off the center are studied. The exact analytical approach is used for the decision of the above-mentioned problems; the method of compensating loads is applied. For this aim the basic and the compensating solutions are received. The basic solution satisfies to the resolving differential equation of the problem under study. The compensating solution satisfies to the corresponding homogeneous equation and this solution amounting to the basic one also satisfies to the boundary conditions. The Nielsen's equation is used for the receiving of the exact solutions expressed in terms of Bessel functions. The equation for determination of frequencies of natural vibrations is obtained.

Keywords: oscillations, circular anisotropic plate, additional mass, point support, Bessel functions.

НЕСИММЕТРИЧНЫЕ КОЛЕБАНИЯ АНИЗОТРОПНОЙ ПЛАСТИНЫ С ПРИСОЕДИНЕННОЙ МАССОЙ

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Аннотация: В работе изучаются несимметричные колебания круглой пластины, сделанной из анизотропного материала. Рассматриваемая пластина имеет дополнительную внецентренно расположенную массу или систему масс, или внецентренную опору точечного типа. Для получения решения поставленной задачи используется точный аналитический метод; используется метод компенсирующих нагрузок (МКН). Для этого строятся основное и компенсирующее решения. Основное решение удовлетворяет разрешающему дифференциальному уравнению. Компенсирующее решение удовлетворяет соответствующему однородному уравнению и совместно с основным решением удовлетворяет граничным условиям. Для получения решения однородного дифференциального уравнения, выраженного в цилиндрических функциях, применяется новый способ, связанный с использованием уравнения Нильсена. Для определения частот собственных колебаний получены соответствующие уравнения. Решения поставленных задач выражены в функциях Бесселя.

Ключевые слова: несимметричные колебания, круглые анизотропные пластины, присоединенные массы, точечные опоры, функции Бесселя.

INTRODUCTION

The analytical method for the solution of unsymmetric vibrations of anisotropic circular plate having an additional point mass attached off the center or a system of masses, also having a point support placed off the center is proposed. The method of compensating loads (MCL) is applied. The sought solutions are obtained in closed form and expressed in terms of Bessel functions.

The problem of oscillations anisotropic or inhomogeneous plates of various geometry is very urgent. The analytical method application, in particular, connected with the theory of special function for plates and shells analysis, is known in literature. The works [1]–[8] are devoted to the plates of variable thickness and shells examination; the solutions are received in terms of hypergeometric and confluent functions. The works [9]–[13] concern to the problems of statics and vibrations of isotropic plate, many of them are obtained in terms of cylindrical functions. In [14], [15] the analytical methods are applied for computation of plates and shells made from orthotropic material. In the monograph [16] for plates of variable thickness of different forms, subjected to the action of complicated loads, the theory of special functions is widely used; the solutions are expressed in terms of hypergeometric and confluent functions, Legendre functions, in orthogonal polynomials.

The present work concerns to unsymmetric oscillations of anisotropic circular plates with additional point mass attached off the center or a system of masses, also a point support placed off the center. The analytical approach and the method of compensating loads (MCL) are used. The new method, suggested in [17], [18], was also utilized. The mentioned method allows to resolve some statics and vibration problems of anisotropic circular plates and to receive the solutions in terms of Bessel functions. This method uses the Nielsen's equation and makes it possible to resolve some statics and vibration problems of anisotropic elastic solids.

The works [20], [21] are devoted to the investigation of symmetric vibrations of circular plates made from orthotropic material. The work [20] received the solution of the problem of the similar plates forced vibrations, the plates under study are subjected to an action of concentrated force $P\sin pt$. In the work [21] the action of loads distributed along concentric circumferences and over ring surfaces are considered.

In [23] the problem of circular orthotropic plates natural oscillations is considered. The plates under study have a point support in the center or a ring support or they have an additional mass in the center or a system of masses.

FORCED VIBRATIONS OF ORTHOTROPIC PLATE SUBJECTED TO AN ACTION OF A CONCENTRATED FORCE

First the auxiliary problem of unsymmetric vibrations of orthotropic plate subjected to an action of concentrated force $P\sin pt$, acting at the point A off the center is considered. Let us denote the distance from the center to the point A by α. We introduce the notation of the angle between the line OA and the fixed radius by θ. Further the method of compensating loads (MCL) [18], [20], [21] is applied. For this aim the basic solution W_0 and the compensating solution W_k

$$W = W_0 + W_k \tag{1}$$

are determined.

