CLASSIFICATION OF INTERNAL RESONANCES IN NONLINEAR FRACTIONALY DAMPED UFLYAND-MINDLIN PLATES

Marina V. Shitikova[1,2], Elena I. Osipova[1]

[1]Voronezh State Technical University, Voronezh, RUSSIA
[2]RAASN Research Institute of Structural Physics, Moscow, RUSSIA

Abstract: In the present paper, the nonlinear free vibrations of fractionally damped plates are studied, equations of motion of which take the rotary inertia and shear deformations into account and involve five coupled nonlinear differential equations in terms of three mutually orthogonal displacements and two angles of rotation. The procedure resulting in decoupling linear parts of equations has been proposed with further utilization of the generalized method of multiple time scales for solving nonlinear governing equations of motion, in so doing the amplitude functions have been expanded into power series in terms of the small parameter and depend on different time scales. The occurrence of the internal or combinational resonances in Uflyand-Mindlin plates has been revealed and classified.

Keywords: Nonlinear elastic Uflyand-Mindlin plate, fractional damping, fractional derivative Kelvin-Voigt model, generalized method of multiple time scales

КЛАССИФИКАЦИЯ ВНУТРЕННИХ РЕЗОНАНСОВ В НЕЛИНЕЙНЫХ ПЛАСТИНКАХ УФЛЯНДА-МИНДЛИНА С ДРОБНЫМ ДЕМПФИРОВАНИЕМ

М.В. Шитикова[1,2], Е.И. Осипова[1]

[1]Воронежский государственный технический университет, Воронеж, Россия
[2]Научно-исследовательский институт строительной физики РААСН, Москва, Россия

Аннотация: В данной работе изучаются нелинейные колебания пластинок на основе моделирования сил внешнего демпфирования с помощь производных дробного порядка. При этом используется система пяти нелиненйых уравнений движения, учитывающая деформации сдвига и силы инерции, относиельно трех перемещений в трех взаимно орто-гональных направлениях и двух углов поворота. В качестве метода решения используется обобщенный метод многих временных масштабов. Выявлены возможные типы внутренних и комбинационных резонансов, которые могут возникать в платинках Уфлянда-Миндлина, и дана их классификация.

Ключевые слова: нелинейно упругая пластинка Уфлянда-Миндлина, демпфирование с помощью дробной производ-ной, модель Кельвина-Фойгта с дробной производной, обобщенный метод многих временных масштабов

1. INTRODUCTION

Recently the interest to nonlinear dynamic response of viscoelastic plates or elastic plates vibrating in a viscoelastic surrounding medium has been greatly renewed due to the appearance of advanced materials exhibiting nonlinear behavior, and a comprehensive review in the field, including experimental results, could be found in [1–7]. In so doing the damping forces are usually taken into account according to the Rayleigh's hypothesis [2,8], resulting in the modal damping [9], i.e. it is assumed that each natural mode of vibrations possesses its own damping coefficient dependent on its natural frequency. For describing the viscoelastic features of plates, the Kelvin-Voigt model [5] or standard linear solid model [6] are of frequent use in engineering practice considering either linear or nonlinear springs in viscoelastic elements [10]. The analysis of free undamped [11] and damped [5] vibrations of nonlinear systems is of great importance

for defining the dynamic system's characteristics dependent on the amplitude-phase relationships and modes of vibration. Moreover, nonlinear vibrations could be accompanied by such a phenomenon as the internal resonance, resulting in strong coupling between the modes of vibrations involved [11–16] and hence in the energy exchange between the interacting modes.

The internal resonance could be observed in the case of some combination of natural frequencies of one and the same type of vibrations. Thus, nonlinear vibrations of rectangular plates, dynamic behavior of which is described by von Karman equations in terms of the plate's deflection and stress function, have been considered in [13] by reducing the governing equations to a set of two modal equations applying the Galerkin procedure. The case of the one-to-one internal resonance (when frequencies of two modes of flexural vibration are equal to each other) accompanied by the external resonance (when the frequency of the harmonic force is close to one of the natural frequency) has been studied.

The one-to-one internal resonance has been investigated also in [14] and [15] for nonlinear vertical vibrations of rectangular plates under the action of harmonic forces acting in the plate's plane [14] and out of the plate's plane [14,15], in so doing a set of three equations in terms of two in-plane displacements and deflection and a set of five equations considering the shear deformations have been used in [14] and [15], respectively. However, considering the inertia forces only for vertical vibrations and utilizing the Galerkin procedure, in both papers a set of two nonlinear equations has been obtained in terms of two flexural modes, which are assumed to be coupled via the one-to-one internal resonance. For the first two natural modes of flexural vibrations, the cases of the 1:2 and 1:3 internal resonances have been also studied in [15]. Another type of the internal resonance has been investigated by Rossikhin and Shitikova [16–20], when one frequency of in-plane vibrations is equal (the 1:1 internal resonance [18,20]) or two times larger (the 1:2 internal resonance [16,19]) than a certain frequency of out-of-plane vibrations. As this takes place, a set of three nonlinear differential equations in terms of three mutually orthogonal displacements has been used considering inertia

of all types of vibrations, what allows the authors to study the combinational resonances of the additive and difference types as well [17, 20–22]. Combinational types of the internal resonance result in the energy exchange between three or more subsystems. It should be noted that investigations in this direction were initiated by Witt and Gorelik [23], who pioneered in the theoretical and experimental analysis of the energy transfer from one subsystem to another using the simplest two-degree-of-freedom mechanical system, as an example.

Moreover, in order to study nonlinear free damped vibrations of a thin plate, the viscoelastic Kelvin-Voigt model involving fractional derivative [24] has been utilized, since this model possesses the advantage over the conventional Kelvin-Voigt model [11–15], because it provides the results matching the experimental data. Thus, for example, experimental data on ambient vibrations study for the Vincent-Thomas [25] and Golden Gate [26] suspension bridges have shown that different modes of vibrations possess different magnitudes of damping coefficients. Besides, the increase in the natural frequency results in the decrease in the damping ratio. In order to lead the theoretical investigation in the agreement with the experiment, in 1998 it was suggested in [27] to utilize the fractional derivatives to describe the processes of internal friction occurring in suspension combined systems, what allowed the authors in a natural way to obtain the damping ratios, which depend on natural frequencies.

Nowadays fractional calculus is widely used for solving linear and nonlinear dynamic problems of structural mechanics, what is evident from numerous studies in the field, the overview of which could be found in the state-of-the-art articles by Rossikhin and Shitikova [28,29], wherein the examples of adopting the fractional derivative Kelvin-Voigt, Maxwell and standard linear solid models are provided for single-mass oscillators, rods, beams, plates, and shells.

In particular, linear vibrations of Kirchhoff-Love plates with Kelvin-Voigt fractional damping were considered for rectangular and circular plates, respectively, in [30] and [31] using one equation for vertical vibrations, while utilizing three equations of in-plane and transverse vibrations in

[8,32], and later multiplate systems were analyzed in [28,33]. It has been proved [29,34] that if viscoelastic properties of plates are described by the Kelvin-Voigt model assuming the Poisson's ratio as the time-independent value (though for real viscoelastic materials the Poisson's ratio is always a time-dependent function [35]), then this case coincides with the case of the dynamic behavior of elastic bodies in a viscoelastic medium. Thus, the authors of [30,31], and not only them, replaced one problem with another, namely: a problem of the dynamic response of viscoelastic Kirchhoff-Love plates in a conventional medium with a problem of dynamic response of elastic Kirchhoff-Love plates in a viscoelastic medium, damping features of which are governed by the fractional derivative Kelvin-Voigt model. The vibration suppression of fractionally damped thin rectangular simply supported plates subjected to a concentrated harmonic loading has been studied recently in [36] in order to minimize the plate deflection at the natural frequencies of the plate, in so doing the vibration suppression is accomplished by attaching multiple absorbers modelled as Kelvin-Voigt fractional oscillators, i.e. generalizing the approach suggested in [28,33].

As for the analysis of nonlinear vibrations of plates, then except the above mentioned papers [16,18–21], the fractional derivative Kelvin-Voigt model was used in [37–42] and fractional derivative standard linear solid model in [7,43,44] but without considering the phenomena of the internal resonance.

Thus, free and forced vertical vibrations of an orthotropic plate have been studied in [37] considering first four modes of flexural vibrations, and during the analysis of force driven vibrations the frequency of a harmonic force was assumed to be equal to one of natural frequencies. The von Karman plate equation with fractional derivative damping was utilized in [38] for analyzing the cases of primary, subharmonic and superharmonic resonance conditions, when the harmonic force frequency, respectively, is approximately equal, three times less or larger than the first or second natural frequency of vertical vibrations. Nonlinear random vibrations of the same plate was studied in [41]. Dynamic nonlinear response to random excitation of a simply supported rectangular plate

based on a foundation, damping features of which are described by the fractional derivative Kelvin-Voigt model, has been considered in [40]. The analysis of chaotic vibrations of simply supported nonlinear viscoelastic plate with fractional derivative Kelvin-Voigt model has been carried out in [42] for the case when the plate is subjected to an in-plane harmonic force in one direction and a transverse harmonic force. The Galerkin decomposition has been used to obtain the modal equation of the system, in so doing the authors restricted themselves only by the first mode. The fractional derivative standard linear solid model has been utilized in [44] for a viscoelastic layer for active damping of geometrically nonlinear vibrations of smart composite plates using the higher order plate theory and finite element method with discretizing the plate by eight-node isoparametric quadrilateral elements.

Recently the approaches suggested in [19,20] for solving the problem on free nonlinear vibrations of elastic plates in a viscoelastic medium, damping features of which are governed by the Riemann-Liouville derivatives of the fractional order, and in [45] for studying the dynamic response of the fractional Duffing oscillator subjected to harmonic loading have been generalized for the case of forced vibrations of a simply-supported nonlinear thin elastic plate under the conditions of different internal resonances, when two or three natural modes corresponding to mutually orthogonal displacements are coupled [46–49].

