

A.A. ILYUSHIN'S FINAL RELATION, ALTERNATIVE EQUIVALENT RELATIONS AND VERSIONS OF ITS APPROXIMATION IN PROBLEMS OF PLASTIC DEFORMATION OF PLATES AND SHELLS

PART 1: A.A. ILYUSHIN'S FINAL RELATION

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Abstract: The finite relationship between the forces and moments of plates and shells in the parametric form of the theory of small elastoplastic deformations is investigated of A.A. Ilyushin, to determine the load-bearing capacity of structures from a material without hardening. A geometric image of the exact yield surface in the space of generalized stresses is obtained. In the first part of the article the conclusion of the final relation is given. In the second and third parts, by introducing other parameters, alternative equivalent dependences of the final relationship have been developed and variants of its approximation for application in computational practice are considered. In the fourth part, additional properties of the final relationship are considered, the possibility and necessity of its use in problems of plastic deformation of plates and shells is shown.

Keywords: the plasticity theory, plastic deformation of plates and shells, surface of fluidity, plasticity condition

КОНЕЧНОЕ СООТНОШЕНИЕ А.А. ИЛЮШИНА, АЛЬТЕРНАТИВНЫЕ ЭКВИВАЛЕНТНЫЕ ЗАВИСИМОСТИ И ВАРИАНТЫ ЕГО АППРОКСИМАЦИИ В ЗАДАЧАХ ПЛАСТИЧЕСКОГО ДЕФОРМИРОВАНИЯ ПЛАСТИН И ОБОЛОЧЕК

ЧАСТЬ 1: КОНЕЧНОЕ СООТНОШЕНИЕ А.А. ИЛЮШИНА

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Аннотация: Выполнено исследование конечного соотношения между силами и моментами пластин и оболочек в параметрическом виде теории малых упругопластических деформаций А.А. Ильюшина, для определения несущей способности конструкций из материала без упрочнения. Получен геометрический образ точной поверхности текучести в пространстве обобщенных напряжений. В первой части статьи приводится вывод конечного соотношения. Во второй и третьей частях введением других параметров разработаны альтернативные эквивалентные зависимости конечного соотношения и рассмотрены варианты его аппроксимации для применения в расчетной практике. В четвертой части рассмотрены дополнительные свойства конечного соотношения, показана возможность и необходимость его использования в задачах пластического деформирования пластин и оболочек.

Ключевые слова: теория пластичности, пластическое деформирование пластин и оболочек, поверхность текучести, условия пластичности

INTRODUCTION

Theory of small elastoplastic deformations Ilyushin was created in connection with the problem of the strength of the projectile while moving in the barrel of the gun. All calculations were carried out by the methods of the theory of elasticity, although a small residual plastic deformation was allowed by normative documents. Together with theorems on simple loading, unloading, and the method of elastic solutions, the theory of A.A. Ilyushin was a powerful apparatus for investigating the strength, deformability and stability of structural elements, structures and machine parts beyond the elastic limit. [1-8]

The theory of elastoplastic deformations of plates and shells is presented by A.A. Ilyushin in [9-14], where on the basis of the methods of the theory of plasticity a finite relationship between forces and moments was obtained to determine the load-bearing capacity of structures from a material without hardening and the limiting state is characterized by the propagation of fluidity throughout the volume.

Since the equations of the theory of plates and shells are formulated in generalized forces and generalized displacements, the conditions of strength and plasticity must also be represented in generalized forces. The transformation of the condition of strength and plasticity from the stress space into the space of generalized stresses is one of the most important and complex problems of the theory of limiting equilibrium of plates and shells [15-16].

The parametric equation of the limiting hypersurface in generalized stresses for thin plates and shells on the basis of the Mises condition of plasticity and the relations of the theory of small elastoplastic deformations was first obtained by A.A. Ilyushin [9,13]. The traditional Kirchhoff-Love hypotheses and the incompressibility of the material are used. Received A.A. Ilyushin's relations are not expressed in explicit form and are complex for solving practical problems. The geometrical image of the exact yield surface in

the literature is absent.

