DOI:10.22337/2587-9618-2020-16-4-14-28

NUMERICAL SOLUTION OF THE PROBLEM OF ISOTROPIC PLATE ANALYSIS WITH THE USE OF B-SPLINE DISCRETE-CONTINUAL FINITE ELEMENT METHOD

Pavel A. Akimov [1, 2, 3, 4], Marina L. Mozgaleva [1], Taymuraz B. Kaytukov [1]

[1] National Research Moscow State University of Civil Engineering, Moscow, RUSSIA

[2] Tomsk State University of Architecture and Civil Engineering, Tomsk, RUSSIA

[3] Peoples' Friendship University of Russia, Moscow, RUSSIA

[4] Russian Academy of Architecture and Construction Sciences, Moscow, RUSSIA

Abstract: Numerical solution of the problem of isotropic plate analysis with the use of B-spline discrete-continual finite element method (specific version of wavelet-based discrete-continual finite element method) is under consideration in the distinctive paper. The original operational continual and discrete-continual formulations of the problem are given, some actual aspects of construction of normalized basis functions of a B-spline are considered, the corresponding local constructions for an arbitrary discrete-continual finite element are described, some information about the numerical implementation and an example of analysis are presented.

Keywords: wavelet-based discrete-continual finite element method, B-spline discrete-continual finite element method, discrete-continual finite element method, finite element method, B-spline, numerical solution, isotropic plate, plate analysis

ЧИСЛЕННОЕ РЕШЕНИЕ ЗАДАЧИ О ПОПЕРЕЧНОМ ИЗГИБЕ ИЗОТРОПНОЙ ПЛАСТИНЫ НА ОСНОВЕ ВЕЙВЛЕТ-РЕАЛИЗАЦИИ ДИСКРЕТНО-КОНТИНУАЛЬНОГО МЕТОДА КОНЕЧНЫХ ЭЛЕМЕНТОВ С ИСПОЛЬЗОВАНИЕМ B-СПЛАЙНОВ

П.А. Акимов [1, 2, 3, 4], М.Л. Мозгалева [1], Т.Б. Кайтуков [1]

[1] Национальный исследовательский Московский государственный строительный университет, г. Москва, РОССИЯ

[2] Томский государственный архитектурно-строительный университет, г. Томск, РОССИЯ

[3] Российский университет дружбы народов, г. Москва, РОССИЯ

[4] Российская академия архитектуры и строительных наук, г. Москва, РОССИЯ

Аннотация: В настоящей статье рассматривается численное решение задачи о поперечном изгибе изотропной пластины на основе вейвлет-реализации дискретно-континуального метода конечных элементов с использованием B-сплайнов. Приведены исходные операторные континуальная и дискретно-континуальная постановки задачи, рассмотрены некоторые актуальные вопросы построения нормализованных базисных функций B-сплайна, описаны соответствующие локальные построения для произвольного дискретно-континуального конечного элемента, представлены некоторые сведения о численной реализации и пример расчета.

Ключевые слова: вейвлет-реализация метода конечных элементов, дискретно-континуальный метод конечных элементов, метод конечных элементов, B-сплайны, численное решение, изотропная пластина, изгиб изотропной пластины

INTRODUCTION

As is known [1], the B-spline in a given simple knot sequence can be constructed by employing piecewise polynomials between the knots and joining them together at the knots [1, 2].

For instance, compared with commonly used Daubechies wavelets [3-7] B-spline wavelet on interval (BSWI) has explicit expressions, facilitating the calculation of coefficient integration and differentiation [1, 2]. Besides, the multiresolution and localization properties of BSWI can also supply some superiority for engineering structural analysis [1, 2]. The early applications of spline can be found in papers of H. Antes [8], J.G. Han [9, 10, 26], Y. Huang [9, 10], W.X. Ren [9, 10]. The spline wavelet finite element method was further developed in papers of D.P. Chen [27], X.F. Chen [11, 12, 14-17, 22, 23, 25], H.B. Dong [22], J.G. Han [24], Y.M. He [16], Z.H. He [17], Z.J. He [11, 12, 14-16, 22, 23, 25], Y. Huang [24, 26], Z.S. Jiang [21], B. Li [12, 14, 16, 22], M. Liang [18, 20], J.Q. Long [19], G. Ma [19], T. Matsumoto [19, 21], S.T. Mau [29], H.H. Miao [14], Q,M. Mo [17], T.H.H. Pian [27-29], K.Y. Qi [22], W.X. Ren [24, 26], K. Sumihara [28], P. Tong [29], Y.W. Wang [21], J.W. Xiang [11-13, 16-21], Z.B. Yang [14, 15, 23], X.W. Zhang [15, 23, 25], Y.H. Zhang [11], Y.T. Zhong [13].

Generally the structural analysis normally require accurate computer-intensive calculations using numerical (discrete) methods. The field of application of discrete-continual finite element method (DCFFEM), proposed by A.B. Zolotov [32] and P.A. Akimov [30-32] comprises structures with regular (in particular, constant or piecewise constant) physical and geometrical parameters in some dimension (so-called “basic” direction (dimension)). Considering problems remain continual along “basic” direction while along other directions DCFEM presupposes finite element approximation. Solution of corresponding resultant multipoint boundary problems [33] for systems of ordinary differential equations with piecewise constant coefficients and immense number of unknowns is the most time-consuming stage of the computing, especially if we take into account the limitation in performance of personal computers, contemporary software and necessity to obtain correct semianalytical solution in a reasonable time.

High-accuracy solution at all points of the model is not required normally, it is necessary to find only the most accurate solution in some pre-known domains. Generally the choice of these domains is a priori data with respect to the structure being modelled. Designers usually choose domains with the so-called edge effect (with the risk of significant stresses that could potentially lead to the destruction of structures, etc.) and regions which are subject to specific operational requirements. It is obvious that the stress-strain state in such domains is of paramount importance. Specified factors along with the obvious needs of the designer or researcher to reduce computational costs by application of DCFEM cause considerable urgency of constructing of special algorithms for obtaining local solutions (in some domains known in advance) of boundary problems. Wavelet analysis provides effective and popular tool for such researches. Solution of the considering problem within multilevel wavelet analysis is represented as a composition of local and global components. Wavelet-based DCFEM is presented in papers of P.A. Akimov [34-41], M. Aslami [37-39], T.B. Kaytukov, M.L. Mozgaleva [34-41] and O.A. Negrozov [37-39].

The distinctive paper is devoted to numerical solution of the problem of isotropic plate analysis with the use of B-spline DCFEM.

1. FORMULATIONS OF THE PROBLEM

Let the constancy of the parameters of the problem be in the direction corresponding to (main direction). The operational formulation of the problem with the use of so-called method of extended domain [42], taking into account the selection of the main direction, is determined by the equation:

$$L\,y = \widetilde{F}\,, \quad 0 \le x_1 \le \ell_1\,,\; 0 \le x_2 \le \ell_2\,, \quad (1.1)$$

where we have

$$L = -L_4\,\partial_2^4 + L_2\,\partial_2^2 + L_0\,; \qquad (1.2)$$

$$L_4 = \theta D\,; \qquad (1.3)$$

$$L_2 = -[\partial_1^2 \theta D\,\nu + 2\partial_1 \theta D(1-\nu)\partial_1 + \theta D \nu \partial_1^2]\,; \qquad (1.4)$$

$$L_0 = -\partial_1^2 \theta D \partial_1^2\,; \qquad (1.5)$$

$$\widetilde{F} = \theta F + \delta_\Gamma f\,; \qquad (1.6)$$

$$\theta(x_1,x_2) = \begin{cases} 1, & 0<x_1<\ell_1 \wedge 0<x_2<\ell_2 \\ 0, & \neg(0<x_1<\ell_1 \wedge 0<x_2<\ell_2); \end{cases} \qquad (1.7)$$

$$\delta_\Gamma(x_1,x_2) = \partial\theta/\partial\overline{n}\,; \qquad (1.8)$$

Ω is the domain, occupied by plate; ℓ_1, ℓ_2 are corresponding dimensions of extended domain (linear dimensions of plate); $x = (x_1, x_2)$; x_1, x_2 are Cartesian coordinates; $\theta(x_1, x_2)$ is characteristic function of domain Ω; $\delta_\Gamma = \delta_\Gamma(x_1, x_2)$ is the delta function of boundary $\Gamma = \partial\Omega$; $\overline{n} = [n_1\, n_2]^T$ is boundary normal vector; y is deflection of plate; D is flexural rigidity of plate; ν is Poisson's ratio of plate; F is the load in domain Ω; $\overline{f}$ is the corresponding boundary load; $\partial_s = \partial / \partial x_s$, $s = 1, 2$. Let us introduce the following notations

$$y_1 = y\,,\quad y_2 = \partial_2 y = y_1'\,,\quad y_3 = \partial_2^2 y = y_2'\,,\quad y_4 = \partial_2^3 y = y_3'\,. \qquad (1.9)$$