In the present paper, the procedure proposed in [20] for solving the problem of free nonlinear vibrations of elastic plates in a fractional derivative viscoelastic medium, when the damped motion is described by a set of three nonlinear equations, has been extended for the case of free vibrations of a simply-supported fractionally damped nonlinear thin elastic plate, the motion of which is described by five equations involving shear deformations and rotary inertia.

2. PROBLEM FORMULATION

In order to consider free damped vibrations of a nonlinear simply-supported rectangular plate, first we recall the equations of motion of a nonlinear elastic rectangular plate, which take into account shear deformations and rotary inertia [50]

$$\frac{\partial^2 u}{\partial x^2} + \frac{1-\mu}{2}\frac{\partial^2 u}{\partial y^2} + \frac{1+\mu}{2}\frac{\partial^2 v}{\partial x \partial y} + \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2} +$$
$$+ \frac{1+\mu}{2}\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x \partial y} + \frac{1-\mu}{2}\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial y^2} = \frac{\rho(1-\mu^2)}{E}\frac{\partial^2 u}{\partial t^2},$$

(1)

$$\frac{\partial^2 v}{\partial y^2} + \frac{1-\mu}{2}\frac{\partial^2 v}{\partial x^2} + \frac{1+\mu}{2}\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial y^2} +$$
$$+ \frac{1+\mu}{2}\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x \partial y} + \frac{1-\mu}{2}\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x^2} = \frac{\rho(1-\mu^2)}{E}\frac{\partial^2 v}{\partial t^2},$$

(2)

$$k^2 \frac{1-\mu}{2}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial \psi_x}{\partial x} + \frac{\partial \psi_y}{\partial y}\right) +$$
$$+ \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\left[\frac{\partial u}{\partial x} + \mu\frac{\partial v}{\partial y}\right] + \frac{1-\mu}{2}\frac{\partial w}{\partial y}\left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right]\right) +$$
$$+ \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial y}\left[\mu\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right] + \frac{1-\mu}{2}\frac{\partial w}{\partial x}\left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right]\right) =$$
$$= \frac{\rho(1-\mu^2)}{E}\frac{\partial^2 w}{\partial t^2},$$

(3)

$$\frac{\partial^2 \psi_x}{\partial x^2} + \frac{1-\mu}{2}\frac{\partial^2 \psi_x}{\partial y^2} + \frac{1+\mu}{2}\frac{\partial^2 \psi_y}{\partial x \partial y} -$$
$$-6k^2\frac{1-\mu}{h^2}\left(\frac{\partial w}{\partial x} + \psi_x\right) = \frac{\rho(1-\mu^2)}{E}\frac{\partial^2 \psi_x}{\partial t^2},$$

(4)

$$\frac{\partial^2 \psi_y}{\partial y^2} + \frac{1-\mu}{2}\frac{\partial^2 \psi_y}{\partial x^2} + \frac{1+\mu}{2}\frac{\partial^2 \psi_x}{\partial x \partial y} -$$
$$-6k^2\frac{1-\mu}{h^2}\left(\frac{\partial w}{\partial y} + \psi_y\right) = \frac{\rho(1-\mu^2)}{E}\frac{\partial^2 \psi_y}{\partial t^2},$$

(5)

subjected to the initial

$$u\big|_{t=0} = v\big|_{t=0} = w\big|_{t=0} = 0,$$
$$\dot{u}\big|_{t=0} = \dot{v}\big|_{t=0} = \dot{w}\big|_{t=0} = 0,$$
$$\psi_x\big|_{t=0} = \dot{\psi}_x\big|_{t=0} = 0,$$
$$\psi_y\big|_{t=0} = \dot{\psi}_y\big|_{t=0} = 0,$$

(6)

as well as the boundary conditions (a) along the y-axis direction

$$w\big|_{x=0} = w\big|_{x=a} = 0, \qquad u\big|_{x=0} = u\big|_{x=a} = 0,$$
$$\frac{\partial u}{\partial x}\bigg|_{x=0} = \frac{\partial u}{\partial x}\bigg|_{x=a} = 0,$$
$$\frac{\partial^2 w}{\partial x^2}\bigg|_{x=0} = \frac{\partial^2 w}{\partial x^2}\bigg|_{x=a} = 0,$$
$$\frac{\partial^2 \psi_x}{\partial x^2}\bigg|_{x=0} = \frac{\partial^2 \psi_x}{\partial x^2}\bigg|_{x=a} = 0,$$

(7)

and (b) along the x-axis direction

$$w\big|_{y=0} = w\big|_{y=b} = 0, \qquad v\big|_{y=0} = v\big|_{y=b} = 0,$$
$$\frac{\partial v}{\partial y}\bigg|_{y=0} = \frac{\partial v}{\partial y}\bigg|_{y=b} = 0,$$
$$\frac{\partial^2 w}{\partial y^2}\bigg|_{y=0} = \frac{\partial^2 w}{\partial y^2}\bigg|_{y=b} = 0,$$
$$\frac{\partial^2 \psi_y}{\partial y^2}\bigg|_{y=0} = \frac{\partial^2 \psi_y}{\partial y^2}\bigg|_{y=b} = 0,$$

(8)

where $u = u(x, y, t)$, $v = v(x, y, t)$ and $w = w(x, y, t)$ are the displacements of points located in the plate's middle surface in the x-, y-, and z-directions, respectively, $\psi_x(x, y, t)$ and $\psi_y(x, y, t)$ are the angles of rotation of the normal to the middle surface and in the plane tangent to the lines z and x, k is the shear coefficient, μ is the Poisson's ratio, a and b are the plate's dimensions along the x- and y-axes, respectively, h is its thickness, and t is the time.

Let us rewrite equations (1)-(8) in the dimensionless form introducing the following dimensionless values:

$$u^* = \frac{u}{a}, \quad v^* = \frac{v}{a}, \quad w^* = \frac{w}{a},$$
$$x^* = \frac{x}{a}, \quad y^* = \frac{y}{b},$$
$$t^* = \frac{t}{a}\sqrt{\frac{E}{(1-\mu^2)\rho}}.$$

(9)

Substituting then (9) in (1)-(8), omitting asterisks for ease of presentation, and introducing the forces of resistance of the surrounding medium, resulting in damped vibrations, as it was suggested in [16,18], yield

$$u_{,xx} + \frac{1-\mu}{2}\beta_1^2 u_{,yy} + \frac{1+\mu}{2}\beta_1 v_{,xy} +$$
$$+ w_{,x}\left(w_{,xx} + \frac{1-\mu}{2}\beta_1^2 w_{,yy}\right) + \qquad (10)$$
$$+ \frac{1+\mu}{2}\beta_1^2 w_{,y}\,w_{,xy} = \ddot{u} + \chi_1 D^\gamma u,$$

$$\beta_1^2 v_{,yy} + \frac{1-\mu}{2}v_{,xx} + \frac{1+\mu}{2}\beta_1 u_{,xy} +$$
$$+ \beta_1 w_{,y}\left(\beta_1^2 w_{,yy} + \frac{1-\mu}{2}w_{,xx}\right) + \qquad (11)$$
$$+ \frac{1+\mu}{2}\beta_1 w_{,x}\,w_{,xy} = \ddot{v} + \chi_2 D^\gamma v,$$

$$k^2\frac{1-\mu}{2}\left(w_{,xx} + \beta_1^2 w_{,yy} + \psi_{x,x} + \beta_1\psi_{y,y}\right) +$$
$$+ w_{,xx}\left(u_{,x} + \mu\beta_1 v_{,y}\right) +$$
$$+ \beta_1^2 w_{,yy}\left(\mu u_{,x} + \beta_1 v_{,y}\right) + (1-\mu)\beta_1 w_{,xy}\left(\beta_1 u_{,y} + \mu v_{,x}\right)$$
$$+ w_{,x}\left(u_{,xx} + \frac{1-\mu}{2}\beta_1^2 u_{,yy} + \frac{1+\mu}{2}\beta_1 v_{,xy}\right)$$
$$+ \beta_1 w_{,y}\left(\frac{1-\mu}{2}v_{,xx} + \beta_1^2 v_{,yy} + \frac{1+\mu}{2}\beta_1 u_{,xy}\right) = \qquad (12)$$
$$= \ddot{w} + \chi_3 D^\gamma w,$$

$$\psi_{x,xx} + \frac{1-\mu}{2}\beta_1^2\psi_{x,yy} + \frac{1+\mu}{2}\beta_1\psi_{y,xy} -$$
$$-6k^2\frac{1-\mu}{\beta_2^2}\left(w_{,x} + \psi_x\right) = \ddot{\psi}_x + \chi_4 D^\gamma\psi_x, \qquad (13)$$

$$\psi_{y,yy} + \frac{1-\mu}{2}\psi_{y,xx} + \frac{1+\mu}{2}\beta_1\psi_{x,xy} -$$
$$-6k^2\frac{1-\mu}{\beta_2^2}\left(\beta_1 w_{,y} + \psi_y\right) = \ddot{\psi}_y + \chi_5 D^\gamma\psi_y, \qquad (14)$$

where $\beta_1 = a/b$ and $\beta_2 = h/a$ are the parameters defining the dimensions of the plate, χ_i ($i = 1, 2, ..., 5$) are damping coefficients, overdots denote time-derivatives, lower indices after a comma label the derivatives with respect to the corresponding coordinates, and D^γ

is the Riemann-Liouville fractional derivative [51] defined as

$$D^\gamma F = \frac{\partial}{\partial t}\int_0^t \frac{F(t-t')\,dt'}{\Gamma(1-\gamma)t'^\gamma}. \qquad (15)$$