Similar relations with the introduction of other parameters were obtained in the works of V.V. Rozhdestvensky [17], G.S. Shapiro [18], P.G. Hodge [19-22], D.C. Drucker, H.G. Hopkins [23], D.C. Drucker [24], D.C. Drucker, R.T. Shield [25], E.T. Onat [26,28], E.T. Onat, W. Prager [27] and other authors. Detailed reviews of literature on this topic can be found in [15-16], as well as in the works of N. Jones [29-30] and Yu.V. Nemirovsky, TP Romanova [31].

In the works of M.I. Erhov [32-33], on the basis of the two-layer cross section model and the flow conditions of R. Mises, a finite relationship between the internal forces and the moments of ideally plastic plates and shells is obtained on the assumption that the strain intensity within the layer is constant in the plastic region. Here is a schematic model of the exact yield surface and the proposed version of its approximation. A similar model of the approximation of the cross-section of a homogeneous shell by a two-layered cross section was used by V.I. Rosenblum [34-37], Yu.N. Rabotnov [38]. This approach and its various variants were used by other authors.

If the shell material is ideally plastic and satisfies to a condition of fluidity of Mises, for a plastic condition $\sigma_i = \sigma_s = \text{const}$. In this case in purely plastic areas of a shell the right parts of determining relations for the generalised pressure will be uniform functions of a zero order concerning six parameters ε_i, χ_i .

From this necessity of existence of a final relation which plays a role of a condition of fluidity follows and connects values of efforts and the moments in purely plastic areas of a shell [9,39]. Owing to noted property of uniformity of the equations in purely plastic areas ε_i, χ_i it is possible to replace deformation components ε_i, χ_i in the corresponding speeds $\dot{\varepsilon}_i, \dot{\chi}_i$.

The definition of the ultimate load reduces to the construction of internal stress fields, moments, displacements, and velocities of dis-

placements of the middle surface that satisfy equilibrium equations in the plastic regions, the final relationship, the dependencies between the velocities of displacements of the middle surface and the deformation rates that determine the relations for generalized stresses. In rigid regions, the velocities must vanish or correspond with the rigid displacement with joints, and the forces and moments must satisfy the equilibrium conditions and do not contradict the final relation. The specified static and kinematic boundary conditions must also be satisfied [9, 39].

The final relation corresponding to the defining equations [9] has a very complex structure and is not explicitly expressed. For an approximate analysis, it is approximated by a quadratic dependence [9-10, 32-37], which corresponds to the particular case [9], while the bilinear form

$$P_{\varepsilon\chi} = \varepsilon_1\chi_1 + \varepsilon_2\chi_2 + \frac{1}{2}\varepsilon_1\chi_2 + \frac{1}{2}\varepsilon_2\chi_1 + \varepsilon_{11}\chi_{12} = 0 :$$

$$\frac{1}{N_s^2} \left(N_1^2 - N_1 N_2 + N_2^2 + 3N_{12}^2 \right) +$$

$$+ \frac{1}{N_s^2} \left(M_1^2 - M_1 M_2 + M_2^2 + 3M_{12}^2 \right) = 1,$$

$$N_s = \sigma_s h, \quad M_s = \frac{\sigma_s h^2}{4}.$$

For the axisymmetric problem, the following approximations are also used.

1. *A semilinear final relation* [19, 32-37], which corresponds to the linearization of the previous relation

$$n^2 + m^2 = 1, \quad n = \max \left\{ \left| \frac{N_1}{N_s} \right|, \left| \frac{N_2}{N_s} \right|, \left| \frac{N_1 - N_2}{N_s} \right| \right\},$$

$$m = \max \left\{ \left| \frac{M_1}{M_s} \right|, \left| \frac{M_2}{M_s} \right|, \left| \frac{M_1 - M_2}{M_s} \right| \right\}.$$

2. *The final relationship with a limited interaction of forces and moments* [19, 32-37], which

does not take into account the interaction of membrane and bending force factors, and others. The degree of approximation of these relations to the exact one [9] depends on the ratio $0 \leq P_{\varepsilon\chi}^2 \leq P_\varepsilon \cdot P_\chi$. Meanwhile, elementary analysis shows that in the center of a flexible circular plate or a slender axisymmetric shell is always satisfied $P_{\varepsilon\chi}^2 = P_\varepsilon \cdot P_\chi \neq 0$.