Thus we can rewrite (1.1):

$$-L_4 y_4' + L_2 y_3 + L_0 y_1 = \widetilde{F}\,. \qquad (1.10)$$

Finally we obtain system of differential equations with operational coefficients:

$$\begin{bmatrix} y_1' \\ y_2' \\ y_3' \\ y_4' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ L_4^{-1}L_0 & 0 & L_4^{-1}L_2 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -L_4^{-1}\widetilde{F} \end{bmatrix}, \qquad (1.11)$$

or

$$\overline{U}' = \widetilde{L}\,\overline{U} + \widetilde{\overline{F}}\,, \qquad (1.12)$$

where

$$\widetilde{L} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ L_4^{-1}L_0 & 0 & L_4^{-1}L_2 & 0 \end{bmatrix}; \qquad (1.13)$$

$$\widetilde{\overline{F}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -L_4^{-1}\widetilde{F} \end{bmatrix}; \quad \overline{U} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}. \qquad (1.14)$$

The system of equations (1.11) is supplemented by boundary conditions, which are set in sections with coordinates $x_2^1 = 0$ and $x_2^2 = \ell_2$.

2. SOME ASPECTS OF THE CONSTRUCTION OF NORMALIZED BASIS FUNCTIONS OF THE B-SPLINE

The construction of B-spline basic functions is determined by the recursive Cox-de Boer formulas [1]:

$$k=1: \quad \varphi_{i,1}(t) = \begin{cases} 1\,, & x_i \le t < x_{i+1} \\ 0\,, & t < x_i \vee t \ge x_{i+1} \end{cases}, \qquad (2.1)$$

$$k \ge 2: \quad \varphi_{i,k}(t) = \frac{(t-x_i)\varphi_{i,k-1}(t)}{x_{i+k-1}-x_i} + \frac{(x_{i+k}-t)\varphi_{i+1,k-1}(t)}{x_{i+k}-x_{i+1}}. \qquad (2.2)$$

We will consider such a construction for the case $x_i = i$ are integers. Let us note that,

$$\varphi_{i,k}(t) = \varphi_{0,k}(t-i)$$

and therefore, recursive formulas (2.1)-(2.2) can be represented in the form

$$k=1: \qquad \varphi_{0,1}(t) = \begin{cases} 1\,, & 0 \le t < 1 \\ 0\,, & t<0 \vee t \ge 1; \end{cases} \qquad (2.3)$$

$$k \ge 2: \qquad \varphi_{0,k}(t) = \frac{1}{k-1}[t\cdot\varphi_{0,k-1}(t) + (k-t)\varphi_{0,k-1}(t-1)]. \qquad (2.4)$$

The function $\varphi_{0,1}(t)$ can be represented by formula

$$\varphi_{0,1}(t) = \frac{1}{2}[\mathrm{sign}(t) - \mathrm{sign}(t-1)]\,. \qquad (2.5)$$

Let us denote by the operator of the first difference. Then we have

$$\varphi_{0,1}(t) = -\frac{1}{2}\Delta_1 \operatorname{sign}(t). \qquad (2.6)$$

We can substitute formula (2.5) into (2.4) in order to determine $\varphi_{0,2}(t)$:

$$\begin{aligned}\varphi_{0,2}(t) &= 1\cdot[t\cdot\varphi_{0,1}(t) + (2-t)\varphi_{0,1}(t-1)] = \\ &= \frac{1}{2}\{t\cdot[\operatorname{sign}(t) - \operatorname{sign}(t-1)] + \\ &(2-t)[\operatorname{sign}(t-1) - \operatorname{sign}(t-2)]\} = \\ &= \frac{1}{2}[t\operatorname{sign}(t) - 2(t-1)\operatorname{sign}(t-1) + \\ &(t-2)\operatorname{sign}(t-2)] = \frac{1}{2}[|t| - 2|t-1| + |t-2|.\end{aligned}$$

Let us denote by Δ_2 the operator of the second difference. Then we have

$$\varphi_{0,2}(t) = \frac{1}{2}[|t| - 2|t-1| + |t-2| = \frac{1}{2}\Delta_2 |t-1|. \qquad (2.7)$$

We can define function $\varphi_{0,3}(t)$:

$$\varphi_{0,3}(t) = \frac{1}{2}[t\cdot\varphi_{0,2}(t) + (3-t)\varphi_{0,2}(t-1)].$$

Omitting intermediate calculations, we get

$$\begin{aligned}\varphi_{0,3}(t) &= \frac{1}{4}[t\cdot|t| - 3(t-1)|t-1| + \\ &+ 3(t-2)|t-2| - (t-3)|t-3|] = \\ &= -\frac{1}{2!}\frac{1}{2}\Delta_1\Delta_2((t-1)|t-1|). \qquad (2.8)\end{aligned}$$

Based on formulas (2.8) and (2.4), we can define the function

$$\varphi_{0,4}(t) = \frac{1}{3}[t\cdot\varphi_{0,3}(t) + (4-t)\varphi_{0,3}(t-1)].$$

Omitting intermediate calculations, as a result we get

$$\begin{aligned}\varphi_{0,4}(t) &= \\ &= \frac{1}{2\cdot 3}\cdot\frac{1}{2}[t^2\cdot|t| - 4(t-1)^2|t-1| + \\ &+ 6(t-2)^2|t-2| - 4(t-3)^2|t-3| + \\ &+ (t-4)^2|t-4|] = \\ &= \frac{1}{3!}\frac{1}{2}(\Delta_2)^2((t-2)^2|t-2|). \qquad (2.9)\end{aligned}$$

It can be proved that for even $k = 2m$ we have

$$\varphi_{0,k}(t) = \frac{1}{(2m-1)!}\frac{1}{2}(\Delta_2)^m((t-m)^{2m-2}|t-m|) \qquad (2.10)$$

and for odd (uneven) $k = 2m + 1$ we have

$$\varphi_{0,k}(t) = -\frac{1}{(2m)!}\frac{1}{2}\Delta_1(\Delta_2)^m((t-m)^{2m-1}|t-1|). \qquad (2.11)$$

Note that $\varphi_{0,k}(t)$ is a polynomial of degree $k – 1$ with bounded support and, as follows from the difference operator, this support is equal to the interval $[0, k]$. In addition, we should note the following property of B-spline basis functions:

$$\sum_i \varphi_{0,k}(t-i) \equiv 1 \text{ for arbitrary } t. \qquad (2.12)$$

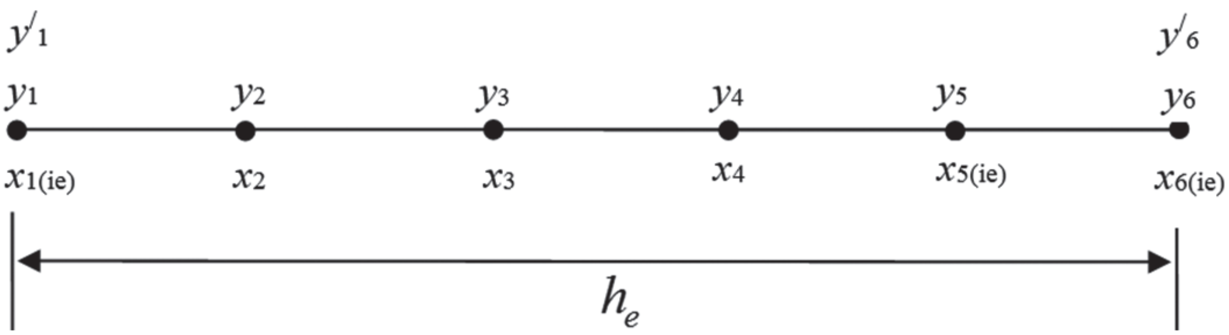


Figure 3.1. Finite element discretication (sample)

3. SOME GENERAL ASPECTS OF FINITE ELEMENT APPROXIMATION

The discrete component of the numerical solution is represented by the direction along the axis corresponding to x_1. The fulfillment within an element (interval) for all components of a vector function $\overline{U}$ (see (1.14)) is the same. Therefore, let us use the following notation for simplicity:

$$x = x_1,\ \ell = \ell_1,\ y = y_1,\ j = 1, 2, 3, 4. \qquad (3.1)$$

Let us divide the interval $(0, \ell)$ segment into N_e parts (elements). Therefore $h_e = \ell/N_e$ is the length of the element. Besides, let us also divide each element into N_k parts, for example, $N_k = 5$ (see Figure 3.1).