3. METHOD OG SOLUTION

Let us seek the solution of equations (10)–(14) in the form of expansions in terms of eigen modes of vibration

$$u(x,y,t) = \sum_{m=1}^\infty\sum_{n=1}^\infty x_{1mn}(t)\eta_{1mn}(x,y),$$
$$v(x,y,t) = \sum_{m=1}^\infty\sum_{n=1}^\infty x_{2mn}(t)\eta_{2mn}(x,y),$$
$$w(x,y,t) = \sum_{m=1}^\infty\sum_{n=1}^\infty x_{3mn}(t)\eta_{3mn}(x,y), \qquad (16)$$
$$\psi_x(x,y,t) = \sum_{m=1}^\infty\sum_{n=1}^\infty x_{4mn}(t)\eta_{4mn}(x,y),$$
$$\psi_y(x,y,t) = \sum_{m=1}^\infty\sum_{n=1}^\infty x_{5mn}(t)\eta_{5mn}(x,y),$$

where $x_{imn}(t)$ ($i = 1, 2, ..., 5$) are the generalized displacements corresponding to the plate's in-plane displacements, its deflection and angles of rotation, while the eigen forms satisfying the boundary conditions (7)-(8) have the form

$$\eta_{1mn}(x,y) = \eta_{4mn}(x,y) = \cos\pi mx\sin\pi ny,$$
$$\eta_{2mn}(x,y) = \sin\pi mx\cos\pi ny, \qquad (17)$$
$$\eta_{3mn}(x,y) = \eta_{5mn}(x,y) = \sin\pi mx\sin\pi ny.$$

Substituting (16) and (17) in equations (10)-(14), multiplying then (10)-(14) by $n_{imn}(x,y)$, respectively, integrating over x and y, and applying the condition of orthogonality of the eigen modes within the domains $0 \leq x,y \leq 1$, we are led to a set of coupled nonlinear second-order differential equations in $x_{imn}(t)$

$$\ddot{x}_{1mn} + \chi_1 D^\gamma x_{1mn} + x_{1mn}S_{11}^{mn} + x_{2mn}S_{12}^{mn} = -F_{1mn}, \qquad (18)$$

$$\ddot{x}_{2mn} + \chi_2 D^\gamma x_{2mn} + x_{1mn}S_{21}^{mn} + x_{2mn}S_{22}^{mn} = -F_{2mn}, \qquad (19)$$

$$\ddot{x}_{3mn} + \chi_3 D^\gamma x_{3mn} + x_{3mn}s_{33}^{mn} + x_{4mn}s_{34}^{mn} + x_{5mn}s_{35}^{mn} = -F_{3mn}$$
(20)

$$\ddot{x}_{4mn} + \chi_4 D^\gamma x_{4mn} + x_{3mn}s_{43}^{mn} + x_{4mn}s_{44}^{mn} + x_{5mn}s_{45}^{mn} = 0,$$
(21)

$$\ddot{x}_{5mn} + \chi_5 D^\gamma x_{5mn} + x_{3mn}s_{53}^{mn} + x_{4mn}s_{54}^{mn} + x_{5mn}s_{55}^{mn} = 0,$$
(22)

where

$$S_{11}^{mn} = \pi^2\left(m^2 + \frac{1-\mu}{2}\beta_1 n^2\right),$$

$$S_{12}^{mn} = S_{21}^{mn} = \pi^2 \frac{1+\mu}{2}\beta_1 mn,$$
(23)

$$S_{22}^{mn} = \pi^2\left(\beta_1 n^2 + \frac{1-\mu}{2}m^2\right),$$

$$s_{33}^{mn} = k^2\frac{1-\mu}{2}\pi^2\left(m^2 + \beta_1 n^2\right),$$

$$s_{34}^{mn} = k^2\frac{1-\mu}{2}\pi m,$$

$$s_{35}^{mn} = k^2\frac{1-\mu}{2}\pi\beta_1 n,$$

$$s_{43}^{mn} = 6k^2\frac{1-\mu}{\beta_2^2}\pi m,$$

$$s_{53}^{mn} = 6k^2\frac{1-\mu}{\beta_2^2}\pi\beta_1 n,$$
(24)

$$s_{44}^{mn} = \pi^2\left(m^2 + \frac{1-\mu}{2}\beta_1 n^2\right) + 6k^2\frac{1-\mu}{\beta_2^2},$$

$$s_{45}^{mn} = s_{54}^{mn} = \pi^2\beta_1\frac{1+\mu}{2}mn,$$

$$s_{55}^{mn} = \pi^2\left(\beta_1 n^2 + \frac{1-\mu}{2}m^2\right) + 6k^2\frac{1-\mu}{\beta_2^2}.$$

Nonlinear parts of equations (18)-(20) have the form

$$F_{1mn} = \sum_{m_1}\sum_{n_1}\sum_{m_2}\sum_{n_2} x_{3m_1 n_1}x_{3m_2 n_2}A_{mn}^{m_1 n_1 m_2 n_2},$$

$$F_{2mn} = \sum_{m_1}\sum_{n_1}\sum_{m_2}\sum_{n_2} x_{3m_1 n_1}x_{3m_2 n_2}B_{mn}^{m_1 n_1 m_2 n_2},$$

$$F_{3mn} = \sum_{m_1}\sum_{n_1}\sum_{m_2}\sum_{n_2}\left[x_{3m_1 n_1}x_{1m_2 n_2}C_{mn}^{m_1 n_1 m_2 n_2} + \right.$$

$$\left. + x_{3m_1 n_1}x_{2m_2 n_2}D_{mn}^{m_1 n_1 m_2 n_2}\right],$$

where

$$A_{mn}^{m_1 n_1 m_2 n_2} = m_1\pi^3\left(m_2^2 + \frac{1-\mu}{2}\beta_1^2 n_2^2\right)a_{1mn}^{m_1 n_1 m_2 n_2} -$$

$$- \frac{1+\mu}{2}\pi^3\beta_1^2 n_1 m_2 n_2 a_{2mn}^{m_1 n_1 m_2 n_2},$$

$$B_{mn}^{m_1 n_1 m_2 n_2} = \beta_1 n_1\pi^3\left(\beta_1^2 n_2^2 + \frac{1-\mu}{2}m_2^2\right)a_{3mn}^{m_1 n_1 m_2 n_2} -$$

$$- \frac{1+\mu}{2}\pi^3\beta_1 m_1 m_2 n_2 a_{4mn}^{m_1 n_1 m_2 n_2},$$

$$C_{mn}^{m_1 n_1 m_2 n_2} = \pi^3 m_2\left(m_1^2 + \mu\beta_1^2 n_1^2\right)a_{5mn}^{m_1 n_1 m_2 n_2} +$$

$$+ (1-\mu)\pi^3\beta_1^2 m_1 n_1 n_2 a_{6mn}^{m_1 n_1 m_2 n_2}$$

$$- \pi^3 m_1\left(m_2^2 + \frac{1-\mu}{2}\beta_1^2 n_2^2\right)a_{7mn}^{m_1 n_1 m_2 n_2} -$$

$$- \frac{1+\mu}{2}\pi^3\beta_1^2 n_1 m_2 n_2 a_{8mn}^{m_1 n_1 m_2 n_2},$$

$$D_{mn}^{m_1 n_1 m_2 n_2} = \pi^3\beta_1 n_2\left(\beta_1^2 n_1^2 + \mu m_1^2\right)a_{5mn}^{m_1 n_1 m_2 n_2} +$$

$$+ (1-\mu)\pi^3\beta_1 m_1 n_1 m_2 a_{6mn}^{m_1 n_1 m_2 n_2} -$$

$$- \pi^3\beta_1 n_1\left(\beta_1^2 n_2^2 + \frac{1-\mu}{2}m_2^2\right)a_{8mn}^{m_1 n_1 m_2 n_2} -$$

$$- \frac{1+\mu}{2}\pi^3\beta_1 m_1 m_2 n_2 a_{7mn}^{m_1 n_1 m_2 n_2},$$

$$a_{1mn}^{m_1 n_1 m_2 n_2} = \int_0^1\int_0^1 \sin\pi m_1 x\sin\pi n_1 x\cos\pi m_2 x\times$$

$$\times\sin\pi n_2 y\cos\pi mx\sin\pi ny\,dxdy,$$

$$a_{2mn}^{m_1 n_1 m_2 n_2} = \int_0^1\int_0^1 \cos\pi m_1 x\cos\pi n_1 x\sin\pi m_2 x\times$$

$$\times\cos\pi n_2 y\cos\pi mx\sin\pi ny\,dxdy,$$

$$a_{3mn}^{m_1 n_1 m_2 n_2} = \int_0^1\int_0^1 \sin\pi m_1 x\cos\pi n_1 x\sin\pi m_2 x\times$$

$$\times\sin\pi n_2 y\sin\pi mx\cos\pi ny\,dxdy,$$

$$a_{4mn}^{m_1 n_1 m_2 n_2} = \int_0^1\int_0^1 \cos\pi m_1 x\sin\pi n_1 x\cos\pi m_2 x\times$$

$$\times\cos\pi n_2 y\sin\pi mx\cos\pi ny\,dxdy,$$

$$a_{5mn}^{m_1 n_1 m_2 n_2} = \int_0^1 \int_0^1 \sin \pi m_1 x \sin \pi n_1 x \sin \pi m_2 x \times$$
$$\times \sin \pi n_2 y \sin \pi m x \sin \pi n y \, dx \, dy,$$

$$a_{6mn}^{m_1 n_1 m_2 n_2} = \int_0^1 \int_0^1 \cos \pi m_1 x \cos \pi n_1 x \cos \pi m_2 x \times$$
$$\times \cos \pi n_2 y \sin \pi m x \sin \pi n y \, dx \, dy,$$

$$a_{7mn}^{m_1 n_1 m_2 n_2} = \int_0^1 \int_0^1 \cos \pi m_1 x \sin \pi n_1 x \cos \pi m_2 x \times$$
$$\times \sin \pi n_2 y \sin \pi m x \sin \pi n y \, dx \, dy,$$

$$a_{8mn}^{m_1 n_1 m_2 n_2} = \int_0^1 \int_0^1 \sin \pi m_1 x \cos \pi n_1 x \sin \pi m_2 x \times$$
$$\times \cos \pi n_2 y \sin \pi m x \sin \pi n y \, dx \, dy.$$

The analysis of the structure of equations (18)-(22) shows that equations (18) and (19) are coupled with each other via linear terms and with equation (20) in terms of nonlinear terms $F_{jmn} (j = 1,2,3)$. Equations (21) and (22) are coupled with each other and with Eq. (20) only via linear terms. Thus, the linearized equations (18)-(22) are decoupled in two linear subsystems.

