EXAMPLE OF GRAGUAL TRANSFORMATION OF STIFFNESS MATRIX AND MAIN SET OF EQUATIONS AT ADDITIONAL FINITE ELEMENT METHOD

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Abstract: The paper considers the example of gradual transformation of the stiffness matrix and the main set of equations at Additional Finite Element Method (AFEM). It is corresponded to the increase of load and the ideal failure model of structure. AFEM uses the additional design diagrams and additional finite elements (AFE) for this operation. This process is illustrated by the transformation of design diagram of bended concrete console from the beginning of its loading to the collapse. The structure reveals four physical nonlinear properties before the ultimate limit state. Every nonlinear property appears under the action of corresponded load. The stiffness matrix and the set of equations are changed under influence of the value of load and the presence of observed nonlinear properties at this moment.

Keywords: Additional finite element method, Finite element method, Stiffness matrix, Set of equations, Additional design diagram, Additional finite element, Ideal failure model

ПРИМЕР ПОСТЕПЕННОГО ПРЕОБРАЗОВАНИЯ МАТРИЦЫ ЖЕСТКОСТИ И ОСНОВНОЙ СИСТЕМЫ УРАВНЕНИЙ МЕТОДА ДОПОЛНИТЕЛЬНЫХ КОНЕЧНЫХ ЭЛЕМЕНТОВ

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Аннотация: В статье рассматривается пример постепенного преобразования матрицы жесткости и основной системы уравнений метода дополнительных конечных элементов (МДКЭ). Это преобразование происходит в соответствии с ростом нагрузки и идеальной моделью разрушения конструкции. Для выполнения этой операции МДКЭ использует дополнительные расчетные схемы из дополнительных конечных элементов (ДКЭ). Для иллюстрации этого процесса рассмотрено изменение расчетной схемы изгибаемой бетонной консоли от начала нагружения до разрушения. Эта конструкция проявляет четыре физически нелинейных свойства к моменту достижения ею предельного состояния. Каждое нелинейное свойство появляется при действии соответствующей нагрузки. Матрица жесткости и система уравнений меняются в зависимости от величины нагрузки и наличия тех нелинейных свойств, которые наблюдаются в этот момент.

Ключевые слова: метод дополнительных конечных элементов, метод конечных элементов, матрица жесткости, система уравнений, дополнительная расчетная схема, дополнительный конечный элемент, идеальная модель разрушения

INTRODUCTION

Some characteristics are necessary when Additional Finite Element Method (AFEM) is used for analysis at limit states of structures with several physical nonlinear properties:

1) The number of all nonlinear properties;
2) The sequence of its appearance before ultimate limit state;
3) The way of taking into account for each nonlinear property;
4) The stress-strain state when each nonlinear property is appeared.

These factors act at the initial design diagram, the stiffness matrix and the main set of algebraic equations. Also they determine of the way of nonlinear analysis. Thus, the problem of mathematic description of this process is appeared. This description must correspond to the logic of AFEM and FEM, the character of observed nonlinear properties and the requirement of limit state analysis. The developed AFEM is destined for decision of this problem. The examples are necessary for verification and realization of its algorithms.

1. GENERAL INFORMATION OF AFEM

Additional Finite Element Method (AFEM) [1] is suggested by author as the variant of the development of Finite Element Method (FEM) [2, 3]. It is destined for analysis of structures with several (n) nonlinear properties at ultimate limit state (state of ultimate equilibrium). It adds the some elements of the Method of Limit States (Ultimate Equilibrium) [4, 5] and the Method of Elastic Decisions [6, 7] to the usual sequence of solving problems by FEM. AFEM is numerical method for combination and development of three science directions:

1) The mathematic basis for several (n) transformations of main set of equations and extension of possibilities of FEM for solving of n-nonlinear problems;
2) The decision of the problems of structural mechanics for analysis of structures at limit states as n-nonlinear systems;
3) The analysis of real structures at limit states as n-nonlinear systems.

The example is given for application AFEM to nonlinear analysis of plane reinforced concrete structure with four nonlinear properties. It corresponds to normative requirements [8 - 10].

2. PROBLEM AND WAY OF DECISION

The nonlinear analysis at limit state is considered for the bended console.

This structure reaches its limit state under increased load gradually.