Let us use the following notation system: i_e is the element number; $x_1(i_e)$ is the coordinate of the starting point of the i_e-th element; $x_6(i_e)$ is the coordinate of the end point of the i_e-th element. We y_i and $y'_i = \partial_1 y(x_i)$, $i = 1{,}6$ take as unknowns at the boundary points. Besides, we take y_i, $i = 2, 3, 4, 5$ as unknowns and at the inner points. Thus, the number of unknowns per element with such a approximation is equal to

$$N = N_k - 1 + 2 \cdot 2 = N_k + 3 = 8.$$

4. LOCAL CONSTRUCTIONS FOR ARBITRARY FINITE ELEMENT

Let us introduce local coordinates:

$$t = \frac{x - x_{1(i_e)}}{h_e},\ x_{1(i_e)} \le x \le x_{6(i_e)},\ 0 \le t \le 1. \qquad (4.1)$$

In this case, we have the following relations:

$$\begin{cases} x = x_{1(i_e)} \Rightarrow t = 0 \\ x = x_2 \Rightarrow t = 0.2 \\ x = x_3 \Rightarrow t = 0.4 \\ x = x_4 \Rightarrow t = 0.6 \\ x = x_5 \Rightarrow t = 0.8 \\ x = x_{6(i_e)} \Rightarrow t = 1 \end{cases}, \qquad \begin{matrix} \dfrac{d}{dx} = \dfrac{d}{dt} \cdot \dfrac{dt}{dx} = \dfrac{1}{h_e}\dfrac{d}{dt} \\ \dfrac{d^p}{dx^p} = \dfrac{1}{h_e^p}\dfrac{d^p}{dt^p} \end{matrix},$$

$$dx = h_e \cdot dt. \qquad (4.2)$$

Since the number of unknowns on the element is equal to $N = 8$, we use a B-spline of the seventh degree in order to represent the unknown deflection function.

Let us use the following notation:

$$\varphi(t) = \varphi_{0,8}(t+4);$$

$$\begin{aligned} \varphi(t) = \frac{1}{7!}\frac{1}{2}(\Delta_2)^4(t^6 \mid t \mid) = \\ = \frac{1}{2 \cdot 7!}[(t+4)^6 \mid t+4 \mid - \\ - 8(t+3)^6 \mid t+3 \mid + \\ + 28(t+2)^6 \mid t+2 \mid - \\ - 56(t+1)^6 \mid t+1 \mid + 70t^6 \mid t \mid - \\ - 56(t-1)^6 \mid t-1 \mid + 28(t-2)^6 \mid t-2 \mid - \\ - 8(t-3)^6 \mid t-3 \mid + (t-4)^6 \mid t-4 \mid]. \end{aligned} \qquad (4.3)$$

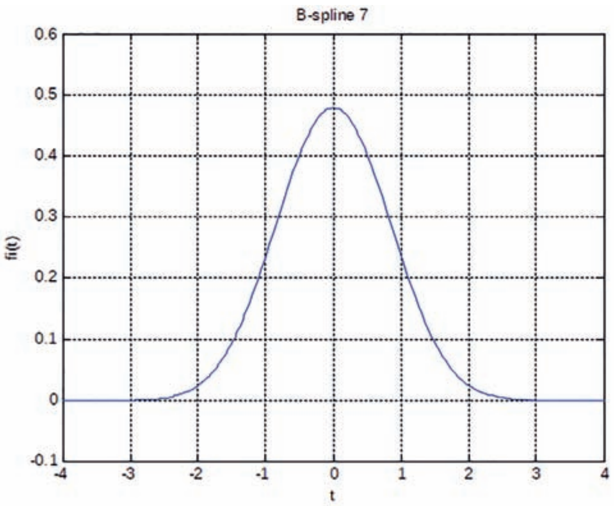


Figure 4.1. B-spline of the seventh order $\varphi(t) = \varphi_{0,8}(t+4)$

Let us use the following notation system:

$$\begin{gathered} \varphi_1(t) = \varphi(t+3),\ \varphi_2(t) = \varphi(t+2), \\ \varphi_3(t) = \varphi(t+1),\ \varphi_4(t) = \varphi(t), \\ \varphi_5(t) = \varphi(t-1), \\ \varphi_6(t) = \varphi(t-2),\ \varphi_7(t) = \varphi(t-3), \\ \varphi_8(t) = \varphi(t-4),\ 0 \le t \le 1. \end{gathered} \qquad (4.4)$$

We represent the unknown deflection function in the form

$$y(x)=w(t)=\sum_{k=1}^{N}\alpha_k\varphi_k(t),\quad x_{1(i_e)}\le x\le x_{6(i_e)},\quad 0\le t\le 1.\quad (4.5)$$

We have to consider bilinear forms with allowance for relations (4.2) in order to construct local stiffness matrices corresponding to the operators L_0, L_2 and L_4 (see (1.3)-(1.5)):

$$B_0(y,z)=<L_0y,z>=-\int_{x_{1(i_e)}}^{x_{5(i_e)}}\frac{d^2}{dx^2}\theta D\frac{d^2y}{dx^2}\cdot z\,dx=$$

$$=-\theta_{i_e}D_{i_e}\int_{x_{1(i_e)}}^{x_{5(i_e)}}\frac{d^2y}{dx^2}\cdot\frac{d^2z}{dx^2}\,dx=$$

$$=-\frac{1}{h_e^3}\theta_{i_e}D_{i_e}\int_0^1\frac{d^2w}{dt^2}\cdot\frac{d^2v}{dt^2}\,dt=B_0(w,v);\ (4.6)$$

$$B_4(y,z)=<L_4y,z>=\theta_{i_e}D_{i_e}\int_{x_{1(i_e)}}^{x_{5(i_e)}}y\cdot z\,dx=$$

$$=h_e\theta_{i_e}D_{i_e}\int_0^1 w\cdot v\,dt=B_4(w,v);\quad (4.7)$$

$$B_2(y,z)=<L_2y,z>=<L_{21}y,z>+$$
$$+<L_{22}y,z>+<L_{23}y,z>,\quad (4.8)$$

where

$$<L_{21}y,z>=-\theta_{i_e}D_{i_e}\nu_{i_e}\int_{x_{1(i_e)}}^{x_{5(i_e)}}\frac{d^2y}{dx^2}z\,dx=$$

$$=-\frac{1}{h_e}\theta_{i_e}D_{i_e}\nu_{i_e}\int_0^1\frac{d^2w}{dt^2}\cdot v\,dt=B_{21}(w,v);\quad (4.9)$$

$$<L_{23}y,z>=-\theta_{i_e}D_{i_e}\nu_{i_e}\int_{x_{1(i_e)}}^{x_{5(i_e)}}y\cdot\frac{d^2z}{dx^2}\,dx=$$

$$=-\frac{1}{h_e}\theta_{i_e}D_{i_e}\nu_{i_e}\int_0^1 w\cdot\frac{d^2v}{dt^2}\,dt=B_{23}(w,v);\quad (4.10)$$

$$<L_{22}y,z>=-2\int_{x_{1(i_e)}}^{x_{5(i_e)}}\frac{d}{dx}\theta D(1-\nu)\frac{dy}{dx}\cdot z\,dx=$$

$$=2\theta_{i_e}D_{i_e}(1-\nu_{i_e})\int_{x_{1(i_e)}}^{x_{5(i_e)}}\frac{dy}{dx}\cdot\frac{dz}{dx}\,dx=$$

$$=\frac{1}{h_e}2\theta_{i_e}D_{i_e}(1-\nu_{i_e})\int_0^1\frac{dw}{dt}\cdot\frac{dv}{dt}\,dt=B_{22}(w,v).$$
$$(4.11)$$

for the following type of functions

$$y(x)=w(t)=\sum_{k=1}^{N}\alpha_k\varphi_k(t),$$
$$z(x)=v(t)=\sum_{k=1}^{N}\beta_k\varphi_k(t),$$
$$x_{1(i_e)}\le x\le x_{6(i_e)},\ 0\le t\le 1\ (4.12)$$