The basic solution W_0 satisfies to the differential equation, describing the problem, and shows the peculiarities of the external loads. The compensating solution W_k satisfies to the corresponding homogeneous equation and amounting to W_0 satisfies to the boundary conditions.

For the receiving of the solution of the resolving differential equation the Nielsen's equation is used. This method is described in detail in [20], [21]. Using the mentioned approach we can write the following expression for the basic solution in terms of Bessel functions:

$$W_0 = -\frac{P}{8Dn_2b^2}\left[Y_0(z)+\frac{2}{\pi}K_0(z)\right], \tag{2}$$

here $b = \sqrt[4]{\frac{\gamma h}{gn_2D}p_S^2}$,

where D is the cylindrical rigidity, γ – the volume weight, p_S is the circular frequency of natural vibrations, S is the member of nodal circles.

$$E_r = \frac{E}{n_1},\ E_\theta = En_2,\ \sigma_r = \frac{\sigma}{n^2},\ \sigma_\theta = \sigma,$$
$$n^2 = n_1n_2,$$

here $z = \sqrt{\alpha^2 + x^2 - 2\alpha x\cos(\theta-\varphi)}$.

Using the formulae of the Bessel functions addition [9], [21] we can write when $x \geq \alpha$:

$$W_0 = -\frac{P}{8Dn_2b^2} \times \\ \times 2\sum{}' \left[Y_n(x)J_n(\alpha) + \frac{2}{\pi}K_n(x)I_n(\alpha) \right] \cos n(\theta - \varphi). \quad (3)$$

The symbol $'$ indicates that when summing the series the term with zero index multiplies on $\frac{1}{2}$.

Let us write the compensating solution in the form of series:

$$W_k = \sum [A_n J_n(x) + B_n I_n(x)] \cos n(\theta - \varphi), \quad (4)$$

where J_n, I_n, Y_n, K_n are the Bessel functions.

The coefficients A_n and B_n are determined from the boundary conditions.

Let us assume that the contour of the plate under study is clamped; the reduced radius is equal to β. When $x = \beta$

$$W = 0, \ \frac{\partial W}{\partial x} = 0, \quad (5)$$

where W is determined from the formula (1).

The sum of the basic and the compensating solutions introduces in the conditions (5). Then the coefficients at the cosines of the identical arguments are equated and we receive the following equations for determination of the coefficients A_n and B_n:

$$-\frac{P}{4Dn_2b^2}\left[Y_n(\beta)J_n(\alpha) + \frac{2}{\pi}K_n(\beta)I_n(\alpha) \right] + \\ + A_n J_n(\beta) + B_n I_n(\beta) = 0, \quad (6)$$

$$-\frac{P}{4Dn_2b^2}\left[Y_n'(\beta)J_n(\alpha) + \frac{2}{\pi}K_n'(\beta)I_n(\alpha) \right] + \\ + A_n J_n'(\beta) + B_n I_n'(\beta) = 0. \quad (7)$$

Solving the system (6), (7) and using the expression for Bessel functions Wronskian [9], [23], we get:

$$A_n = \frac{P}{4Dn_2b^2} \frac{J_n(\alpha)[-Y_n(\beta)I_n'(\beta) + Y_n'(\beta)I_n(\beta)]\frac{2}{\pi\beta}I_n(\alpha)}{I_n(\beta)J_n'(\beta) - J_n(\beta)I_n'(\beta)}, \quad (8)$$

$$B_n = \frac{P}{4Dn_2b^2} \frac{\frac{2}{\pi}I_n(\alpha)[J_n'(\beta)K_n(\beta) - J_n(\beta)K_n'(\beta)] - \frac{2}{\pi\beta}J_n(\alpha)}{I_n(\beta)J_n'(\beta) - J_n(\beta)I_n'(\beta)}. \quad (9)$$