3.1. Solution of the eigen value problem and decoupling the equations of motion

To determine the natural frequencies of linear vibrations ω_{imn} $(i = 1,2,3,4,5)$, it is a need to solve the linear eigen value problem. The characteristic equation of the linearized equations (18) and (19) has the form

$$\omega_{mn}^4 - \omega_{mn}^2 (S_{11}^{mn} + S_{22}^{mn}) + S_{11}^{mn} S_{22}^{mn} - S_{12}^{mn} S_{21}^{mn} = 0, \quad (25)$$

the solution of which gives the natural frequencies of in-plane vibrations

$$\omega_{1mn}^2 = \pi^2 \left(m^2 + \beta_1^2 n^2 \right),$$
$$\omega_{2mn}^2 = \frac{1-\mu}{2} \pi^2 \left(m^2 + \beta_1^2 n^2 \right), \qquad (26)$$

which coincide with those obtained in [16,19].
The linearized set of equations (20)-(22) provides the following frequency equation:

$$\omega_{mn}^6 + e_2^{mn} \omega_{mn}^4 + e_1^{mn} \omega_{mn}^2 + e^{mn} = 0, \qquad (27)$$

where

$$e^{mn} = s_{33}^{mn} s_{45}^{mn} s_{54}^{mn} + s_{44}^{mn} s_{35}^{mn} s_{53}^{mn} + s_{55}^{mn} s_{34}^{mn} s_{43}^{mn} -$$
$$- s_{33}^{mn} s_{44}^{mn} s_{55}^{mn} - s_{34}^{mn} s_{53}^{mn} s_{54}^{mn} - s_{43}^{mn} s_{35}^{mn} s_{45}^{mn},$$

$$e_1^{mn} = s_{33}^{mn} s_{44}^{mn} + s_{33}^{mn} s_{55}^{mn} + s_{44}^{mn} s_{55}^{mn} - s_{34}^{mn} s_{43}^{mn} -$$
$$- s_{35}^{mn} s_{53}^{mn} - s_{45}^{mn} s_{54}^{mn},$$

$$e_2^{mn} = -s_{33}^{mn} - s_{44}^{mn} - s_{55}^{mn}.$$

The solution of equation (27) results in three sets of natural frequencies, ω_{3mn}, ω_{4mn} and ω_{5mn}, and the least of them, ω_{3mn}, corresponds to the frequency of flexural vibrations. It is defined as

$$\omega_{3mn}^2 = \frac{1}{4\beta_2^2} \Big\{ 12k^2(1-\mu) +$$
$$+ \beta_2^2 \pi^2 \left(2 + k^2(1-\mu) \right) \left(m^2 + \beta_1^2 n^2 \right) -$$
$$- \Big[\big[12k^2(1-\mu) + \beta_2^2 \pi^2 \left(2 + k^2(1-\mu) \right) \left(m^2 + \beta_1^2 n^2 \right) \big]^2 -$$
$$- 8\beta_2^4 k^2 (1-\mu) \pi^4 \left(m^2 + \beta_1^2 n^2 \right)^2 \Big]^{1/2} \Big\}. \qquad (28)$$

The other two roots of equation (27) correspond to the high frequency vibrations and have the form

$$\omega_{4mn}^2 = \frac{1-\mu}{2} \left[\frac{12}{\beta_2^2} k^2 + \pi^2 \left(m^2 + \beta_1^2 n^2 \right) \right], \qquad (29)$$

$$\omega_{5mn}^2 = \frac{1}{4\beta_2^2} \{ 12k^2(1-\mu) +$$
$$+ \beta_2^2 \pi^2 \left(2 + k^2(1-\mu) \right) \left(m^2 + \beta_1^2 n^2 \right) +$$
$$+ \Big[\big[12k^2(1-\mu) + \beta_2^2 \pi^2 \left(2 + k^2(1-\mu) \right) \left(m^2 + \beta_1^2 n^2 \right) \big]^2$$
$$- 8\beta_2^4 k^2 (1-\mu) \pi^4 \left(m^2 + \beta_1^2 n^2 \right)^2 \Big]^{1/2} \}. \qquad (30)$$

The natural frequencies correspond to mutually orthogonal eigen vectors

$$L_{mn}^{I} \{L_{imn}^{I}\}, \quad L_{mn}^{II} \{L_{imn}^{II}\} \quad (i = 1, 2), \qquad (31)$$

$$L_{mn}^{III} \{L_{imn}^{III}\}, \quad L_{mn}^{IV} \{L_{imn}^{IV}\}, \quad L_{mn}^{V} \{L_{imn}^{V}\} \quad (i = 3, 4, 5). \quad (32)$$

Following [20], let us expand the matrices S_{ij}^{mn} $(i, j = 1,2)$, S_{ij}^{mn} $(i,j = 3,4,5)$ and generalized displacements x_{imn} entering in equations (18)-(22) in terms of the eigen vectors (31) and (32)

$$S_{ij}^{mn} = \omega_{1mn}^2 L_{imn}^{I} L_{jmn}^{I} + \omega_{2mn}^2 L_{imn}^{II} L_{jmn}^{II},$$
$$x_{imn} = X_{1mn} L_{imn}^{I} + X_{2mn} L_{imn}^{II} \qquad (i = 1, 2), \tag{33}$$

$$s_{ij}^{mn} = \omega_{3mn}^2 L_{imn}^{III} L_{jmn}^{III} + \omega_{4mn}^2 L_{imn}^{IV} L_{jmn}^{IV} + \omega_{5mn}^2 L_{imn}^{V} L_{jmn}^{V}, \tag{34}$$

$$x_{imn} = X_{3mn} L_{imn}^{III} + X_{4mn} L_{imn}^{IV} + X_{5mn} L_{imn}^{V} \quad (i = 3, 4, 5) \tag{35}$$

Now substituting expansions (33)-(35) in Eqs. (18)-(22) and then multiplying (18)-(19) successively by L_{imn}^{I}, L_{imn}^{II}, and (20)-(22) successively by L_{imn}^{III}, L_{imn}^{IV}, and finally by L_{imn}^{V} with due account for the conditions of orthogonality of the eigen vectors

$$\begin{aligned} L_{imn}^{K} L_{imn}^{N} &= 0 \quad \text{at} \quad K \neq N \\ L_{imn}^{K} L_{imn}^{K} &= 1 \quad (K, N = I, II, III, IV, V), \end{aligned} \tag{36}$$

we are led to the following set of equations of motion:

$$\ddot{X}_{1mn} + \chi_1 D^\gamma X_{1mn} + \omega_{1mn}^2 X_{1mn} = -\sum_i^2 F_{imn} L_{imn}^{I}, \tag{37}$$

$$\ddot{X}_{2mn} + \chi_2 D^\gamma X_{2mn} + \omega_{2mn}^2 X_{2mn} = -\sum_i^2 F_{imn} L_{imn}^{II}, \tag{38}$$

$$\ddot{X}_{3mn} + \chi_3 D^\gamma X_{3mn} + \omega_{3mn}^2 X_{3mn} = -F_{3mn} L_{3mn}^{III}, \tag{39}$$

$$\ddot{X}_{4mn} + \chi_4 D^\gamma X_{4mn} + \omega_{4mn}^2 X_{4mn} = 0, \tag{40}$$

$$\ddot{X}_{5mn} + \chi_5 D^\gamma X_{5mn} + \omega_{5mn}^2 X_{5mn} = 0, \tag{41}$$

in terms of new generalized displacements X_{jmn}

$$X_{1mn} = x_{1mn} L_{1mn}^{I} + x_{2mn} L_{2mn}^{I}, \tag{42}$$

$$X_{2mn} = x_{1mn} L_{1mn}^{II} + x_{2mn} L_{2mn}^{II}, \tag{42}$$

$$X_{3mn} = x_{3mn} L_{3mn}^{III} + x_{4mn} L_{2mn}^{III} + x_{5mn} L_{5mn}^{III}, \tag{43}$$

$$X_{4mn} = x_{3mn} L_{3mn}^{IV} + x_{4mn} L_{2mn}^{IV} + x_{5mn} L_{5mn}^{IV}, \tag{44}$$

$$X_{5mn} = x_{3mn} L_{3mn}^{V} + x_{4mn} L_{2mn}^{V} + x_{5mn} L_{5mn}^{V}. \tag{45}$$

It should be emphasized that the left-hand side parts of (37)-(41) are linear and independent of each other, while equations (37)-(39) are coupled only by non-linear terms in their right-hand sides.

Moreover, the set of equations (37)-(41) is decoupled into three subsystems, namely: the first subset compiles three nonlinear fractional derivative equations (37)-(39), the second and the third subsystems involve one linear fractional derivative equation each, i.e. equations (40) and (41), respectively. Thus, in order to find a solution, it is need to examine each subsystem.

3.2. Analysis of the reduced equations of motion

Equations (40) and (41) describe free damped vibrations of a linear oscillator with a viscoelastic resistance force modelled in terms of the fractional derivative Kelvin-Voigt model [24]. For the case of weak damping, i.e. when $\chi_i = \varepsilon \alpha_i$ or $\chi_i = \varepsilon^2 \alpha$ with $0 < \varepsilon = 1$, approximate analytical solutions of equations similar to (40) and (41) have been found in [28,52] utilyzing the fractional derivative expansion method [27], which is the extension of the multiple time scales procedure [53]. The case of ε-order damping and the half-derivative, i.e. when the order of the fractional derivative is $\gamma = 1/2$, was treated in [54] using the averaging perturbation technique.

Free damped vibrations of a linear fractional derivative Kelvin-Voigt oscillator in a medium with finite viscosity, i.e. without any restrictions on the magnitude of the damping coefficient χ_i, have been studied analytically in [24,52] utilizing the construction of the Green function, which was proposed for the first time for such fractional derivative equations by Professor Yury Rossikhin in his PhD thesis [55] in 1970 and then published in 1971 in the pioneer paper [56]. Further this procedure was generalized for dynamics of linear oscillators, beams, plates and shells using different fractional operator models, and their overview could be found in [24,28,29].