It gradually reveals four nonlinear properties:
1) The plasticity;
2) The partial unload due to the redistribution of stresses after the cracking;
3) The presence of the cracking;
4) The ultimate limit state before the collapse.

The realization of nonlinear analysis at limit state demands one linear analysis and four nonlinear ones depending on the number of nonlinear properties at this step of loading. The way of these analyses is:

1) The initial linear analysis without any nonlinear properties;
2) The plastic analysis with one nonlinear property;
3) The analysis with taking into account of two nonlinear properties: the plasticity and the partial unload due to redistribution of stresses after cracking;
4) The analysis with taking into account of three nonlinear properties: the plasticity, the partial unload due to the redistribution of stresses and the presence of the cracking;
5) The analysis with taking into account of four nonlinear properties: the plasticity, the partial unload due to the redistribution of stresses, the cracking and limit state.

3. GROWTH OF LOAD AND FORM OF MAIN SET OF EQUATIONS

The growth of load P and appearance of nonlinear properties are the main factors for realization of analysis of structure at limit state.

These factors influence over transformation of the design diagram, the stiffness matrix of structure and the main set of algebraic equations.

3.1. Growth of load and nonlinear properties
There is the condition of analysis at first limit state for guarantee the bearing capacity of structure:

$$P_{max} \leq P_{lim}. \qquad (1)$$

Where P_{max} = maximal value of external static load which is equal to minimal kinematic one. P_{lim} = minimal internal resistance of the structure to this external load.

The external load P changes from 0 to P_{lim} gradually ($P \rightarrow P_{lim}$):

$$P_1 = 0 < P_2 = P_3 < P_4 = P_{max} = P_{lim}. \quad (2)$$

Where P_i = the intermediate value of load P when i-th nonlinear property is appeared (i changes from 1 to $n = 4$).

The first nonlinear property is plasticity ($i = 1$). It is observed from load $P = 0$ to load $P = P_{lim}$, i.e. all time of loading. It is only nonlinear property under load $P_1 = 0 < P < P_2$. The second nonlinear property is the partial unload ($i = 2$) due to redistribution of stresses after the cracking. It is appeared under load $P = P_2 = P_3$ together the crack simultaneously. It is manifested from load $P = P_2 = P_3$ to load $P = P_4$, i.e. interval of load $P_2 = P_3 < P < P_4 = P_{lim}$. The third nonlinear property is the existence of crack ($i = 3$). It is observed during interval of load $P_3 < P < P_4 = P_{lim}$. The last nonlinear property is ultimate limit state ($i = n = 4$). It is occurred under load $P = P_4 = P_{max} = P_{lim}$. Thus, the way (2) of the growth of load P is:

$$P_1 = 0 \rightarrow P_2 = P_3 \rightarrow P_4 = P_{max} = P_{lim}. \quad (3)$$

The condition (3) is the first for the formation of the main set of equations.

3.2. Transformation of design diagram

The condition (3) demands the gradual transformation of the design diagram of structure in the nest sequence:

1) The initial linear design diagram of structure without nonlinear properties ($i = 0$) under load $P = P_1 = 0$;
2) The design diagram of structure with plastic property ($i=1$) only under load $P_1=0<P< P_2$;
3) The design diagram of structure with two nonlinear properties under load $P = P_2 = P_3$: plasticity ($i = 1$) and the partial unload ($i=2$) due to redistribution of stresses after the cracking;

4) The design diagram of structure with three nonlinear properties under load $P_3 < P < P_4$: plasticity ($i = 1$), the partial unload ($i = 2$) due to redistribution of stresses and the cracking ($i = 3$);
5) The design diagram of structure with four nonlinear properties under load $P = P_4 = P_{max} = P_{lim}$: plasticity ($i = 1$), the partial unload ($i = 2$) due to redistribution of stresses, the existence of crack ($i = 3$) and limit state ($i = 4$).

In nonlinear analysis of structure at limit state the initial design diagram gradually takes the three intermediated forms and fifth at last: 1) $\rightarrow$ 2) $\rightarrow$ 3) $\rightarrow$ 4) $\rightarrow$ 5). The fifth last form is *ideal failure model* or design diagram of structure at limit state. It is necessary for realization of nonlinear analysis by AFEM [11].

Also, when AFEM is used for nonlinear analysis, all five forms of design diagram must have the identical characteristics, for example the same number of nodes points, view and number of finite elements (FE's). It is necessary for the definition of stiffness matrixes of all forms of design diagram.