In the works of V.I. Korolev [40] and P.M. Ogibalov [41] deduces the derivation of the finite relation AA. Ilyushin and solve the problem for the simplest complex stress state of shells at $P_\chi \neq 0, P_\varepsilon \neq 0, P_{\varepsilon\chi} = 0$.

The purpose of this article is to investigate the final relationship of AA. Ilyushin, obtaining a geometric image of the exact yield surface, alternative dependencies and variants of its approximation.

In the first part of the paper, with some abbreviations, the derivation of the final relation presented in §24-26 [9] is given. In contrast to [9], the designations of stresses, forces and shear forces in the shell sections have been changed

$$\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{zx}, \tau_{zy}$$

$N_1, N_2, N_{12}, n_1, n_2, n_{12}, Q_1, Q_2$, while the numbering of formulas, tables, graphs and references to formulas are completely preserved. In the second and third parts, alternative equivalent dependencies of the final relationship are developed and variants of its approximation are considered for application in computational practice. In the fourth part, additional properties of the final relationship are considered, the possibility and necessity of its use in problems of plastic deformation of plates and shells is shown.

1.1. The connection between internal forces, moments and deformations of the shell on the basis of the theory of small elastoplastic deformations

Intensity of deformations, according to (4.7):

$$e_i = \frac{2}{\sqrt{3}} \sqrt{P_\varepsilon - 2zP_{\varepsilon\chi} + z^2P_\chi},$$

$$P_\varepsilon = \varepsilon_1^2 + \varepsilon_1\varepsilon_2 + \varepsilon_2^2 + \varepsilon_{12}^2, \quad P_\chi = \chi_1^2 + \chi_1\chi_2 + \chi_2^2 + \chi_{12}^2,$$

$$P_{\varepsilon\chi} = \varepsilon_1\chi_1 + \varepsilon_2\chi_2 + \frac{1}{2}\varepsilon_1\chi_2 + \frac{1}{2}\varepsilon_2\chi_1 + \varepsilon_{11}\chi_{12}. \quad (4.19)$$

The stresses according to (4.2):

$$S_x = \sigma_x - \frac{1}{2}\sigma_y = \frac{\sigma_i}{e_i}(\varepsilon_1 - z\chi_1),$$

$$S_y = \sigma_y - \frac{1}{2}\sigma_x = \frac{\sigma_i}{e_i}(\varepsilon_2 - z\chi_2), \quad (4.20)$$

$$S_{xy} = \tau_{xy} = \frac{2\sigma_i}{3e_i}(\varepsilon_{12} - z\chi_{12}),$$

And σ_i there is a certain function e_i , the voltage τ_{xz} , τ_{yz} , σ_z is small in comparison with the main ones. If the shell is thin enough and the ratio of its thickness to the characteristic radius of curvature can be neglected, we obtain the following five expressions for the forces:

$$N_1 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_x dz, \quad N_2 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_y dz, \quad N_{12} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{xy} dz,$$

$$Q_1 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{zx} dz, \quad Q_2 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{zy} dz. \quad (4.21)$$

The shearing forces Q_1 , Q_2 , despite the small stresses τ_{zx} , τ_{zy} , are not equal to zero, and are determined from the equilibrium equations. Similarly, one can write formulas for bending and twisting moments

$$M_1 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_x z dz, \quad M_2 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_y z dz, \quad M_{12} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{xy} z dz. \quad (4.22)$$

For simplification of calculations instead of forces N_1 , N_2 , N_{12} it is convenient to enter their linear combinations

$$S_1 = N_1 - \frac{1}{2}N_2 = \int_{-\frac{h}{2}}^{\frac{h}{2}} S_x dz, \quad S_2 = N_2 - \frac{1}{2}N_1 = \int_{-\frac{h}{2}}^{\frac{h}{2}} S_y dz,$$

$$\frac{2}{3}S_{12} = N_{12} = \int_{-\frac{h}{2}}^{\frac{h}{2}} S_{xy} dz, \quad (4.23)$$