As for the first subsystem (37)-(39) involving three nonlinear equations with fractional derivative terms, then it has the similar structure as the set of three governing equations considered previously but ignoring the influence of the rotary inertia and shear deformations [19].

Following [19,20] it could be shown that the solution of equations (37)-(39) could be constructed using the

generalized method of multiple time scales suggested in [27]. We will not repeat this procedure, since it is described in detail in [20,57], and it could be easily adopted to equations (37)-(39) within an accuracy of coefficients.

Thus, it has been revealed that nonlinear vibrations of the plate could be accompanied by different types of the internal resonance when two or more modes could be coupled, resulting in the energy exchange between the coupled modes. Moreover, its type depends on the order of smallness of the viscosity involved into consideration. Thus, it has been found that at the ε – order, damped vibrations could be accompanied by the following types of the internal resonance:

the two-to-one internal resonance (2:1), when one natural frequency is twice the other natural frequency,

$$\omega_1 = 2\omega_3 \ (\omega_1 \neq \omega_2, \ 2\omega_3 \neq \omega_2), \qquad (47)$$

$$\omega_2 = 2\omega_3 \ (\omega_1 \neq \omega_2, \ 2\omega_3 \neq \omega_1), \qquad (48)$$

the one-to-one-to-two internal resonance (1:1:2), that is,

$$\omega_1 = \omega_2 = 2\omega_3 ; \qquad (49)$$

at the ε^2 -order, damped vibrations could be accompanied by the following types of the internal resonance:

the one-to-one internal resonance (1:1)

$$\omega_1 = \omega_2 \ (\omega_3 \neq \omega_1, \ \omega_3 \neq \omega_2), \qquad (50)$$

$$\begin{aligned} \omega_1 = \omega_3 \ (\omega_2 \neq \omega_1, \ \omega_2 \neq \omega_3), \\ \omega_2 = \omega_3 \ (\omega_1 \neq \omega_2, \ \omega_1 \neq \omega_3), \end{aligned} \qquad (51)$$

the one-to-one-to-one internal resonance (1:1:1)

$$\omega_1 = \omega_2 = \omega_3 , \qquad (52)$$

the combinational resonance of the additive-difference type

$$\omega_1 = \omega_2 + 2\omega_3 , \qquad (53)$$

$$\begin{aligned} \omega_1 = 2\omega_3 - \omega_2, \\ \omega_1 = \omega_2 - 2\omega_3, \end{aligned} \qquad (54)$$

where ω_1 and ω_2 are the frequencies of certain modes of in-plane vibrations in the x- and y- axes, respectively, and ω_3 is the frequency of a certain mode of out-of-plane vibrations.

For each type of the resonance, the nonlinear sets of resolving equations in terms of amplitudes and phase differences could be obtained using the same procedure as in [20]. The influence of viscosity on the energy exchange mechanism is revealed by the fact that each mode is characterized by its damping coefficient connected with the natural frequency by the exponential relationship with a negative fractional exponent. Thus, during free vibrations of the plate with internal resonances three regimes could be observed: stationary (absence of damping at $\gamma = 0$), quasistationary (damping is defined by an ordinary derivative at $\gamma = 1$), and transient (damping is defined by a fractional derivative at $0 < \gamma < 1$).

4. ANALYSIS OF SPECTRA OF NATURAL FREQUENCIES

In order to show that the phenomenon of internal resonance could be very critical, since in the thin plate under consideration the internal resonance is always present, it is a need to analyze the spectra of natural frequencies.

Thus, natural frequencies of vibrations ω_{imn} (i = 1,2,...,5) calculated according to (26) and (28)-(30), as well as frequency of vertical flexural vibrations without shear deformations and rotary inertia calculated via the formula [20]

$$\bar{\omega}_{3mn}^2 = \frac{\beta_2^2}{12}\pi^4 \left(m^2 + \beta_1^2 n^2 \right)^2 \qquad (55)$$

are given in Tables 1-3 for a square plate, i.e. at $\beta_1 = a/b = 1$, at $\beta_2 = h/a = 0.1$ and 0.025, respectively. Reference to Tables 1-3 shows the influence of the shear deformations and rotary inertia on the frequencies of flexural vibrations, in so doing the thicker the plate, the more difference between the frequencies ω_3 and $\bar{\omega}_3$. Thus, for example, for the square plate the frequency of the fundamental mode at $m = 1$, $n = 1$ calculated by the classical theory at $\beta_2 = 0.1$, 0.05 and 0.025 is reduced, respectively, by 3.51, 1.05 and 0.7% as compared with that calculated by the refined theory. This difference increases for more high frequencies, what is evident from Table 4. Natural frequencies for a rectangular plate at $\beta_1 = 0.5$ and $\beta_2 = 0.05$ are presented in Table 5. The influence of the ratio of the plate's dimensions on natural frequencies is seen from Table 6, whence it follows

that the difference between the frequencies according to classical and refined theories increases with the increase in plate's length.

From Tables 1-3 and 5 it is seen that the internal resonances of all types (47)-(54) could take place, and the occurrence of this or that case depends on the dimensions of the plate, i.e. on magnitudes of the coefficients β_1 and β_2.

As soon as the case of the internal resonance is revealed, then the further treatment of nonlinear

Table 1. Natural frequencies of vibrations ω_{imn} $(i = 1, 2, ..., 5)$ at $\beta_1 = 1$ and $\beta_2 = 0.1$.

m	n	ω_{1mn}	ω_{2mn}	$\omega_{3mn}/\bar{\omega}_{3mn}$	ω_{4mn}	ω_{5mn}
1	1	4.443	2.628	0.550/0.570	18.892	19.370
1	2	7.023	4.156	1.313/1.425	19.164	20.298
2	1	7.023	4.156	1.313/1.425	19.164	20.298
2	2	8.886	5.257	2.017/2.279	19.433	21.164
1	3	9.935	5.877	2.458/2.849	19.610	21.715
3	1	9.935	5.877	2.458/2.849	19.610	21.715
2	3	11.327	6.701	3.080/3.704	19.873	22.500
3	3	13.329	7.885	4.043/5.128	20.302	23.731
1	4	12.953	7.663	3.857/4.843	20.217	23.491
2	4	14.050	8.312	4.405/5.698	20.472	24.198
3	4	15.708	9.293	5.263/7.123	20.889	25.318
4	4	17.772	10.514	6.368/9.117	21.460	26.784
1	5	16.019	9.471	5.427/7.408	20.972	25.534
2	5	16.918	10.009	5.907/8.262	21.217	26.169
3	5	18.319	10.831	6.667/9.687	21.621	27.184
4	5	20.116	11.901	7.660/11.681	22.173	28.531
5	5	22.214	13.142	8.838/14.246	23.510	30.155

Table 2. Natural frequencies of vibrations ω_{imn} $(i = 1, 2, ..., 5)$ at $\beta_1 = 1$ and $\beta_2 = 0.05$.

m	n	ω_{1mn}	ω_{2mn}	$\omega_{3mn}/\bar{\omega}_{3mn}$	ω_{4mn}	ω_{5mn}
1	1	4.443	2.628	0.282/0.285	37.509	37.755
1	2	7.023	4.156	0.697/0.712	37.647	38.253
2	2	8.886	5.257	1.101/1.140	37.784	38.740
1	3	9.935	5.877	1.365/1.423	37.915	39.060
2	3	11.327	6.701	1.753/1.852	38.012	39.530
3	3	13.329	7.885	2.381/2.564	38.238	40.296
1	4	12.953	7.663	2.257/2.422	38.193	40.145
2	4	14.050	8.312	2.627/2.849	38.329	40.596
3	4	15.708	9.293	3.224/3.561	38.553	41.332
4	4	17.772	10.514	4.030/4.559	38.866	42.329
1	5	16.019	9.471	3.341/3.704	38.598	41.476
2	5	16.918	10.009	3.689/4.131	38.732	41.906
3	5	18.319	10.831	4.253/4.843	38.954	42.607
4	5	20.116	11.901	5.017/5.841	39.264	43.560
5	5	22.214	13.142	5.956/7.123	39.658	44.743

Table 3. Natural frequencies of vibrations ω_{imn} $(i = 1, 2, ..., 5)$ at $\beta_1 = 1$ and $\beta_2 = 0.025$.

m	n	ω_{1mn}	ω_{2mn}	$\omega_{3mn} / \bar{\omega}_{3mn}$	ω_{4mn}	ω_{5mn}
1	1	4.443	2.628	0.142/0.143	74.879	75.006
1	2	7.023	4.156	0.354/0.356	74.948	75.257
2	2	8.886	5.257	0.565/0.570	75.018	75.509
1	3	9.935	5.877	0.685/0.712	75.064	75.677
2	3	11.327	6.701	0.913/0.926	75.133	75.927
3	3	13.329	7.885	1.257/1.282	75.247	76.341
1	4	12.953	7.663	1.188/1.210	75.225	76.258
2	4	14.050	8.312	1.394/1.425	75.293	76.505
3	4	15.708	9.293	1.732/1.781	75.408	76.914
4	4	17.772	10.514	2.201/2.279	75.568	77.480
1	5	16.019	9.471	1.800/1.852	75.431	76.995
2	5	16.918	10.009	2.001/2.066	75.500	77.238
3	5	18.319	10.831	2.334/2.422	75.614	77.640
4	5	20.116	11.901	2.795/2.920	75.774	78.198
5	5	22.214	13.142	3.378/3.561	75.979	78.905

Table 4. Difference in vertical frequencies of flexural vibrations $\delta = \left[(\omega_3 - \bar{\omega}_3) / \bar{\omega}_3 \right] 100\%$ at $\beta_1 = 1$ for plates of different thickness.

	$m = 1$, $n = 1$			$m = 5$, $n = 5$		
β_2	0.1	0.05	0.025	0.1	0.05	0.025
δ, %	3.51	1.05	0.70	61.19	16.38	5.14

equations (37)-(39) could be carried out by the procedure developed in [27] within an accuracy of the coefficients.