3.3. Transformation of stiffness matrix

The fulfillment of the condition (1) requires the definition of minimal value of internal resistance P_{lim}. Usually this minimum corresponds to the minimum stiffness of structure due to negative influence of each i-th nonlinear property. For considered example i changes from 1 to $n = 4$. The stiffness matrix is changed gradually from initial value K to its minimal value K_{min} due to these defects:

$$K \rightarrow K_1 \rightarrow K_2 \rightarrow K_3 \rightarrow K_4 = K_{min} = K_{lim}. \quad (4)$$

Where K = stiffness matrix of structure without nonlinear properties ($i = 0$) under the load $P = P_1 = 0$;

K_1 = stiffness matrix of structure with plastic property ($i=1$) under the load $P_1 = 0 < P < P_2$;

K_2 = stiffness matrix of structure with plastic property ($i=1$) and the partial unload ($i=2$) due to redistribution of stresses after cracking under the load $P = P_2 = P_3$;

K_3 = stiffness matrix of structure with plastic property ($i = 1$), the partial unload ($i = 2$) due to redistribution of stresses and the cracking ($i = 3$) under the load $P_3 < P < P_4$;

K_4 = stiffness matrix of structure with plastic property ($i = 1$), the partial unload ($i = 2$) due to redistribution of stresses, the existence of cracking ($i = 3$) and limit state ($i = 4$) under the load $P = P_4$;

K_{min} = stiffness matrix of structure with $n=4$ nonlinear properties at moment of its minimal internal resistance to external load $P=P_4=P_{max}$;

K_{lim} = stiffness matrix of structure at limit state under the load $P=P_4=P_{max}=P_{lim}$, when its design diagram is *ideal failure model.*

The condition (4) is the second for the formation of the main set of equations. All matrices K, K_1, K_2, K_3, K_4 (K_{min}, K_{lim}) must have the equal dimensions and the same filling for computer realization of analysis by AFEM.

3.4. Required transformation of the set of algebraic equations

The main operation of analysis by FEM and AFEM is the solving of the set of equations:

$$K_{nonl}V = P . \qquad (5)$$

Where P = matrix of external load;
V = matrix of unknown node displacements;
K_{nonl} = stiffness matrix of structure with nonlinear properties. This matrix is changed in accordance with the degree of its influence. The stiffness matrix K_{nonl} is formed from coefficients of stiffness matrices of the separate finite elements (FE's).

The set of equations (5) solves one time in linear analysis because of matrix $K_{nonl} = K = const$ due to the absence of nonlinear properties.

In nonlinear analysis this set of equation must be solved by iterative process because of $K_{nonl} \neq K \neq const$. In this process matrix K turns into matrix K_{nonl} gradually. The transformation of the set of equation (5) is connected with difficulties in presence of several (n) of physical nonlinear properties due to its different causes. When $n =$

4 this transformation must go under the condition (3) for right part and the condition (4) for the left one of the set of equations (5):
Under the load $P = P_1 = 0$ and $i = 0$

$$KV = P . \qquad (6)$$

Under the load $P_1 = 0 < P < P_2$ and $i = 1$:

$$K_1V = P. \qquad (7)$$

Under the load $P = P_2 = P_3$ and $i = 2$

$$K_2V = P. \qquad (8)$$

Under the load $P_3 < P < P_4$ and $i = 3$

$$K_3V = P . \qquad (9)$$

Under the load $P = P_4 = P_{max} = P_{lim}$ and $i = 4$

$$K_4V = P. \qquad (10)$$

Thus the initial form of the set equations (6) takes the requirement forms (7), (8), (9) and (10) gradually according to the value of load P.
In limit state of structure (see (1)) the set of equations (10) must became

$$K_{lim}V = P_{lim} . \qquad (11)$$

This description (11) corresponds to the next view of expression (1)

$$P_{lim} = P_{max} . \qquad (12)$$

Method of Limit States guarantees the appearance the equality (12) for formula (1) in one case from million ones (see s. 3.1).