Let us substitute (4.12) into (4.6)-(4.11):

$$B_0(w,v)=-\frac{1}{h_e^3}\theta_{i_e}D_{i_e}\int_0^1\frac{d^2w}{dt^2}\cdot\frac{d^2v}{dt^2}\,dt=$$

$$=-\frac{\theta_{i_e}D_{i_e}}{h_e^3}\sum_{i=1}^{N}\sum_{j=1}^{N}\alpha_i\beta_j\int_0^1\varphi_i''(t)\varphi_j''(t)\,dt=$$

$$=-\frac{\theta_{i_e}D_{i_e}}{h_e^3}(K^0_{\alpha\beta}\bar{\alpha},\bar{\beta}),\quad (4.13)$$

where

$$K^0_{\alpha\beta}(i,j)=\int_0^1\varphi_i''(t)\varphi_j''(t)\,dt;\quad \varphi''=\frac{d^2\varphi}{dt^2};\quad (4.14)$$

$$B_4(w,v)=h_e\theta_{i_e}D_{i_e}\int_0^1 w\cdot v\,dt=$$

$$=\theta_{i_e}D_{i_e}h_e\sum_{i=1}^{N}\sum_{j=1}^{N}\alpha_i\beta_j\int_0^1\varphi_i(t)\varphi_j(t)\,dt=$$

$$=h_e\theta_{i_e}D_{i_e}(K^4_{\alpha\beta}\bar{\alpha},\bar{\beta}),\quad (4.15)$$

where

$$K^4_{\alpha\beta}(i,j)=\int_0^1\varphi_i(t)\varphi_j(t)\,dt;\quad (4.16)$$

$$B_{21}(w,v)=-\frac{1}{h_e}\theta_{i_e}D_{i_e}\nu_{i_e}\int_0^1\frac{d^2w}{dt^2}\cdot v\,dt=$$

$$=-\frac{\theta_{i_e}D_{i_e}\nu_{i_e}}{h_e}\sum_{i=1}^{N}\sum_{j=1}^{N}\alpha_i\beta_j\int_0^1\varphi_i''(t)\varphi_j(t)\,dt=$$

$$=-\frac{\theta_{i_e}D_{i_e}\nu_{i_e}}{h_e}(K^{21}_{\alpha\beta}\bar{\alpha},\bar{\beta}),\quad (4.17)$$

where

$$K^{21}_{\alpha\beta}(i,j)=\int_0^1 \varphi''_i(t)\varphi_j(t)\,dt\,; \qquad (4.18)$$

$$B_{23}(w,v)=-\frac{1}{h_e}\theta_{i_e}D_{i_e}\nu_{i_e}\int_0^1 w\cdot\frac{d^2v}{dt^2}\,dt=$$
$$=-\frac{\theta_{i_e}D_{i_e}\nu_{i_e}}{h_e}\sum_{i=1}^{N}\sum_{j=1}^{N}\alpha_i\beta_j\int_0^1\varphi_i(t)\varphi''_j(t)\,dt=$$
$$=-\frac{\theta_{i_e}D_{i_e}\nu_{i_e}}{h_e}(K^{23}_{\alpha\beta}\overline{\alpha},\overline{\beta}), \quad (4.19)$$

where

$$K^{23}_{\alpha\beta}(i,j)=\int_0^1\varphi_i(t)\varphi''_j(t)\,dt=K^{21}_{\alpha\beta}(j,i)\,; \quad (4.20)$$

$$B_{22}(w,v)=\frac{1}{h_e}2\theta_{i_e}D_{i_e}(1-\nu_{i_e})\int_0^1\frac{dw}{dt}\cdot\frac{dv}{dt}\,dt=$$
$$=2\frac{\theta_{i_e}D_{i_e}(1-\nu_{i_e})}{h_e}\sum_{i=1}^{N}\sum_{j=1}^{N}\alpha_i\beta_j\int_0^1\varphi'_i(t)\varphi'_j(t)\,dt=$$
$$=2\frac{\theta_{i_e}D_{i_e}(1-\nu_{i_e})}{h_e}(K^{22}_{\alpha\beta}\overline{\alpha},\overline{\beta}), \quad (4.21)$$

where

$$K^{22}_{\alpha\beta}(i,j)=\int_0^1\varphi'_i(t)\varphi'_j(t)\,dt\,, \quad \varphi'=\frac{d\varphi}{dt}. \quad (4.22)$$

Let us define the parameters α_k and β_k through the nodal unknowns on the element:

$$\begin{cases} y_1=w(0)=\sum_{k=1}^{N}\alpha_k\varphi_k(0) \\ \dfrac{dy_1}{dx}=\dfrac{1}{h_e}w'(0)=\dfrac{1}{h_e}\sum_{k=1}^{N}\alpha_k\varphi'_k(0) \\ y_2=w(0.2)=\sum_{k=1}^{N}\alpha_k\varphi_k(0.2) \\ y_3=w(0.4)=\sum_{k=1}^{N}\alpha_k\varphi_k(0.4) \\ y_4=w(0.6)=\sum_{k=1}^{N}\alpha_k\varphi_k(0.6) \\ y_5=w(0.8)=\sum_{k=1}^{N}\alpha_k\varphi_k(0.8) \\ y_6=w(1)=\sum_{k=1}^{N}\alpha_k\varphi_k(1) \\ \dfrac{dy_6}{dx}=\dfrac{1}{h_e}w'(1)=\dfrac{1}{h_e}\sum_{k=1}^{N}\alpha_k\varphi'_k(1) \end{cases} \qquad (4.23)$$

Therefor we have

$$\overline{y}^{i_e}=T\overline{\alpha}\,, \qquad (4.24)$$

where (see also (4.27))

$$\overline{y}^{i_e}=[y_1\ \ \frac{dy_1}{dx}\ \ y_2\ \ y_3\ \ y_4\ \ y_5\ \ y_6\ \ \frac{dy_6}{dx}]^{\mathrm{T}}\ ; \quad (4.25)$$

$$\overline{\alpha}=[\alpha_1\ \ \alpha_2\ \ \alpha_3\ \ \alpha_4\ \ \alpha_5\ \ \alpha_6\ \ \alpha_7\ \ \alpha_8]^{\mathrm{T}}\ ; \quad (4.26)$$

$$D=diag(1\ \ 1/h_e\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1/h_e)\,. \quad (4.27)$$

$$T=D\begin{bmatrix} \varphi_1(0) & \varphi_2(0) & \varphi_3(0) & \varphi_4(0) & \varphi_5(0) & \varphi_6(0) & \varphi_7(0) & \varphi_8(0) \\ \varphi'_1(0) & \varphi'_2(0) & \varphi'_3(0) & \varphi'_4(0) & \varphi'_5(0) & \varphi'_6(0) & \varphi'_7(0) & \varphi'_8(0) \\ \varphi_1(0.2) & \varphi_2(0.2) & \varphi_3(0.2) & \varphi_4(0.2) & \varphi_5(0.2) & \varphi_6(0.2) & \varphi_7(0.2) & \varphi_8(0.2) \\ \varphi_1(0.4) & \varphi_2(0.4) & \varphi_3(0.4) & \varphi_4(0.4) & \varphi_5(0.4) & \varphi_6(0.4) & \varphi_7(0.4) & \varphi_8(0.4) \\ \varphi_1(0.6) & \varphi_2(0.6) & \varphi_3(0.6) & \varphi_4(0.6) & \varphi_5(0.6) & \varphi_6(0.6) & \varphi_7(0.6) & \varphi_8(0.6) \\ \varphi_1(0.8) & \varphi_2(0.8) & \varphi_3(0.8) & \varphi_4(0.8) & \varphi_5(0.8) & \varphi_6(0.8) & \varphi_7(0.8) & \varphi_8(0.8) \\ \varphi_1(1) & \varphi_2(1) & \varphi_3(1) & \varphi_4(1) & \varphi_5(1) & \varphi_6(1) & \varphi_7(1) & \varphi_8(1) \\ \varphi'_1(1) & \varphi'_2(1) & \varphi'_3(1) & \varphi'_4(1) & \varphi'_5(1) & \varphi'_6(1) & \varphi'_7(1) & \varphi'_8(1) \end{bmatrix}, \quad (4.28)$$

Similarly, we get

$$\overline{z}^{i_e}=T\overline{\beta}\,. \qquad (4.29)$$

From (4.23) and (4.28) it follows

$$\overline{\alpha}=T^{-1}\overline{y}^{i_e}\,; \quad \overline{\beta}=T^{-1}\overline{z}^{i_e}\,. \qquad (4.30)$$

We have the following chain of equalities

$$(K_{\alpha\beta}\overline{\alpha},\overline{\beta}) = (K_{\alpha\beta}T^{-1}\overline{y}^{i_e},T^{-1}\overline{z}^{i_e}) = \\ = ((T^{-1})^{\mathrm{T}}K_{\alpha\beta}T^{-1}\overline{y}^{i_e},\overline{z}^{i_e}). \quad (4.31)$$

Therefore, substituting (4.30) sequentially in (4.13), (4.15), (4.17), (4.19), (4.21), we obtain local stiffness matrices $K_0^{i_e}$, $K_4^{i_e}$, $K_{21}^{i_e}$, $K_{23}^{i_e}$, $K_{22}^{i_e}$, $K_2^{i_e}$, corresponding to the operators L_0, L_4, L_{21}, L_{23}, L_{22}, L_2.