CONCLUSION

In the present paper, the nonlinear free vibrations of fractionally damped plates are studied, equations of motion of which take the rotary inertia and shear deformations into account and involve five coupled nonlinear differential equations in terms of three mutually orthogonal displacements and two angles of rotation. The procedure resulting in decoupling linear parts of equations has been adopted with further utilization of the generalized method of multiple time scales for solving nonlinear governing equations of motion, in so doing the amplitude functions have been expanded into power series in terms of the small parameter and depend on different time scales.

Numerical analysis of the natural frequency spectra reveals the possibility of the occurrence of different internal and combinational resonances.

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REFERENCES

1. **Sathyamoorty M.** Nonlinear vibration analysis of plates: A review and survey of current developments // Applied Mechanics Reviews, 1987, Vol. 40(11), pp. 1553–1561.

Table 5. Natural frequencies of vibrations ω_{imn} $(i=1,2,...,5)$ at $\beta_1=0.5$ and $\beta_2=0.05$.

m	n	ω_{1mn}	ω_{2mn}	$\omega_{3mn}/\bar{\omega}_{3mn}$	ω_{4mn}	ω_{5mn}
1	1	3.685	2.531	0.177/0.178	37.474	37.629
1	2	5.211	3.580	0.282/0.285	37.509	37.755
2	1	6.550	4.129	0.594/0.605	37.612	38.129
2	2	7.370	5.062	0.697/0.712	37.647	38.253
1	3	7.171	4.675	1.266/1.318	37.841	38.940
3	1	9.602	5.860	0.456/0.463	37.566	37.963
2	3	8.714	6.109	0.866/0.830	37.704	38.457
3	2	10.142	6.611	1.365/1.425	37.875	39.059
3	3	11.055	7.593	1.528/1.603	37.932	39.257
1	4	9.264	5.839	0.697/0.712	37.647	38.253
4	1	12.699	7.653	2.164/2.315	38.159	40.031
2	4	10.423	7.159	1.101/1.140	37.784	38.740
4	2	13.101	8.258	2.257/2.422	38.193	40.145
3	4	12.324	8.639	1.753/1.852	38.012	39.530
4	3	13.780	9.125	2.417/2.600	38.250	40.334
4	4	14.740	10.125	2.626/2.849	38.329	40.596
1	5	11.409	7.051	1.001/1.033	37.750	38.619
5	1	15.814	9.470	3.253/3.597	38.564	41.368
2	5	12.330	8.236	1.397/1.460	37.887	39.099
5	2	16.134	9.971	3.341/3.704	38.598	41.476
3	5	13.880	9.686	2.038/2.172	38.114	39.878
5	3	16.674	10.726	3.487/3.882	38.654	41.656
4	5	15.969	11.169	2.898/3.170	38.430	40.930
5	4	17.437	11.647	3.688/4.131	38.732	41.906
5	5	18.425	12.656	3.945/4.452	38.832	42.223

Table 6. Difference in vertical frequencies of flexural vibrations $\delta=\left[(\omega_3-\bar{\omega}_3)/\bar{\omega}_3\right]100\%$ at $\beta_2=0.05$ for plates of different length.

	$m=1$, $n=1$			$m=5$, $n=5$		
β_1	0.5	1	2	0.5	1	2
δ, %	0.56	1.05	2.11	11.39	16.38	29.75

2. **Amabili M.** Nonlinear vibrations of rectangular plates with different boundary conditions: theory and experiments // Computers and Structures, 2004, Vol. 82, pp. 2587–2605.
3. **Amabili M.** Nonlinear vibrations and stability of shells and plates. London: Cambridge University Press, 2008.
4. **Breslavsky I.D., Amabili M., Legrand M.** Physically and geometrically non-linear vibrations of thin rectangular plates // International Journal of Non-Linear Mechanics, 2014, Vol. 58, pp. 30-40.
5. **Amabili M.** Nonlinear vibrations of viscoelastic rectangular plates // Journal of Sound and Vibration, 2016, Vol. 362, pp. 142-156.
6. **Amabili M.** Nonlinear damping in nonlinear vibrations of rectangular plates: Derivation from viscoelasticity and experimental validation //

Journal of Mechanics and Physics of Solids, 2018, Vol. 118, pp. 275–295.

7. **Amabili M.** Nonlinear damping in large-amplitude vibrations: modelling and experiments // Nonlinear Dynamics, 2018, Vol. 93, pp. 5–18.

8. **Rossikhin Yu.A., Shitikova M.V.** Thin bodies embedded in fractional derivative viscoelastic medium, Dynamic response. In: Encyclopedia of Continuum Mechanics (Altenbach H., chsner A.,eds.), Vol. 3, pp. 2512–2518. Berlin-Heidelberg: Springer, 2019.

9. **Clough R.W., Penzien J.** Dynamics of structures. New York: McGraw-Hill, 1975.

10. **(Stevanovic) Hedrih K.R., Simonovic J.D.** Structural analogies on systems of deformable bodies coupled with non-linear layers // International Journal of Non-Linear Mechanics, 2015, Vol. 73, pp. 18–24.

11. **Ribeiro P., Petyt M.** Nonlinear free vibration of isotropic plates with internal resonance // International Journal of Non-Linear Mechanics, 2000, Vol. 35, 263–278.

12. **Nayfeh A.H.** Nonlinear interaction: Analytical, computational, and experimental methods. New York: Wiley, 2000.

13. **Chang S.I., Bajaj A.K., Krousgrill C.M.** Non-linear vibrations and chaos in harmonically excited rectangular plates with one-to-one internal resonance // Nonlinear Dynamics, 1993, Vol. 4, pp. 433–460.

14. **Anlas G., Elbeyli O.** Nonlinear vibrations of a simply supported rectangular metallic plate subjected to transverse harmonic excitation in the presence of a one-to-one internal resonance // Nonlinear Dynamics, 2002, Vol. 30, 1–28.

15. **Hao Y.X., Zhang W., Ji X.L.** Nonlinear dynamic response of functionally graded rectangular plates under different internal resonances // Mathematical Problems in Engineering, 2010, Vol. 2010, Article ID 738648.

16. **Rossikhin Yu.A., Shitikova M.V.** Free damped non-linear vibrations of a viscoelastic plate under the two-to-one internal resonance // Materials Science Forum, 2003, Vol. 440–441, 29–36.

17. **Rossikhin Yu.A., Shitikova M.V., Ovsjannikova E.I.** Free damped vibrations of a nonlinear rectangular thin plate under the conditions of internal combinational resonance. In: Nonlinear Acoustics at the Beginning of the 21st Century, Proceedings of the 16th International Symposium on Nonlinear Acoustics (O.V. Rudenko and O.A. Sapozhnikov, eds.), August 19-23, 2002, Moscow, Russia, Vol.2, pp. 693–696.

18. **Rossikhin Yu.A., Shitikova M.V.** Analysis of free non-linear vibrations of a viscoelastic plate under the conditions of different internal resonances // International Journal of Non-Linear Mechanics, 2006, Vol. 2, 313–325.

19. **Rossikhin Yu.A., Shitikova M.V.** A new approach for studying nonlinear dynamic response of a thin fractionally damped plate with 2:1 and 2:1:1 internal resonances. In: Shell and Membrane Theories in Mechanics and Biology: From Macro- to Nanoscale Structures (H. Altenbach and G.I. Mikhasev, Eds.). Advanced Structured Materials, Vol. 45, Chapter 15, pp. 267–288. Berlin-Hiedelberg: Springer, 2015.

20. **Rossikhin Yu.A., Shitikova M.V., Ngenzi J.Cl.** A new approach for studying nonlinear dynamic response of a thin plate with internal resonance in a fractional viscoelastic medium // Shock and Vibration, 2015, Vol. 2015, Article ID 795606.

21. **Rossikhin Yu.A., Shitikova M.V., Ngenzi J.Cl.** Phenomenological analysis of the additive combinational internal resonance in nonlinear vibrations of fractionally damped thin plates // WSEAS Transactions of Applied and Theoretical Mechanics, 2015, Vol. 10, pp. 260–276.

22. **Rossikhin Yu.A., Shitikova M.V., Ngenzi J.Cl.** Fractional calculus application in problems of non-linear vibrations of thin plates with combinational internal resonances. // Procedia Engineering, 2016, Vol. 144, pp. 849–858.

23. **Witt A.A., Gorelik G.S.** Oscillations of an elastic pendulum as an example of the oscillations of two parametrically coupled linear systems // Journal of Technical Physics, 1933, № 3, pp. 294–307.

24. **Rossikhin Yu.A., Shitikova M.V.** Applications of fractional calculus to dynamic problems of linear and nonlinear hereditary mechanics of solids // Applied Mechanics Reviews, 1997, Vol. 50, pp. 15–67.

25. **Abdel-Ghaffar A.M., Housner G.W.** Ambient vibration tests of suspension bridge // ASCE

Journal of Engineering Mechanics, 1978, Vol. 104, pp. 983–999.

26. **Abdel-Ghaffar A.M., Scanlan R.H.** Ambient vibration studies of Golden Gate bridge: I. Suspended structure // ASCE Journal of Engineering Mechanics, 1985, Vol. 111, pp. 463–482.

27. **Rossikhin Yu.A., Shitikova M.V.** Application of fractional calculus for analysis of nonlinear damped vibrations of suspension bridges // ASCE Journal of Engineering Mechanics, 1998, Vol. 124, pp. 1029–1036.

28. **Rossikhin Yu.A., Shitikova M.V.** Application of fractional calculus for dynamic problems of solid mechanics: novel trends and recent results // Applied Mechanics Reviews, 2010, Vol. 63, Article ID 01081.

29. **Rossikhin Yu.A., Shitikova M.V.** Fractional calculus in structural mechanics. In: Baleanu, D., Lopes, A.M. (eds.) Handbook of Fractional Calculus with Applications. Vol 7, Applications in Engineering, Life and Social Sciences, Part A., pp. 159–192. Berlin: De Gruyter, 2019.

30. **(Stevanovic) Hedrih K.** Partial fractional differential equations of creeping and vibrations of plate and their solutions (First part) // Journal of Mechanical Behavior of Materials, 2005, Vol. 16, pp. 305–314.