4. SET OF EQUATIONS AT AFEM

The Additional Finite Element Method (AFEM) was suggested by author [1] as the variant of the Finite Element Method (FEM) for analysis of structures with several nonlinear properties at

limit states. It is numerical combination of the three effective methods of structural analysis: FEM, Method of Elastic Decisions and Limit State Method. It solves the problem of analysis of structure at limit states according to failure model, when nonlinear properties and defects are revealed due to increase of load. AFEM uses the additional finite elements and additional design diagrams for gradually transformation of main set of equations [12].

4.1. Transformation of design diagram by means of additional design diagrams

The example is illustrated the action of additional design diagrams at the initial design diagram for bending console in plane stress-state (see table 1). The initial design diagram consists of 8 triangular deep beam finite elements with liner properties (p.1 table 1). AFEM uses four additional design diagrams for transformation of the initial design diagram into ideal failure model of console (see s. 3.2):

1) The initial design diagram of structure without nonlinear properties $(i = 0)$ under load $P = P_1 = 0$ transforms into the design diagram of structure with first $(i = 1)$ nonlinear property (plasticity) by means of the first additional design diagram under load $P_1 = 0 < P < P_2$ (p. 2 table 1);

2) The design diagram of structure with one $(i=1)$ nonlinear property (plasticity) under load $P_1 = 0 < P < P_2$ transforms into the design diagram of structure with two $(i = 2)$ nonlinear properties (the plasticity $(i = 1)$ and partial unload $(i = 2)$ due to redistribution of stresses after cracking) by means of the second additional design diagram under load $P = P_2 = P_3$ (p. 3 table 1);

3) The design diagram of structure with two $(i=2)$ nonlinear properties (the plasticity $(i=1)$ and the partial unload $(i=2)$) under load $P=P_2=P_3$ transforms into the design diagram of structure with three $(i=3)$ nonlinear properties (the plasticity $(i=1)$, the partial unload $(i=2)$ and the cracking $(i=3)$) by means of the third additional design diagram under load $P_3<P<P_4$ (p. 4 table 1);

4) The design diagram of structure with three $(i=3)$ nonlinear properties (the plasticity $(i=1)$, the partial $(i=2)$ unload and the cracking $(i=3)$) under load $P_3 < P < P_4$ transforms into the design diagram of structure with four nonlinear properties ((the plasticity $(i=1)$, the partial $(i=2)$ unload, the cracking $(i=3)$ and limit state $(i=4)$) or ideal failure model by means of the fourth additional design diagram under load $P = P_4= P_{max} = P_{lim}$ (p. 5 table 1).

Every additional design diagram may be compared with empty space imbedded in the initial design diagram. It is filled negative stiffness for taking into account of only one nonlinear property. It consists of corresponding additional finite elements (AFE-s) (see s. 4.5). Additional design diagrams are basic for realization of nonlinear analysis at limit state due to fulfillment of conditions (3) and (4).

4.2. Transformation of initial stiffness matrix by means of stiffness matrices of additional design diagrams

The condition (4) is realized due to using of stiffness matrices of additional design diagrams.

Under application of AFEM the next equation is correct at the moment of limit state of structure with four nonlinear properties:

$$K_{lim}= K+\Delta K_1+ \Delta K_2+ \Delta K_3 + \Delta K_4 . \quad (13)$$

Where $\Delta K_1, \Delta K_2, \Delta K_3, \Delta K_4$ = stiffness matrices of the first, the second, the third and the fourth additional design diagrams consisting of additional finite elements (AFE's) for taking into account the first, the second, the third and the fourth nonlinear property respectively.

The stiffness matrices of additional design diagrams are destined for fulfillment of condition (4) and may be defined according to next formulas (see s. 3.3):

$$\Delta K_1 = K_1 - K , \quad (14)$$
$$\Delta K_2 = K_2 - K_1 , \quad (15)$$
$$\Delta K_3 = K_3 - K_2 , \quad (16)$$
$$\Delta K_4 = K_4 - K_3. \quad (17)$$

The next way is used for the gradual transformation of the stiffness matrix K of initial design diagram of structure without nonlinear properties into stiffness matrix $K_{lim} = K_{min} = K_4$ of ideal of failure model or design diagram of structure at limit state (see table 1):