$$K_0^{i_e} = -\frac{\theta_{i_e}D_{i_e}}{h_e^3}(T^{-1})^{\mathrm{T}}K_{\alpha\beta}^0T^{-1}; \quad (4.32)$$

$$K_4^{i_e} = h_e\theta_{i_e}D_{i_e}(T^{-1})^{\mathrm{T}}K_{\alpha\beta}^4T^{-1}; \quad (4.33)$$

$$K_{21}^{i_e} = -\frac{\theta_{i_e}D_{i_e}\nu_{i_e}}{h_e}(T^{-1})^{\mathrm{T}}K_{\alpha\beta}^{21}T^{-1}; \quad (4.34)$$

$$K_{23}^{i_e} = -\frac{\theta_{i_e}D_{i_e}\nu_{i_e}}{h_e}(T^{-1})^{\mathrm{T}}K_{\alpha\beta}^{23}T^{-1} = (K_{21}^{i_e})^{\mathrm{T}}; \quad (4.35)$$

$$K_{22}^{i_e} = 2\frac{\theta_{i_e}D_{i_e}(1-\nu_{i_e})}{h_e}(T^{-1})^{\mathrm{T}}K_{\alpha\beta}^{22}T^{-1}. \quad (4.36)$$

Since $L_2 = L_{21} + L_{22} + L_{23}$ the corresponding local stiffness matrix has the form:

$$K_2^{i_e} = K_{21}^{i_e} + K_{22}^{i_e} + K_{23}^{i_e}. \quad (4.37)$$

5. INFORMATION ABOUT NUMERICAL IMPLEMENTATION

The presented algorithm can be implemented using MATLAB tools. The MATLAB system has convenient functions for working with polynomials. Moreover, the main parameter of these functions is the vector of coefficients of the polynomial. To determine the coefficients of basic polynomials φ_k on an interval [0 1], we can firstly determine their values at eight points of the interval $t = [t_1, t_2, ..., t_8]$, $t_i \in [0\ \ 1]$, $i = 1, 2, ..., 8$:

$$F_k(i) = \varphi_k(t_i),\ i = 1, 2, ..., 8,\ k = 1, 2, ..., 8.$$

Then, using the `polyfit` function, we define their coefficient vector:

```
pk=polyfit(t,Fk,7)
```

This function is used to determine the coefficients of the optimal polynomial using the least squares method. In the considering case, we are looking for a polynomial of the 7th degree (i.e. we have to define 8 coefficients of polynomial, according to its 8 values), therefore, we get a polynomial passing through the given values.
In order to calculate the derivatives we can sequentially use the `polyder` function:

`dpk=polyder(pk)`
is the vector of coefficients φ'_k;
`d2pk=polyder(dpk)`
is the vector of coefficients φ''_k.

In order to calculate the product of polynomials we can use the conv function:

`pij=conv(pi, pj)`
is the vector of coefficients $\varphi_i\varphi_j$;
`d20pij=conv(d2pi, pj)`
is the vector of coefficients $\varphi''_i\varphi_j$;
`d02pij=conv(pi, d2pj)`
is the vector of coefficients $\varphi_i\varphi''_j$;
`dpij=conv(dpi, dpj)`
is the vector of coefficients $\varphi'_i\varphi'_j$;
`d2pij=conv(d2pi, d2pj)`
is the vector of coefficients $\varphi''_i\varphi''_j$.

In order to calculate the antiderivative of a polynomial we can use the `polyint` function:

`Pi=polyint(pi)`
is the vector of coefficients $\int\varphi_i dt$;
`Pij=polyint(pij)`
is the vector of coefficients $\int\varphi_i\varphi_j dt$;
`d20Pij=polyint(d20pij)`
is the vector of coefficients $\int\varphi''_i\varphi_j dt$;
`d02Pij=polyint(d02pij)`
is the vector of coefficients $\int\varphi_i\varphi''_j dt$;
`dPij=polyint(dpij)`
is the vector of coefficients $\int\varphi'_i\varphi'_j dt$;
`d2Pij=polyint(d2pij)`
is the vector of coefficients $\int\varphi''_i\varphi''_j dt$.

Then the calculation of matrices $K^{0}_{\alpha\beta}(i,j)$, $K^{4}_{\alpha\beta}(i,j)$, $K^{21}_{\alpha\beta}(i,j)$, $K^{23}_{\alpha\beta}(i,j)$, $K^{22}_{\alpha\beta}(i,j)$ can be done in the following algorithm:

$K^{0}_{\alpha\beta}(i,j) =$ `polyval(d2Pij,1)-polyval(d2Pij,0);`

$K^{4}_{\alpha\beta}(i,j) =$ `polyval(Pij,1)-polyval(Pij,0),`

$K^{21}_{\alpha\beta}(i,j) =$ `polyval(d20Pij,1)-polyval(d20Pij,0),`

$K^{23}_{\alpha\beta}(i,j) =$ `polyval(d02Pij,1)-polyval(d02Pij,0),`

$K^{22}_{\alpha\beta}(i,j) =$ `polyval(dPij,1)-polyval(dPij,0),`

where the function `polyval (p, t)` allows researcher to calculate the values of a polynomial with a vector of coefficients p at a given point t.

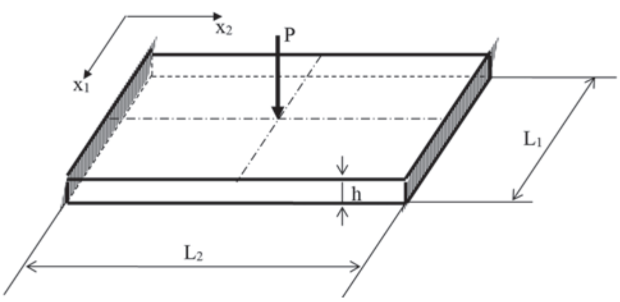


Figure 6.1. Example of analysis

6. EXAMPLE OF ANALYSIS

Let us consider the problem of the bending of a thin plate rigidly fixed along the side faces under the influence of a load concentrated in the center as an example (Figure 6.1).
Let us consider the following geometric parameters: $L_1 = 0.9$ m, $L_2 = 1.0$ m, $h = 0.05$ m is the thickness.
Let us consider the following design parameters of material of plate: coefficient of elasticity $E = 3000 \cdot 10^4$ kN/m^2, Poisson's ratio $\nu = 0.16$.
Let external load parameter be equal to $P = 1$ kN.
Let the number of elements be equal to $N_e = 4$.

Then we have the following element length:

$$h_e = L_1 / N_e = 0{,}9/4 = 0{,}225.$$

Distance between the coordinates of the nodes is equal to

$$h_p = h_e / 5 = 0{,}225/5 = 0{,}045.$$

The number of nodal unknowns for each component of the vector function y_j, $j = 1, 2, 3, 4$ is equal to

$$N_g = N_p + 2N_b = 4 \cdot (5 - 1) + 2 \cdot (4 + 1) = 26,$$

where $N_p = N_e(N_k - 1)$ is the total number of internal nodes for all elements; $N_b = N_e + 1$ is the total number of border nodes for all elements.
The total number of unknowns is equal to

$$N_U = 4N_g = 4 \cdot 26 = 104.$$

Let us conventional finite element method (FEM) for comparison. Unknown functions on an element within FEM are represented as a cubic parabola and at a node each unknown function is represented by two unknown nodal quantities: the nodal value of the unknown function itself and its first derivative in the discrete direction. In this case the total number of nodal points in the discrete direction corresponding to x_1 is equal to

$$N_1 = L_1 / N_p + 1 = 0{,}9 / 0{,}045 + 1 = 21.$$

The number of nodal unknowns for each component of the vector function y_j, $j = 1, 2, 3, 4$ is equal to

$$N_g = 2N_1 = 42.$$

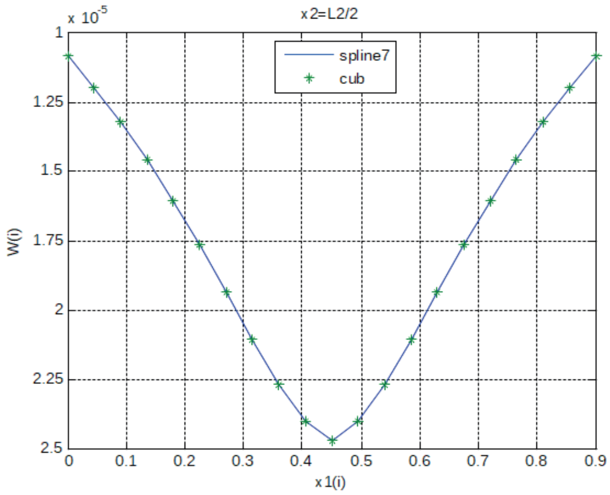
The total number of unknowns is equal to