31. **Ingman D., Suzdalnitsky J.** Response of viscoelastic plate to impact // ASME Journal of Vibration and Acoustics, 2008, Vol. 130, Article ID 011010.

32. **Rossikhin Yu.A., Shitikova M.V.** Analysis of damped vibrations of linear viscoelastic plates with damping modeled with fractional derivatives // Signal Processing, 2006, Vol. 86, pp. 2703–2711.

33. **(Stevanovic) Hedrih K.** Dynamics of coupled systems // Nonlinear Analysis: Hybrid Systems, 2008, Vol. 2, pp. 310–334.

34. **Rossikhin Yu.A., Shitikova M.V., Trung P.T.** Application of the fractional derivative Kelvin–Voigt model for the analysis of impact response of a Kirchhoff-Love plate // WSEAS Transaction on Mathematics, 2016, Vol. 15, pp. 498–501.

35. **Hilton H.H.** Implications and constraints of time-independent Poisson ratios in linear isotropic and anisotropic viscoelasticity // Journal of Elasticity, 2001, Vol 63, pp. 221-251.

36. **Ari M., Faal R.T., Zayernouri M.** Vibrations suppression of fractionally damped plates using multiple optimal dynamic vibration // International Journal of Computational Mathematics, 2020. DOI:10.1080/00207160.2019.1594792

37. **Mashrouteh S.** Nonlinear vibration analysis of viscoelastic plates with fractional damping. Master Thesis, University of Ontario, Institute of Technology, 2017.

38. **Permoon M.R., Haddadpour H., Javadi M.** Nonlinear vibration of fractional viscoelastic plate: primary, subharmonic, and superharmonic response // International Journal of Non-Linear Mechanics, 2018, Vol 99, pp. 154–164.

39. **Babouskos N.G., Katsikadelis J.T.** Nonlinear vibrations of viscoelastic plates of fractional derivative type: An AEM solution // The Open Mechanics Journal, 2010, Vol. 4, pp. 8–20.

40. **Hosseinkhani A., Younesian D., Farhangdoust S.** Dynamic analysis of a plate on the generalized foundation with fractional damping subjected to random excitation // Mathematical Problems in Engineering, 2018, Vol. 2018, Paper ID 3908371.

41. **Malara G., Spanos P.D.** Nonlinear random vibrations of plates endowed with fractional derivative elements // Probabalistic Engineering Mechanics, 2018, Vol. 54, pp. 2–8.

42. **Nwagoum Tuwa P.R., Miwadinou C.H., Monwanou A.V., Chabi Orou J.B., Woafo P.** Chaotic vibrations of nonlinear viscoelastic plate with fractional derivative model and subjected to parametric and external excitations // Mechanics Reserch Communications, 2019, Vol. 97, pp. 8–15.

43. **Litewka P., Lewandowski R.** Steady-state non-linear vibrations of plates using Zener material model with fractional derivative // Computational Mechanics, 2017, Vol. 60, pp. 333–354.

44. **Datta P., Ray M.C.** Fractional order derivative model of viscoelastic layer for active damping of geometrically nonlinear vibrations of smart composite plates // CMC, 2015, Vol. 49-50(1), pp. 47–80.

45. **Rossikhin Yu.A., Shitikova M.V., Shcheglova T.A.** Forced vibrations of a nonlinear oscillator

with weak fractional damping // Journal of Mechanics of Materials and Structures, 2009, Vol. 4(9), pp. 1619–1636.

46. **Shitikova M.V., Rossikhin Yu.A., Kandu V.** Interaction of internal and external resonances during force driven vibrations of a nonlinear thin plate embedded into a fractional derivative medium // Procedia Engineering, 2017, Vol. 199, pp. 832–837.

47. **Shitikova M.V., Kandu V.V.** Force driven nonlinear vibrations of a thin plate in one-to-one internal resonance in a fractional viscoelastic medium (in Russian) // News of Higher Educational Institutions. Construction. 2018, Issue 12, pp. 9–22.

48. **Shitikova M.V., Kandu V.V.** Force driven nonlinear vibrations of a thin plate with 1:1:1 internal resonance in a fractional viscoelastic medium // Journal of Physics: Conference Series, 2019, Vol. 1203, Article ID 012003.

49. **Shitikova M.V., Kandu V.V.** Force driven nonlinear vibrations of a thin plate with 1:1 internal resonance in a fractional viscoelastic medium // IOP Conference Series: Material Science Engineering, 2019, Vol. 489 Article ID 012043.

50. **Volmir A.S.** Nonlinear dynamics of plates and shells. Moscow: Nauka, 1972.

51. **Samko S.G., Kilbas A.A., Marichev O.I.** Fractional integrals and derivatives. Theory and applications. Amsterdam: Gordon and Breach Science Publishers, 1993.

52. **Rossikhin Yu.A., Shitikova M.V.** New approach for the analysis of damped vibrations of fractional oscillators // Shock and Vibration, 2009, Vol. 16(4), pp. 365–387.

53. **Nayfeh A.H.** Perturbation Methods. New York: Wiley, 1973.

54. **Wahi P., Chatterjee A.** Averaging oscillators with small fractional damping and delayed terms // Nonlinear Dynamics, 2004, Vol. 38, pp. 3–22.

55. **Rossikhin Yu.A.** Dynamic problems of linear viscoelasticity connected with investigation of the relaxation-retardation spectra. PhD thesis, Voronezh Polytechnical Institute, 1970.

56. **Meshkov S.I., Pachevskaja G.N., Postnikov V.S., Rossikhin Yu.A.** Integral representation of -functions and their application to problems in linear viscoelasticity // International Journal of Engineering Science, 1971, Vol. 9, pp. 387–398.

57. **Shitikova M.V.** The fractional derivative expansion method in nonlinear dynamic analysis of structures // Nonlinear Dynamics, 2020, Vol. 99(1), pp. 109–122.

СПИСОК ЛИТЕРАТУРЫ

1. **Sathyamoorty M.** Nonlinear vibration analysis of plates: A review and survey of current developments // Applied Mechanics Reviews, 1987, Vol. 40(11), pp. 1553–1561.

2. **Amabili M.** Nonlinear vibrations of rectangular plates with different boundary conditions: theory and experiments // Computers and Structures, 2004, Vol. 82, pp. 2587–2605.

3. **Amabili M.** Nonlinear vibrations and stability of shells and plates. London: Cambridge University Press, 2008.

4. **Breslavsky I.D., Amabili M., Legrand M.** Physically and geometrically non-linear vibrations of thin rectangular plates // International Journal of Non-Linear Mechanics, 2014, Vol. 58, pp. 30–40.

5. **Amabili M.** Nonlinear vibrations of viscoelastic rectangular plates // Journal of Sound and Vibration, 2016, Vol. 362, pp. 142–156.

6. **Amabili M.** Nonlinear damping in nonlinear vibrations of rectangular plates: Derivation from viscoelasticity and experimental validation // Journal of Mechanics and Physics of Solids, 2018, Vol. 118, pp. 275–295.

7. **Amabili M.** Nonlinear damping in large-amplitude vibrations: modelling and experiments // Nonlinear Dynamics, 2018, Vol. 93, pp. 5–18.

8. **Rossikhin Yu.A., Shitikova M.V.** Thin bodies embedded in fractional derivative viscoelastic medium, Dynamic response. In: Encyclopedia of Continuum Mechanics (Altenbach H., chsner A.,eds.), Vol. 3, pp. 2512–2518. Berlin-Heidelberg: Springer, 2019.

9. **Клаф Р., Пензиен Д.** Динамика сооружений. М.: Стройиздат, 1979.

10. **(Stevanovic) Hedrih K.R., Simonovic J.D.** Structural analogies on systems of deformable bodies coupled with non-linear layers // International Journal of Non-Linear Mechanics, 2015, Vol. 73, pp. 18–24.

11. **Ribeiro P., Petyt M.** Nonlinear free vibration of isotropic plates with internal resonance // International Journal of Non-Linear Mechanics, 2000, Vol. 35, 263–278.

12. **Nayfeh A.H.** Nonlinear interaction: Analytical, computational, and experimental methods. New York: Wiley, 2000.

13. **Chang S.I., Bajaj A.K., Krousgrill C.M.** Nonlinear vibrations and chaos in harmonically excited rectangular plates with one-to-one internal resonance // Nonlinear Dynamics, 1993, Vol. 4, pp. 433–460.

14. **Anlas G., Elbeyli O.** Nonlinear vibrations of a simply supported rectangular metallic plate subjected to transverse harmonic excitation in the presence of a one-to-one internal resonance // Nonlinear Dynamics, 2002, Vol. 30, 1–28.

15. **Hao Y.X., Zhang W., Ji X.L**. Nonlinear dynamic response of functionally graded rectangular plates under different internal resonances // Mathematical Problems in Engineering, 2010, Vol. 2010, Article ID 738648.

16. **Rossikhin Yu.A., Shitikova M.V.** Free damped non-linear vibrations of a viscoelastic plate under the two-to-one internal resonance // Materials Science Forum, 2003, Vol. 440–441, 29–36.

17. **Rossikhin Yu.A., Shitikova M.V., Ovsjannikova E.I.** Free damped vibrations of a nonlinear rectangular thin plate under the conditions of internal combinational resonance. In: Nonlinear Acoustics at the Beginning of the 21st Century, Proceedings of the 16th International Symposium on Nonlinear Acoustics (O.V. Rudenko and O.A. Sapozhnikov, eds.), August 19–23, 2002, Moscow, Russia, Vol.2, pp. 693–696.

18. **Rossikhin Yu.A., Shitikova M.V.** Analysis of free non-linear vibrations of a viscoelastic plate under the conditions of different internal resonances // International Journal of Non-Linear Mechanics, 2006, Vol. 2, 313–325.

19. **Rossikhin Yu.A., Shitikova M.V.** A new approach for studying nonlinear dynamic response of a thin fractionally damped plate with 2:1 and 2:1:1 internal resonances. In: Shell and Membrane Theories in Mechanics and Biology: From Macro- to Nanoscale Structures (H. Altenbach and G.I. Mikhasev, Eds.). Advanced Structured Materials, Vol. 45, Chapter 15, pp. 267–288. Berlin-Hiedelberg: Springer, 2015.