UNSYMMETRIC VIBRATIONS OF ANISOTROPIC PLATE WITH ADDITIONAL JOINT MASS OR WITH SEVERAL MASSES

Further the problem of oscillations of anisotropic circular plate with additional mass M in the point $(\alpha, 0)$ is considered. The equation for frequencies [22] has the following form:

$$M\omega^2 W(\alpha,0;\alpha,0) = 1. \quad (10)$$

Substituting instead W the expression received above for the forced vibrations consideration, we can receive the corresponding equation for frequencies. Setting the right part of the received equation equal to zero we obtain the equation for frequencies for the case when the joint mass attached off the center is extremely great. This fact is identical for the case of the plate with single rigid support of the point type placed off the center. We mark that for the case of the zero term of the series representing basic and compensating solutions, we get the frequency equation for the plate having ring intermediate support.

The natural oscillations of the circular orthotropic plate with several additional masses ($n = 1,2,3,...$) applied in the points with the coordinates (α_n, θ_n) are examined. For this case we have the following equation for frequencies:

$$\begin{vmatrix} \omega^2 M_1 w(\alpha_1,\theta_1;\alpha_1,\theta_1;\omega)-1 & \ldots & \omega^2 M_l w(\alpha_1,\theta_1;\alpha_l,\theta_l;\omega) \\ \omega^2 M_1 w(\alpha_2,\theta_2;\alpha_1,\theta_1;\omega) & \ldots & \omega^2 M_l w(\alpha_2,\theta_2;\alpha_l,\theta_l;\omega) \\ \ldots\ldots & \ldots & \ldots\ldots \\ \omega^2 M_1 w(\alpha_l,\theta_l;\alpha_1,\theta_1;\omega) & \ldots & \omega^2 M_l w(\alpha_l,\theta_l;\alpha_l,\theta_l;\omega)-1 \end{vmatrix} = 0 . \qquad (11)$$

For the study of the problem of oscillation of orthotropic circular plate with elastic supports of the point type the frequency equation has the form which is similar to (11). However, we must assume in the frequency equation that $M_n = -\dfrac{D_n}{\omega^2}$, where D_n is the rigidity of the support.

When solving the vibration problem of the orthotropic circular plate with several additional masses uniformly distributed along a circumference the system under study will have the same number of axes of symmetry as the number of joint masses. In this case the frequency equation becomes more simple.

Let us consider the following computation example: the forced vibration of the circular orthotropic plate which contour is clamped and the radius is equal to b. This plate is loaded by the force $P\sin pt$ which applied at the distance a = 0,5b from the center.

For computation it is assumed that the plate's radius b = 3m, the plate's thickness h = 0,2m, modulus of elasticity $E = 2 \cdot 10^6 t/m^2$, Poisson's ratio $\sigma = \frac{1}{6}$; we put $n_2 = 1$.

The calculations show that it is sufficient to retain four terms of the series.

The results of evaluation are given in the table 1.

Table 1. The values of the deflections

Point (x; ϕ)	0; 0	0,2; 0	0; 0	2,0; 0	0,2; $\frac{\pi}{2}$	1,0; $\frac{\pi}{2}$	2,0; $\frac{\pi}{2}$	0,2; π	1,2; π	2,0; π
w	0,3071	0,3475	0,3895	0,0013	0,2916	0,1448	0,000	0,2641	0,0981	0,0003

The first line shows the coordinates of the points where the plate's deflections are determined. In this table the fixed value $\dfrac{P}{8Dn_2b^2}$ is omitted.

We note that the function w reduces to infinity when

$$I_n(\beta)J_n'(\beta) - I_n'(\beta)J_n(\beta) = 0 . \qquad (12)$$

In this case we have the phenomenon of resonance; the corresponding frequency of forced vibrations is

$$P = \sqrt{\frac{Dn_2 g}{\gamma h}} \frac{\beta_n^2}{b^2},$$

where β_n are the roots of the equation (12).

CONCLUSIONS

The exact analytical solutions of the problem of unsymmetric vibrations of the circular plate made from orthotropic material are obtained. An action of a force applied in arbitrary point, an action of a joint additional mass attached off the center or a system of masses, an influence of a point support, placed of the center are considered.

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