$$N_U = 4N_g = 4 \cdot 42 = 168.$$

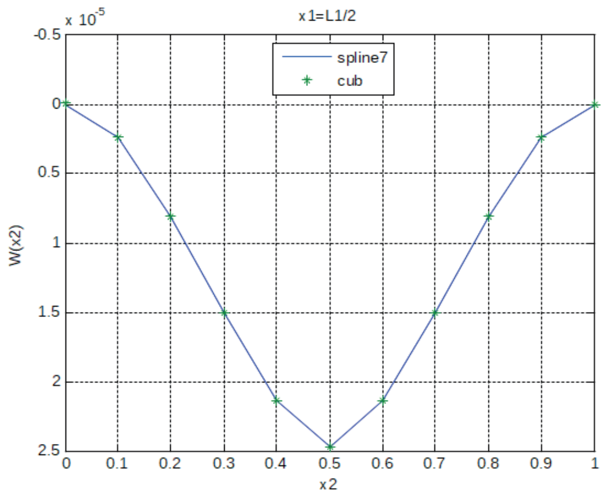
The graphical comparison of results of analysis is presented at Figures 6.2, 6.3, where `spline7` corresponds to deflection values, obtained using 7th order B-splines; `cub` corresponds to deflection values obtained using the traditional finite element method; $h_1 = h_p = 0{,}045$ and $h_2 = 0{,}1$ are steps for printing results along directions corresponding to x_1 and x_2.

As researcher can see, the results obtained are almost completely identical. And at the same time, the algorithm of the discrete-continual finite element method based on the use of B-splines leads to a significant reduction in the number of unknowns. The difference is equal to

$$4N_p = 4N_e(N_k - 1) = 4 \cdot 4 \cdot (5 - 1) = 64.$$



Figures 6.2. Comparison of the results of analysis in the middle sections along direction



Figures 6.3. Comparison of the results of analysis in the middle sections along direction

ACKNOWLEDGEMENTS

The distinctive research work was carried out at the expense of the State program of the Russian Federation “Scientific and technological development of the Russian Federation” and the Program for Fundamental Research of State Academies of Science for 2013–2020, as part of the Plan for Fundamental Scientific Research of the Ministry of Construction and Housing and Communal Services of the Russian Federation and the Russian Academy of Architecture and Construction Sciences for 2020, within science topic “Research and development of fundamental theoretical foundations of the synthesis of optimal structures as methods for designing structures with predetermined properties”.

REFERENCES

1. **Akimov P.A., Mozgaleva M.L., Kaytukov T.B.** Numerical solution of the problem of beam analysis with the use of B-spline finite element method. // International Journal for Computational Civil and Structural Engineering, 2020, Vol. 16, Issue 3, pp. 12–22.
2. **Li B., Chen X.** Wavelet-based numerical analysis: A review and classification. // Finite Elements in Analysis and Design, 2014, Vol. 81, pp. 14–31.
3. **Daubechies I.** Orthonormal bases of compactly supported wavelets. // Commun. Pure Appl. Math., 1988, Vol. 41, pp. 909–996.
4. **Li B., Cao H.R., He Z.J.** The construction of one-dimensional Daubechies wavelet-based finite elements for structural response analysis. // J. Vibroeng, 2011, vol. 13, pp. 729–738.
5. **Ma J.R., Xue J.J.** A study of the construction and application of a Daubechies wave-let-based beam element. // Finite Elements in Analysis and Design, 2003, Vol. 39, pp. 965–975.
6. **Mozgaleva M.L., Akimov P.A., Kaytukov T.B.** About wavelet-based computational

beam analysis with the use of Daubechies scaling functions. // International Journal for Computational Civil and Structural Engineering, 2019, Vol. 15, Issue 2, pp. 95–110.

7. **Mozgaleva M.L., Akimov P.A., Kaytukov T.B.** Wavelet-based discrete-continual finite element plate analysis with the use of Daubechies scaling functions. // International Journal for Computational Civil and Structural Engineering, 2019, Vol. 15, Issue 3, pp. 96–108.
8. **Antes H.** Bicubic fundamental splines in plate bending. // Int. J. Numer. Methods Eng., 1974, Vol. 8, pp. 503–511.
9. **Han J.G., Ren W.X., Huang Y.** A spline wavelet finite-element method in structural mechanics. // Int. J. Numer. Methods Eng., 2006, Vol. 66, pp. 166–190.
10. **Han J.G., Ren W.X., Huang Y.** A spline wavelet finite element formulation of thin plate bending. // Eng. Comput., 2009, Vol. 25, pp. 319–326.
11. **Xiang J.W., Chen X.F., He Z.J., Zhang Y.H.** A new wavelet-based thin plate element using B-spline wavelet on the interval. // Comput. Math., 2008, Vol. 41, pp. 243–255.
12. **Chen X.F., Xiang J.W., Li B., He Z.J.** A study of multiscale wavelet-based elements for adaptive finite element analysis. // Adv. Eng. Softw., 2010, Vol. 41, pp. 196–205.
13. **Zhong Y.T., Xiang J.W.** Construction of wavelet-based elements for static and stability analysis of elastic problems. // Acta Mech. Solida Sin., 2011, Vol. 24, pp. 355–364.
14. **Yang Z.B., Chen X.F., Li B., He Z.J., Miao H.H.** Vibration analysis of curved shell using b-spline wavelet on the interval (BSWI). // Finite Elements Method and General Shell Theory, CMES85, 2012, pp. 129–155.
15. **Yang Z.B., Chen X.F., Zhang X.W., He Z.J**. Free vibration and buckling analysis of plates using B-spline wavelet on the interval Mindlin element. // Appl. Math. Model., 2013, Vol. 37, pp. 3449-3466.
16. **Xiang J.W., Chen X.F., Li B., He Y.M., He Z.J.** Identification of a crack in a beam based on the finite element method of a B-spline wavelet on the interval. // J. Sound Vibr., 2006, Vol. 296, pp. 1046–1052.
17. **Xiang J.W., Chen X.F., Mo Q.M., He Z.H.** Identification of crack in a rotor system based on wavelet finite element method. // Finite Elem. Anal. Des., 2007, Vol. 43, pp. 1068–1081.
18. **Xiang J.W., Liang M.** A two-step approach to multi-damage detection for plate structures. // Eng. Fract. Mech., 2012, Vol. 91, pp. 73–86.
19. **Xiang J.W., Matsumoto T., Long J.Q., Ma G**. Identification of damage locations based on operating deflection shape. // Nondestruct. Test. Eval., 2012, Vol. 1, pp. 1–15.
20. **Xiang J.W., Liang M.** Wavelet-based detection of beam cracks using modal shape and frequency measurements. // Comput.-Aided Civil Infrastruct. Eng., 2012, Vol. 27, pp. 439–454.
21. **Xiang J.W., Matsumoto T., Wang Y.W., Jiang Z.S.** Detect damages in conical shells using curvature mode shape and wavelet finite element method. // Int. J. Mech. Sci., 2013, Vol. 66, pp. 83–93.
22. **Dong H.B., Chen X.F., Li B., Qi K.Y., He Z.J.** Rotor crack detection based on high-precision modal parameter identification method and wavelet finite element model. // Mech. Syst. Signal Process., 2009, Vol. 23, pp. 869–883.
23. **Chen X.F., Yang Z.B., Zhang X.W., He Z.J.** Modeling of wave propagation in one-dimension structures using B-spline wavelet on interval finite element. // Finite Elem. Anal. Des., 2012, Vol. 51, pp. 1–9.
24. **Han J.G., Ren W.X., Huang Y.** A multivariable wavelet-based finite element method and its application to thick plates. // Finite Elem. Anal. Des., 2005, Vol. 41, pp. 821–833.
25. **Zhang X.W., Chen X.F., He Z.J.** The analysis of shallow shells based on multivariable wavelet finite element method. // Acta Mech. Solida Sin., 2011, Vol. 24, pp. 450–460.
26. **Han J.G., Ren W.X., Huang Y.** A wavelet-based stochastic finite element method of thin plate bending. // Appl. Math. Model., 2007, Vol. 31, pp. 181–193.