20. **Rossikhin Yu.A., Shitikova M.V., Ngenzi J.Cl.** A new approach for studying nonlinear dynamic response of a thin plate with internal resonance in a fractional viscoelastic medium // Shock and Vibration, 2015, Vol. 2015, Article ID 795606.

21. **Rossikhin Yu.A., Shitikova M.V., Ngenzi J.Cl.** Phenomenological analysis of the additive combinational internal resonance in nonlinear vibrations of fractionally damped thin plates // WSEAS Transactions of Applied and Theoretical Mechanics, 2015, Vol. 10, pp. 260–276.

22. **Rossikhin Yu.A., Shitikova M.V., Ngenzi J.Cl.** Fractional calculus application in problems of non-linear vibrations of thin plates with combinational internal resonances. // Procedia Engineering, 2016, Vol. 144, pp. 849–858.

23. **Витт А.А., Горелик Г.С.** Колебания упругого маятника как пример двух параметрически связанных линейных систем // Журнал технической физики, 1933, Т.3, № 2–3. С. 294–307.

24. **Rossikhin Yu.A., Shitikova M.V.** Applications of fractional calculus to dynamic problems of linear and nonlinear hereditary mechanics of solids // Applied Mechanics Reviews, 1997, Vol. 50, pp. 15–67.

25. **Abdel-Ghaffar A.M., Housner G.W.** Ambient vibration tests of suspension bridge // ASCE Journal of Engineering Mechanics, 1978, Vol. 104, pp. 983–999.

26. **Abdel-Ghaffar A.M., Scanlan R.H.** Ambient vibration studies of Golden Gate bridge: I. Suspended structure // ASCE Journal of Engineering Mechanics, 1985, Vol. 111, pp. 463–482.

27. **Rossikhin Yu.A., Shitikova M.V.** Application of fractional calculus for analysis of nonlinear damped vibrations of suspension bridges // ASCE Journal of Engineering Mechanics, 1998, Vol. 124, pp. 1029–1036.

28. **Rossikhin Yu.A., Shitikova M.V.** Application of fractional calculus for dynamic problems of solid mechanics: novel trends and recent results // Applied Mechanics Reviews, 2010, Vol. 63, Article ID 01081.

29. **Rossikhin Yu.A., Shitikova M.V.** Fractional calculus in structural mechanics. In: Baleanu,

D., Lopes, A.M. (eds.) Handbook of Fractional Calculus with Applications. Vol 7, Applications in Engineering, Life and Social Sciences, Part A., pp. 159–192. Berlin: De Gruyter, 2019.

30. **(Stevanovic) Hedrih K.** Partial fractional differential equations of creeping and vibrations of plate and their solutions (First part) // Journal of Mechanical Behavior of Materials, 2005, Vol. 16, pp. 305–314.

31. **Ingman D., Suzdalnitsky J.** Response of viscoelastic plate to impact // ASME Journal of Vibration and Acoustics, 2008, Vol. 130, Article ID 011010.

32. **Rossikhin Yu.A., Shitikova M.V.** Analysis of damped vibrations of linear viscoelastic plates with damping modeled with fractional derivatives // Signal Processing, 2006, Vol. 86, pp. 2703–2711.

33. **(Stevanovic) Hedrih K.** Dynamics of coupled systems // Nonlinear Analysis: Hybrid Systems, 2008, Vol. 2, pp. 310–334.

34. **Rossikhin Yu.A., Shitikova M.V., Trung P.T.** Application of the fractional derivative Kelvin–Voigt model for the analysis of impact response of a Kirchhoff-Love plate // WSEAS Transaction on Mathematics, 2016, Vol. 15, pp. 498–501.

35. **Hilton H.H.** Implications and constraints of time-independent Poisson ratios in linear isotropic and anisotropic viscoelasticity // Journal of Elasticity, 2001, Vol 63, pp. 221–251.

36. **Ari M., Faal R.T., Zayernouri M.** Vibrations suppression of fractionally damped plates using multiple optimal dynamic vibration // International Journal of Computational Mathematics, 2020. DOI: 10.1080/00207160.2019.1594792

37. **Mashrouteh S.** Nonlinear vibration analysis of viscoelastic plates with fractional damping. Master Thesis, University of Ontario, Institute of Technology, 2017.

38. **Permoon M.R., Haddadpour H., Javadi M.** Nonlinear vibration of fractional viscoelastic plate: primary, subharmonic, and superharmonic response // International Journal of Non-Linear Mechanics, 2018, Vol 99, pp. 154–164.

39. **Babouskos N.G., Katsikadelis J.T.** Nonlinear vibrations of viscoelastic plates of fractional derivative type: An AEM solution // The Open Mechanics Journal, 2010, Vol. 4, pp. 8–20.

40. **Hosseinkhani A., Younesian D., Farhangdoust S.** Dynamic analysis of a plate on the generalized foundation with fractional damping subjected to random excitation // Mathematical Problems in Engineering, 2018, Vol. 2018, Paper ID 3908371.

41. **Malara G., Spanos P.D.** Nonlinear random vibrations of plates endowed with fractional derivative elements // Probabalistic Engineering Mechanics, 2018, Vol. 54, pp. 2–8.

42. **Nwagoum Tuwa P.R., Miwadinou C.H., Monwanou A.V., Chabi Orou J.B., Woafo P.** Chaotic vibrations of nonlinear viscoelastic plate with fractional derivative model and subjected to parametric and external excitations // Mechanics Reserch Communications, 2019, Vol. 97, pp. 8–15.

43. **Litewka P., Lewandowski R.** Steady-state non-linear vibrations of plates using Zener material model with fractional derivative // Computational Mechanics, 2017, Vol. 60, pp. 333–354.

44. **Datta P., Ray M.C.** Fractional order derivative model of viscoelastic layer for active damping of geometrically nonlinear vibrations of smart composite plates // CMC, 2015, Vol. 49-50(1), pp. 47–80.

45. **Rossikhin Yu.A., Shitikova M.V., Shcheglova T.A.** Forced vibrations of a nonlinear oscillator with weak fractional damping // Journal of Mechanics of Materials and Structures, 2009, Vol. 4(9), pp. 1619–1636.

46. **Shitikova M.V., Rossikhin Yu.A., Kandu V.** Interaction of internal and external resonances during force driven vibrations of a nonlinear thin plate embedded into a fractional derivative medium // Procedia Engineering, 2017, Vol. 199, pp. 832–837.

47. **Шитикова М.В., Канду В.В.** Численный анализ вынужденных колебаний нелинейных пластинок в вязкоупругой среде при наличии внутреннего резонанса один к одному // Известия вузов. Строительство, 2018, № 12. С. 9–22.

48. **Shitikova M.V., Kandu V.V.** Force driven nonlinear vibrations of a thin plate with 1:1:1 internal resonance in a fractional viscoelastic medium // Journal of Physics: Conference Series, 2019, Vol. 1203, Article ID 012003.

49. **Shitikova M.V., Kandu V.V.** Force driven nonlinear vibrations of a thin plate with 1:1 internal resonance in a fractional viscoelastic medium // IOP Conference Series: Material Science Engineering, 2019, Vol. 489 Article ID 012043.

50. **Вольмир А.С.** Нелинейная динамика пластинок и оболочек. М.: Наука, 1972.

51. **Самко Г.С., Килбас А.А., Маричев О.И.** Дробные интегралы и производные: Теория и приложения. Минск: Наука и техника, 1987.

52. **Rossikhin Yu.A., Shitikova M.V.** New approach for the analysis of damped vibrations of fractional oscillators // Shock and Vibration, 2009, Vol. 16(4), pp. 365–387.

53. **Найфэ А.Х.** Методы возмущений. М.: Мир, 1976.

54. **Wahi P., Chatterjee A.** Averaging oscillators with small fractional damping and delayed terms // Nonlinear Dynamics, 2004, Vol. 38, pp. 3–22.

55. **Россихин Ю.А.** Динамические задачи линейной вязко-упругости, связанные с исследованием ретардационно-ре- лаксационных спектров. Диссертация на соикание ученой степени канд. физ.-мат. наук. Воронеж, 1970.

56. **Meshkov S.I., Pachevskaja G.N., Postnikov V.S., Rossikhin Yu.A**. Integral representation of -functions and their application to problems in linear viscoelasticity // International Journal of Engineering Science, 1971, Vol. 9, pp. 387–398.

57. **Shitikova M.V.** The fractional derivative expansion method in nonlinear dynamic analysis of structures // Nonlinear Dynamics, 2020, Vol. 99(1), pp. 109–122.

Marina V. Shitikova, Advisor of the Russian Academy of Architecture and Construction Sciences, Prof., Dr.Sc., Research Center on Dynamics of Solids and Structures; Voronezh State Technical University; 84, 20-letija Oktyabrya, Voronezh, 394006, Russia; phone +7 (473) 271-52-68; fax +7 (473) 271-52-68; Senior Researcher, RAASN Research Institute of Structural Physics, Moscow, Russia. E-mail: mvs@vgasu.vrn.ru.

Шитикова Марина Вячеславовна, советник РААСН, профессор, доктор физико-математических наук; руководитель международного научного Центра по фундаментальным исследованиям в области естественных и строительных наук; Воронежский государственный технический университет; 394006, Россия, г. Воронеж, ул. 20 лет Октября, д. 84, тел. +7 (473) 271-52-68; факс +7 (473) 271-52-68; Главный научный сотрудник, Научно-исследовательский институт строительной физики РААСН, Москва, Россия. E-mail: mvs@vgasu.vrn.ru.

Elena I. Osipova, Cand. Sc., Research Center on Dynamics of Solids and Structures; Voronezh State Technical University; 84, 20-letija Oktyabrya, Voronezh, 394006, Russia, E-mail: oss@vgasu.vrn.ru.

Осипова Елена Ивановна, кандидат физико-математических наук, доцент кафедры строительной механики, Воронежский государственный технический университет; 394006, Россия, г. Воронеж, ул. 20 лет Октября, д. 84, E-mail: : oss@vgasu.vrn.ru