Under the load $P_1 = 0 < P < P_2$ and $i = 1$

$$K_1 = K + \Delta K_1 . \qquad (18)$$

Under the load $P = P_2 = P_3$ and $i = 2$

$$K_2 = K + \Delta K_1 + \Delta K_2. \qquad (19)$$

Under the load $P_3 < P < P_4$ and $i = 3$

$$K_3 = K + \Delta K_1 + \Delta K_2 + \Delta K_3. \qquad (20)$$

Under the load $P = P_4 = P_{max} = P_{lim}$ and $i = 4$
$$K_4 = K + \Delta K_1 + \Delta K_2 + \Delta K_3 + \Delta K_4 . \qquad (21)$$
The initial stiffness matrix K transforms gradually according to formulas (18) -– (21) by means of stiffness matrices of $\Delta K_1, \Delta K_2, \Delta K_3, \Delta K_4$. The main characteristics of matrices K, K_1, K_2, K_3, K_4, ΔK_1, ΔK_2, ΔK_3 and ΔK_4 are: the same dimensions; the same filling positions; the square; the symmetry; the linearity; the positivity of matrices K, K_1, K_2, K_3, K_4; the negativity of matrices ΔK_1, ΔK_2, ΔK_3, ΔK_4. These characteristics are necessary for application of matrix theory [13]. The fulfillment of the condition (4) demands these characteristics for mathematic realization. Thus steps (18) – (21) are developed on the base of Method Elastic Decisions when the nonlinear stiffness matrix is divided into linear and nonlinear component [14].

4.3. Transformation of the set of algebraic equations
AFEM suggests the using of additional design diagrams consisting of additional finite elements (AFE-s) (see s. 3.4 and s. 4.2). In this case the sets of equations (7) – (10) are formed according to the formulas (18) – (21) under conditions (3) and (4):
Under the load $P_1 < P < P_2$ and $i = 1$

$$(K + \Delta K_1)V = P. \qquad (22)$$

Under the load $P_2 = P = P_3$ and $i = 2$

$$(K + \Delta K_1 + \Delta K_2)V = P . \qquad (23)$$

Under the load $P_3 < P < P_4$ and $i = 3$

$$(K + \Delta K_1 + \Delta K_2 + \Delta K_3)V = P . \qquad (24)$$

Under the load $P = P_4 = P_{max} = P_{lim}$ and $i = 4$

$$(K + \Delta K_1 + \Delta K_2 + \Delta K_3 + \Delta K_4)V = P. \qquad (25)$$

Thus, the algebraic equations (22) – (25) are corresponded to requirement forms for numerical realization of analysis at limit state of structure with four nonlinear properties.

Also the Method of Elastic Decision (Method of Additional loads) may used for the solving of these sets of equations. In this case the formulas (22)–(25) are formed according to the next way:
Under the load $P_1 < P < P_2$ and $i = 1$:

$$KV = P - \Delta K_1 V. \qquad (26)$$

Under the load $P_2 = P = P_3$ and $i = 2$:

$$KV = P - \Delta K_1 V - \Delta K_2 V. \qquad (27)$$

Under $P_3 < P < P_4$ $i = 3$:

$$KV = P - \Delta K_1 V - \Delta K_2 V - \Delta K_3 V . \qquad (28)$$

Under $P = P_4 = P_{max} = P_{lim}$ and $i = 4$:

$$KV = P - \Delta K_1 V - \Delta K_2 V - \Delta K_3 V - \Delta K_4 V . \qquad (29)$$

In relations (26)–(29) values $(-\Delta K_1 V)$, $(-\Delta K_2 V)$, $(-\Delta K_3 V)$ and $(-\Delta K_4 V)$ determines the influence of the first, the second, the third and the fourth nonlinear property respectively. For example the term $(-\Delta K_1 V)$ of the right-hand part of these equations is the additional load which with the main load P must be applied to linear structure to

reach the displacements corresponding to its displacements with the first nonlinear property under the action of the only external load P.

In nonlinear analysis at limit state the sets of algebraic equations (7) – (10) take the forms (22) – (25) or (26) – (29). These forms provide the taking into account the influence of each of four nonlinear property of structure. This way allows the using of different theoretical data [15 – 17] for nonlinear analysis [18 – 20] according to normative rules [8 – 10].

Thus logic of AFEM is corresponded to FEM.

4.4. Two ways for realization of iterative process at AFEM

Solution of the set of algebraic equations by iterative methods is the main step for nonlinear analysis of structures. AFEM suggests two ways for creation of this process [21]. Both ways are based on the decision of the set (6):

$$V = K^{-1}P \qquad , \qquad (30)$$

Where K^{-1} = inverse stiffness matrix K.