27. **Pian T.H.H., Chen D.P.** Alternative ways for formulation of hybrid stress elements. // Int. J. Numer. Methods Eng., 1982, Vol. 18, pp. 1679–1684.
28. **Pian T.H.H., Sumihara K.** Rational approach for assumed stress finite elements. // Int. J. Numer. Methods Eng., 1984, Vol. 20, pp. 1685–1695.
29. **Mau S.T., Tong P., Pian T.H.H.** Finite element solutions for laminated thick plates. // J. Compos. Mater., 1972, Vol. 6, pp. 304–311.
30. **Akimov P.A.** Correct Discrete-Continual Finite Element Method of Structural Analysis Based on Precise Analytical Solutions of Resulting Multipoint Boundary Problems for Systems of Ordinary Differential Equations. // Applied Mechanics and Materials, 2012, Vols. 204–208, pp. 4502–4505.
31. **Akimov P.A., Belostotskiy A.M., Mozgaleva M.L., Mojtaba Aslami, Negrozov O.A.** Correct Multilevel Discrete-Continual Finite Element Method of Structural Analysis. // Advanced Materials Research, 2014, Vol. 1040, pp. 664–669.
32. **Zolotov A.B., Akimov P.A.** Semianalytical Finite Element Method for Two-dimensional and Three-dimensional Problems of Structural Analysis. // Proceedings of the International Symposium LSCE 2002 organized by Polish Chapter of IASS, Warsaw, Poland, 2002, pp. 431–440.
33. **Akimov P.A., Sidorov V.N.** Correct Method of Analytical Solution of Multipoint Boundary Problems of Structural Analysis for Systems of Ordinary Differential Equations with Piecewise Constant Coefficients. // Advanced Materials Research, 2011, Vols. 250–253, pp. 3652–3655.
34. **Akimov P.A., Mozgaleva M.L.** Correct Wavelet-based Multilevel Discrete-Continual Methods for Local Solution of Boundary Problems of Structural Analysis. // Applied Mechanics and Materials, 2013, Vols. 353–356, pp. 3224–3227.
35. **Akimov P.A., Mozgaleva M.L.** Wavelet-based Multilevel Discrete-Continual Finite Element Method for Local Deep Beam Analysis. // Applied Mechanics and Materials, 2013, Vols. 405–408, pp. 3165–3168.
36. **Akimov P.A., Mozgaleva M.L.** Wavelet-based Multilevel Discrete-Continual Finite Element Method for Local Plate Analysis. // Applied Mechanics and Materials, 2013, Vols. 351–352, pp. 13–16.
37. **Akimov P.A., Mozgaleva M.L., Mojtaba Aslami, Negrozov O.A.** Modified Wavelet-based Multilevel Discrete-Continual Finite Element Method for Local Structural Analysis. Part 1: Continual and Discrete-Continual Formulations of the Problems. // Applied Mechanics and Materials, 2014, Vols. 670–671, pp.720–723.
38. **Akimov P.A., Mozgaleva M.L., Mojtaba Aslami, Negrozov O.A.** Modified Wavelet-based Multilevel Discrete-Continual Finite Element Method for Local Structural Analysis. Part 2: Reduced Formulations of the Problems in Haar Basis. // Applied Mechanics and Materials, 2014, Vols. 670–671, pp. 724–727.
39. **Akimov P.A., Mozgaleva M.L., Mojtaba Aslami, Negrozov O.A.** Wavelet-Based Discrete-Continual Finite Element Method of Local Structural Analysis for Two-Dimensional Problems. // Procedia Engineering, 2014, Vol. 91, pp. 8–13.
40. **Mozgaleva M.L., Akimov P.A.** About Verification of Wavelet-Based Discrete-Continual Finite Element Method for Three-Dimensional Problems of Structural Analysis. Part 1: Structures with Constant Physical and Geometrical Parameters Along Basic Direction. // Applied Mechanics and Materials, 2015, Vol. 709, pp. 105–108.
41. **Mozgaleva M.L., Akimov P.A.** About Verification of Wavelet-Based Discrete-Continual Finite Element Method for Three-Dimensional Problems of Structural Analysis. Part 2: Structures with Piecewise Constant Physical and Geometrical Parameters Along Basic Direction. // Applied Mechanics and Materials, 2015, Vol. 709, pp. 109–112.
42. **Akimov P.A., Mozgaleva M.L.** Method of Extended Domain and General Principles of

Mesh Approximation for Boundary Problems of Structural Analysis. // Applied Mechanics and Materials, 2014, Vols. 580–583, pp. 2898–2902.

СПИСОК ЛИТЕРАТУРЫ

1. **Akimov P.A., Mozgaleva M.L., Kaytukov T.B.** Numerical solution of the problem of beam analysis with the use of B-spline finite element method. // International Journal for Computational Civil and Structural Engineering, 2020, Vol. 16, Issue 3, pp. 12–22.
2. **Li B., Chen X.** Wavelet-based numerical analysis: A review and classification. // Finite Elements in Analysis and Design, 2014, Vol. 81, pp. 14–31.
3. **Daubechies I.** Orthonormal bases of compactly supported wavelets. // Commun. Pure Appl. Math., 1988, Vol. 41, pp. 909–996.
4. **Li B., Cao H.R., He Z.J.** The construction of one-dimensional Daubechies wavelet-based finite elements for structural response analysis. // J. Vibroeng, 2011, vol. 13, pp. 729–738.
5. **Ma J.R., Xue J.J.** A study of the construction and application of a Daubechies wave-let-based beam element. // Finite Elements in Analysis and Design, 2003, Vol. 39, pp. 965–975.
6. **Mozgaleva M.L., Akimov P.A., Kaytukov T.B.** About wavelet-based computational beam analysis with the use of Daubechies scaling functions. // International Journal for Computational Civil and Structural Engineering, 2019, Vol. 15, Issue 2, pp. 95–110.
7. **Mozgaleva M.L., Akimov P.A., Kaytukov T.B.** Wavelet-based discrete-continual finite element plate analysis with the use of Daubechies scaling functions. // International Journal for Computational Civil and Structural Engineering, 2019, Vol. 15, Issue 3, pp. 96–108.
8. **Antes H.** Bicubic fundamental splines in plate bending. // Int. J. Numer. Methods Eng., 1974, Vol. 8, pp. 503–511.
9. **Han J.G., Ren W.X., Huang Y.** A spline wavelet finite-element method in structural mechanics. // Int. J. Numer. Methods Eng., 2006, Vol. 66, pp. 166–190.
10. **Han J.G., Ren W.X., Huang Y.** A spline wavelet finite element formulation of thin plate bending. // Eng. Comput., 2009, Vol. 25, pp. 319–326.
11. **Xiang J.W., Chen X.F., He Z.J., Zhang Y.H.** A new wavelet-based thin plate element using B-spline wavelet on the interval. // Comput. Math., 2008, Vol. 41, pp. 243–255.
12. **Chen X.F., Xiang J.W., Li B., He Z.J.** A study of multiscale wavelet-based elements for adaptive finite element analysis. // Adv. Eng. Softw., 2010, Vol. 41, pp. 196–205.
13. **Zhong Y.T., Xiang J.W.** Construction of wavelet-based elements for static and stability analysis of elastic problems. // Acta Mech. Solida Sin., 2011, Vol. 24, pp. 355–364.
14. **Yang Z.B., Chen X.F., Li B., He Z.J., Miao H.H.** Vibration analysis of curved shell using b-spline wavelet on the interval (BSWI). // Finite Elements Method and General Shell Theory, CMES85, 2012, pp. 129–155.
15. **Yang Z.B., Chen X.F., Zhang X.W., He Z.J**. Free vibration and buckling analysis of plates using B-spline wavelet on the interval Mindlin element. // Appl. Math. Model., 2013, Vol. 37, pp. 3449–3466.
16. **Xiang J.W., Chen X.F., Li B., He Y.M., He Z.J.** Identification of a crack in a beam based on the finite element method of a B-spline wavelet on the interval. // J. Sound Vibr., 2006, Vol. 296, pp. 1046–1052.
17. **Xiang J.W., Chen X.F., Mo Q.M., He Z.H.** Identification of crack in a rotor system based on wavelet finite element method. // Finite Elem. Anal. Des., 2007, Vol. 43, pp. 1068–1081.
18. **Xiang J.W., Liang M.** A two-step approach to multi-damage detection for plate structures. // Eng. Fract. Mech., 2012, Vol. 91, pp. 73–86.
19. **Xiang J.W., Matsumoto T., Long J.Q., Ma G**. Identification of damage locations based on operating deflection shape. // Nondestruct. Test. Eval., 2012, Vol. 1, pp. 1–15.