Operations connected with obtaining of this inverse matrix K^{-1} are the most laborious. They take roughly three quarters of time for solving of the set of equations (1). In the first case iterative process is based on (21)–(25) and (30):

Under the load $P_1 < P < P_2$ and $i = 1$

$$V^{(k)} = (K + \Delta K_1^{(k-1)})^{-1} P . \qquad (31)$$

Under $P_2 = P = P_3$ and $i = 2$

$$V^{(k)} = (K + \Delta K_1^{(k-1)} + \Delta K_2^{(k-1)})^{-1} P. \qquad (32)$$

Under $P_3 < P < P_4$ and $i = 3$

$$V^{(k)} = (K + \Delta K_1^{(k-1)} + \Delta K_2^{(k-1)} + \Delta K_3^{(k-1)})^{-1} P . \qquad (33)$$

Under the load $P = P_4 = P_{max} = P_{lim}$ and $i = 4$

$$V^{(k)} = (K + \Delta K_1^{(k-1)} + \Delta K_2^{(k-1)} + \Delta K_3^{(k-1)} + \Delta K_4^{(k-1)})^{-1} P. \qquad (34)$$

Where k, $(k$-1$)$ = moving and previous iterations. This way is very laborious due to the obtaining of inverse matrix K^{-1} at everyone iteration.

The second way for realization of iterative process is based on the next views of the formulas (26) – (29):

Under the load $P_1 < P < P_2$ and $i = 1$

$$KV^{(k)} = P - \Delta K_1^{(k-1)} V^{(k-1)} . \qquad (35)$$

Under the load $P_2 = P = P_3$ and $i = 2$

$$KV^{(k)} = P - \Delta K_1^{(k-1)} V^{(k-1)} - \Delta K_2^{(k-1)} V^{(k-1)} . \qquad (36)$$

Under the load $P_3 < P < P_4$ and $i = 3$)

$$KV^{(k)} = P - \Delta K_1^{(k-1)} V^{(k-1)} - \Delta K_2^{(k-1)} V^{(k-1)} - \Delta K_3^{(k-1)} V^{(k-1)}. \qquad (37)$$

Under the load $P = P_4 = P_{max} = P_{lim}$ and $i = 4$

$$KV^{(k)} = P - \Delta K_1^{(k-1)} V^{(k-1)} - \Delta K_2^{(k-1)} V^{(k-1)} - \Delta K_3^{(k-1)} V^{(k-1)} - \Delta K_4^{(k-1)} V^{(k-1)}. \qquad (38)$$

The iterative process goes in accordance to formulas (35) – (38) and (30):

Under the load $P_1 < P < P_2$ and $i = 1$

$$V^{(k)} = K^{-1}(P - \Delta K_1^{(k-1)} V^{(k-1)}) . \qquad (39)$$

Under the $P_2 = P = P_3$ and $i = 2$:

$$V^{(k)} = K^{-1}(P - \Delta K_1^{(k-1)} V^{(k-1)} - \Delta K_2^{(k-1)} V^{(k-1)}) . \qquad (40)$$

Under $P_3 < P < P_4$ and $i = 3$:

$$V^{(k)} = K^{-1}(P - \Delta K_1^{(k-1)} V^{(k-1)} - \Delta K_2^{(k-1)} V^{(k-1)} - \Delta K_3^{(k-1)} V^{(k-1)}) . \qquad (41)$$

Under the load $P = P_4 = P_{max} = P_{lim}$ and $i = 4$

$$V^{(k)} = K^{-1}(P - \Delta K_1^{(k-1)} V^{(k-1)} - \Delta K_2^{(k-1)} V^{(k-1)} - \Delta K_3^{(k-1)} V^{(k-1)} - \Delta K_4^{(k-1)} V^{(k-1)}) . \qquad (42)$$

The formulas (39) – (42) are the results of solution of the sets of equations (35) – (38). They allow the obtaining of inverse stiffness matrix K^{-}

[1]at the first iteration only when $k = 1$. This advantage is useful when set of equations is solved by means of Gauss Elimination. The second way for creation of iterative process of AFEM is less laborious then the first one.