20. **Xiang J.W., Liang M.** Wavelet-based detection of beam cracks using modal shape and frequency measurements. // Comput.-Aided Civil Infrastruct. Eng., 2012, Vol. 27, pp. 439–454.
21. **Xiang J.W., Matsumoto T., Wang Y.W., Jiang Z.S.** Detect damages in conical shells using curvature mode shape and wavelet finite element method. // Int. J. Mech. Sci., 2013, Vol. 66, pp. 83–93.
22. **Dong H.B., Chen X.F., Li B., Qi K.Y., He Z.J.** Rotor crack detection based on high-precision modal parameter identification method and wavelet finite element model. // Mech. Syst. Signal Process., 2009, Vol. 23, pp. 869–883.
23. **Chen X.F., Yang Z.B., Zhang X.W., He Z.J.** Modeling of wave propagation in one-dimension structures using B-spline wavelet on interval finite element. // Finite Elem. Anal. Des., 2012, Vol. 51, pp. 1–9.
24. **Han J.G., Ren W.X., Huang Y.** A multivariable wavelet-based finite element method and its application to thick plates. // Finite Elem. Anal. Des., 2005, Vol. 41, pp. 821–833.
25. **Zhang X.W., Chen X.F., He Z.J.** The analysis of shallow shells based on multivariable wavelet finite element method. // Acta Mech. Solida Sin., 2011, Vol. 24, pp. 450–460.
26. **Han J.G., Ren W.X., Huang Y.** A wavelet-based stochastic finite element method of thin plate bending. // Appl. Math. Model., 2007, Vol. 31, pp. 181–193.
27. **Pian T.H.H., Chen D.P.** Alternative ways for formulation of hybrid stress elements. // Int. J. Numer. Methods Eng., 1982, Vol. 18, pp. 1679–1684.
28. **Pian T.H.H., Sumihara K.** Rational approach for assumed stress finite elements. // Int. J. Numer. Methods Eng., 1984, Vol. 20, pp. 1685–1695.
29. **Mau S.T., Tong P., Pian T.H.H.** Finite element solutions for laminated thick plates. // J. Compos. Mater., 1972, Vol. 6, pp. 304–311.
30. **Akimov P.A.** Correct Discrete-Continual Finite Element Method of Structural Analysis Based on Precise Analytical Solutions of Resulting Multipoint Boundary Problems for Systems of Ordinary Differential Equations. // Applied Mechanics and Materials, 2012, Vols. 204–208, pp. 4502–4505.
31. **Akimov P.A., Belostotskiy A.M., Mozgaleva M.L., Mojtaba Aslami, Negrozov O.A.** Correct Multilevel Discrete-Continual Finite Element Method of Structural Analysis. // Advanced Materials Research, 2014, Vol. 1040, pp. 664–669.
32. **Zolotov A.B., Akimov P.A.** Semianalytical Finite Element Method for Two-dimensional and Three-dimensional Problems of Structural Analysis. // Proceedings of the International Symposium LSCE 2002 organized by Polish Chapter of IASS, Warsaw, Poland, 2002, pp. 431–440.
33. **Akimov P.A., Sidorov V.N.** Correct Method of Analytical Solution of Multipoint Boundary Problems of Structural Analysis for Systems of Ordinary Differential Equations with Piecewise Constant Coefficients. // Advanced Materials Research, 2011, Vols. 250–253, pp. 3652–3655.
34. **Akimov P.A., Mozgaleva M.L.** Correct Wavelet-based Multilevel Discrete-Continual Methods for Local Solution of Boundary Problems of Structural Analysis. // Applied Mechanics and Materials, 2013, Vols. 353–356, pp. 3224–3227.
35. **Akimov P.A., Mozgaleva M.L.** Wavelet-based Multilevel Discrete-Continual Finite Element Method for Local Deep Beam Analysis. // Applied Mechanics and Materials, 2013, Vols. 405–408, pp. 3165–3168.
36. **Akimov P.A., Mozgaleva M.L.** Wavelet-based Multilevel Discrete-Continual Finite Element Method for Local Plate Analysis. // Applied Mechanics and Materials, 2013, Vols. 351–352, pp. 13–16.
37. **Akimov P.A., Mozgaleva M.L., Mojtaba Aslami, Negrozov O.A.** Modified Wavelet-based Multilevel Discrete-Continual Finite Element Method for Local Structural Analysis. Part 1: Continual and Discrete-Continual Formulations of the Problems. // Applied Mechanics and Materials, 2014, Vols. 670–671, pp.720–723.

38. **Akimov P.A., Mozgaleva M.L., Mojtaba Aslami, Negrozov O.A.** Modified Wavelet-based Multilevel Discrete-Continual Finite Element Method for Local Structural Analysis. Part 2: Reduced Formulations of the Problems in Haar Basis. // Applied Mechanics and Materials, 2014, Vols. 670–671, pp. 724–727.
39. **Akimov P.A., Mozgaleva M.L., Mojtaba Aslami, Negrozov O.A.** Wavelet-Based Discrete-Continual Finite Element Method of Local Structural Analysis for Two-Dimensional Problems. // Procedia Engineering, 2014, Vol. 91, pp. 8–13.
40. **Mozgaleva M.L., Akimov P.A.** About Verification of Wavelet-Based Discrete-Continual Finite Element Method for Three-Dimensional Problems of Structural Analysis. Part 1: Structures with Constant Physical and Geometrical Parameters Along Basic Direction. // Applied Mechanics and Materials, 2015, Vol. 709, pp. 105–108.
41. **Mozgaleva M.L., Akimov P.A.** About Verification of Wavelet-Based Discrete-Continual Finite Element Method for Three-Dimensional Problems of Structural Analysis. Part 2: Structures with Piecewise Constant Physical and Geometrical Parameters Along Basic Direction. // Applied Mechanics and Materials, 2015, Vol. 709, pp. 109–112.
42. **Akimov P.A., Mozgaleva M.L.** Method of Extended Domain and General Principles of Mesh Approximation for Boundary Problems of Structural Analysis. // Applied Mechanics and Materials, 2014, Vols. 580–583, pp. 2898–2902.

Pavel A. Akimov, Full Member of the Russian Academy of Architecture and Construction Sciences, Professor, Dr.Sc.; Acting Rector of National Research Moscow State University of Civil Engineering; Professor of Department of Architecture and Construction, Peoples’ Friendship University of Russia; Professor of Department of Structural Mechanics, Tomsk State University of Architecture and Building; Acting Vice-President of the Russian Academy of Architecture and Construction Sciences; 26, Yaroslavskoe Shosse, Moscow, 129337, Russia; phone: +7(495) 651-81-85; Fax: +7(499) 183-44-38; E-mail: AkimovPA@mgsu.ru, rector@mgsu.ru, pavel.akimov@gmail.com.

Marina L. Mozgaleva, Senior Scientist Researcher, Dr.Sc.; Professor of Department of Applied Mathematics, National Research Moscow State University of Civil Engineering; 26, Yaroslavskoe Shosse, Moscow, 129337, Russia; phone/fax +7(499) 183-59-94; Fax: +7(499) 183-44-38; Email: marina.mozgaleva@gmail.com.

Taymuraz B. Kaytukov, Advisor of the Russian Academy of Architecture and Construction Sciences, Associate Professor, Ph.D.; Vice-Rector, Associate Professor Department of Applied Mathematics, National Research Moscow State University of Civil Engineering; 26, Yaroslavskoe Shosse, Moscow, 129337, Russia; phone: +7(499) 929-52-29; fax: +7(499) 183-59-94; Email: KaytukovTB@mgsu.ru.

Акимов Павел Алексеевич, академик РААСН, профессор, доктор технических наук; временно исполняющий обязанности ректора Национального исследовательского Московского государственного строительного университета; профессор Департамента архитектуры и строительства Российского университета дружбы народов; профессор кафедры строительной механики Томского государственного архитектурно-строительного университета; исполняющий обязанности вице-президента Российской академии архитектуры и строительных наук; 129337, Россия, г. Москва, Ярославское шоссе, дом 26; телефон: +7(495) 651-81-85; факс: +7(499) 183-44-38; Email: AkimovPA@mgsu.ru, rector@mgsu.ru, pavel.akimov@gmail.com.

Мозгалева Марина Леонидовна, старший научный сотрудник, доктор технических наук; профессор кафедры прикладной математики Национального исследовательского Московского государственного строительного университета; 129337, Россия, г. Москва, Ярославское шоссе, дом 26; телефон/факс: +7(499) 183-59-94; Email: marina.mozgaleva@gmail.com.

Кайтуков Таймураз Батразович, советник РААСН, доцент, кандидат технических наук; проректор, профессор кафедры прикладной математики Национального исследовательского Московского государственного строительного университета; 129337, Россия, г. Москва, Ярославское шоссе, дом 26; телефон: +7(499) 929-52-29; факс: +7(499) 183-44-38; Email: KaytukovTB@mgsu.ru.