4.5. Additional finite elements

The condition (4) requires the fulfillment analogous one for every finite element. Due to nonlinear properties the its stiffness matrix gradually decreases from initial value K_e to its minimal value $K_{e,min}$. Usually this minimum corresponds to limit state, when the carrying capacity of finite element is lost and $K_{e,min}=K_{e,lim}= 0$ or close to 0.

If the number of nonlinear properties i changes from 1 to $n = 4$, the next condition is correct

$$K_e \rightarrow K_{e,1} \rightarrow K_{e,2} \rightarrow K_{e,3} \rightarrow K_{e,4}=K_{e,min}=K_{e,lim}=0$$
(43)

Where K_e = stiffness matrix of finite element without nonlinear properties ($i = 0$);

$K_{e,1}$ = stiffness matrix of finite element with plastic property ($i = 1$);

$K_{e,2}$ = stiffness matrix of finite element with plastic property ($i = 1$) and the partial unload ($i = 2$) due to redistribution of stresses after cracking;

$K_{e,3}$ = stiffness matrix of finite element with plastic property ($i = 1$), the partial unload ($i = 2$) due to redistribution of stresses and the cracking ($i = 3$);

$K_{e,4}$ = stiffness matrix of finite element with plastic property ($i = 1$), the partial unload ($i = 2$) due to redistribution of stresses, the existence of cracking ($i= 3$) and limit state ($i = n = 4$);

$K_{e,min}$ = stiffness matrix of finite element with $n=4$ nonlinear properties at moment of its minimal value;

$K_{e,lim}$ = stiffness matrix of finite element at limit state, when its value is closed to 0.

Four additional finite elements (AFE-s) are necessary for fulfillment of the condition (43). They transform gradually the initial finite element with linear properties into the same finite element with all nonlinear ones [1].

The stiffness matrix $\Delta K_{e,1}$ of the first additional finite element for taking into account the plastic property ($i = 1$) is equal

$$\Delta K_{e,1} = K_{e,1} - K_e.$$
(44)

The value $\Delta K_{e,1}$ depends on the level of stress-strain state under load $P_1 = 0 < P < P_4 = P_{max} = P_{lim}$. These additional finite elements are formed the first additional design diagram for taking into account the plasticity in formulas (22) – (25).

The stiffness matrix $\Delta K_{e,2}$ of the second additional finite element for taking into account the partial unload ($i = 2$) due to redistribution of stresses after cracking has next formula:

$$\Delta K_{e,2} = K_{e,2} - K_{e,1}.$$
(45)

The value $\Delta K_{e,2}$ depends on the stress-strain state under load $P=P_2$ when crack is appeared.
These additional finite elements are formed the second additional design diagram for taking into account the partial unload due to redistribution of stresses after cracking in formulas (23) – (25).

The stiffness matrix $\Delta K_{e,3}$ of the third additional finite element for taking into account the existence of cracking ($i = 3$) is defined

$$\Delta K_{e,3} = K_{e,3} - K_{e,2}.$$
(46)

The value $\Delta K_{e,3}$ depends on the level of stress-strain state under load $P_3 < P < P_4 = P_{max} = P_{lim}$.
These additional finite elements are consisted the third additional design diagram for taking into account the existence of cracking in formulas (24) and (25).

The stiffness matrix $\Delta K_{e,4}$ of the fourth additional finite element for taking into account the limit state ($i = 4$) is equal

$$\Delta K_{e,4} = K_{e,4} - K_{e,3}.$$
(47)

The value $\Delta K_{e,4}$ depends on the level of limit stress-strain state under load $P = P_4 = P_{max} = P_{lim}$.
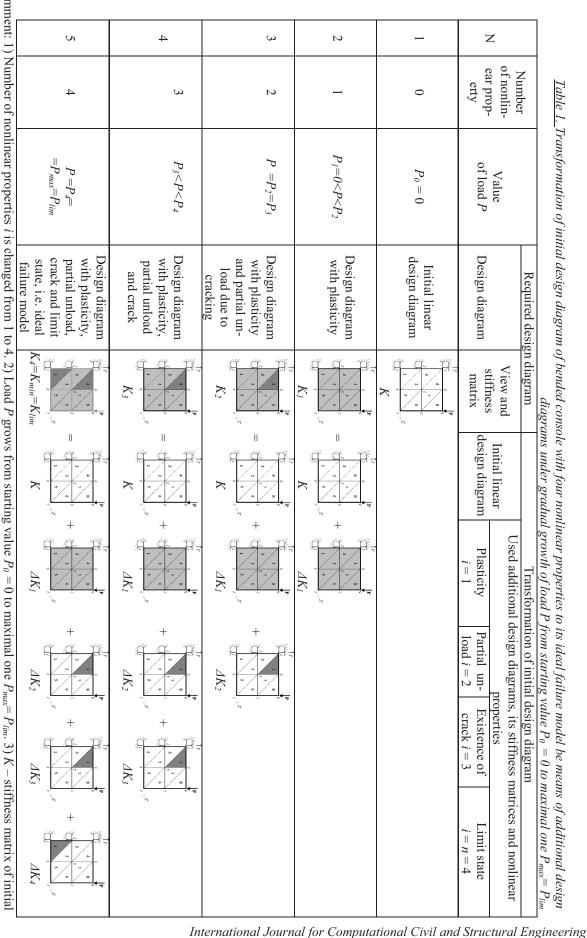
Table 1. Transformation of initial design diagram of bended console with four nonlinear properties to its ideal failure model be means of additional design diagrams under gradual growth of load P from starting value $P_0 = 0$ to maximal one $P_{max} = P_{lim}$

N	Number of nonlinear property	Value of load P	Required design diagram		Transformation of initial design diagram				
			Design diagram	View and stiffness matrix	Used additional design diagrams, its stiffness matrices and nonlinear properties				
					Initial linear design diagram	Plasticity $i = 1$	Partial unload $i = 2$	Existence of crack $i = 3$	Limit state $i = n = 4$
1	0	$P_0 = 0$	Initial linear design diagram	K					
2	1	$P_1 = 0 < P < P_2$	Design diagram with plasticity	K_1	$=$ K	$+$ ΔK_1			
3	2	$P = P_2 = P_3$	Design diagram with plasticity and partial unload due to cracking	K_2	$=$ K	$+$ ΔK_1	$+$ ΔK_2		
4	3	$P_3 < P < P_4$	Design diagram with plasticity, partial unload and crack	K_3	$=$ K	$+$ ΔK_1	$+$ ΔK_2	$+$ ΔK_3	
5	4	$P = P_4 = P_{max} = P_{lim}$	Design diagram with plasticity, partial unload, crack and limit state, i.e. ideal failure model	$K_4 = K_{min} = K_{lim}$	$=$ K	$+$ ΔK_1	$+$ ΔK_2	$+$ ΔK_3	$+$ ΔK_4

Comment: 1) Number of nonlinear properties i is changed from 1 to 4. 2) Load P grows from starting value $P_0 = 0$ to maximal one $P_{max} = P_{lim}$. 3) K – stiffness matrix of initial linear design diagram; K_1, K_2, K_3, K_4, – stiffness matrix of design diagram with taking into account of one, two, three and four nonlinear properties respectively; ΔK_1, ΔK_2, ΔK_3, ΔK_4 – stiffness matrix of the first, the second, the third and the fourth additional design diagram consisting of additional finite elements (AFE's) taking into account the first, the second, thee third and the fourth nonlinear property respectively; K_{min} and K_{lim} – stiffness matrix of structure at limit state (ideal failure modal of structure)

If stiffness matrix of finite element at limit state $K_{e,4} = K_{e,min} = K_{e,lim} = 0$ the stiffness matrix of its additional finite element $\Delta K_{e,4} = - K_{e,3}$.

These additional finite elements are consisted the fourth additional design diagram for taking into account the limit state in formula (25).

The initial design diagram of bended console is transformed to its ideal failure model due to four additional finite elements (table 1).

CONCLUSIONS

Considered example proves the possibility of realization the nonlinear analysis at limit state for bended console according to its ideal failure model by means of Additional Finite Element Method (AFEM).

Additional design diagrams and additional finite elements are used for gradual transformation of the stiffness matrix and the main set of equations.

Next conditions are fulfilled for this process:

1) the correspondence to algorithmic logic of nonlinear analysis due to conservation of main mathematic characteristics of stiffness matrix of structure under the numerical variation of its coefficients;

2) the orientation at gradual achievement of criterion of limit state before collapse;

3) the guarantee the allowance for each of four nonlinear properties at stress-strain state;

4) the creation of iterative process for solving of the main set of equations by two ways: usual manner and use the advantages of Method of Elastic Decisions.

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