And instead of the moments M_1 , M_2 , M_{12} of their combination

$$H_1 = M_1 - \frac{1}{2}M_2 = \int_{-\frac{h}{2}}^{\frac{h}{2}} S_x z dz,$$

$$H_2 = M_2 - \frac{1}{2}M_1 = \int_{-\frac{h}{2}}^{\frac{h}{2}} S_y z dz, \quad (4.24)$$

$$\frac{2}{3}H_{12} = M_{12} = \int_{-\frac{h}{2}}^{\frac{h}{2}} S_{xy} z dz.$$

From (4.23) and (4.20) we have:

$$\begin{aligned}
S_1 &= \varepsilon_1 \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\sigma_i}{e_i} dz - \chi_1 \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\sigma_i}{e_i} z dz, \\
S_2 &= \varepsilon_2 \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\sigma_i}{e_i} dz - \chi_2 \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\sigma_i}{e_i} z dz, \\
S_{12} &= \varepsilon_{12} \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\sigma_i}{e_i} dz - \chi_{12} \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\sigma_i}{e_i} z dz,
\end{aligned} \quad (4.23')$$

And from (4.24) we have:

$$\begin{aligned}
H_1 &= \varepsilon_1 \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\sigma_i}{e_i} z dz - \chi_1 \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\sigma_i}{e_i} z^2 dz, \\
H_2 &= \varepsilon_2 \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\sigma_i}{e_i} z dz - \chi_2 \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\sigma_i}{e_i} z^2 dz, \\
H_{12} &= \varepsilon_{12} \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\sigma_i}{e_i} z dz - \chi_{12} \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\sigma_i}{e_i} z^2 dz.
\end{aligned} \quad (4.24')$$

In formulas (4.23') and (4.24'), there are three types of integrals that are common in shell thickness:

$$J_1 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\sigma_i}{e_i} dz, \quad J_2 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\sigma_i}{e_i} z dz, \quad J_3 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\sigma_i}{e_i} z^2 dz. \quad (4.25)$$

Through them the forces and moments are expressed:

$$\begin{aligned}
\frac{3}{4} N_1 &= \left(\varepsilon_1 + \frac{1}{2} \varepsilon_2 \right) J_1 - \left(\chi_1 + \frac{1}{2} \chi_2 \right) J_2, \\
\frac{3}{4} N_2 &= \left(\varepsilon_2 + \frac{1}{2} \varepsilon_1 \right) J_1 - \left(\chi_2 + \frac{1}{2} \chi_1 \right) J_2, \\
\frac{3}{2} N_{12} &= \varepsilon_{12} J_1 - \chi_{12} J_2, \\
\frac{3}{4} M_1 &= \left(\varepsilon_1 + \frac{1}{2} \varepsilon_2 \right) J_2 - \left(\chi_1 + \frac{1}{2} \chi_2 \right) J_3, \\
\frac{3}{4} M_2 &= \left(\varepsilon_2 + \frac{1}{2} \varepsilon_1 \right) J_2 - \left(\chi_2 + \frac{1}{2} \chi_1 \right) J_3, \\
\frac{3}{2} M_{12} &= \varepsilon_{12} J_2 - \chi_{12} J_3.
\end{aligned} \quad (4.26) \quad (4.27)$$

Since in (4.25) σ_i there is a given function of e_i , and its form for each material becomes known in particular problems, it is natural to get rid of integration with respect to z and proceed from (4.19) to integrate over e_i .

Multiplying J_1 by P_ε , J_2 by $-2P_{\varepsilon\chi}$, J_3 by J_χ and by adding the results, we get:

$$J_1 P_\varepsilon - 2J_2 P_{\varepsilon\chi} + J_3 P_\chi = \frac{3}{4} \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_i e_i dz. \quad (4.28)$$

Differentiating (4.19) with respect to z , we find:

$$\frac{3}{4} e_i de_i = (z P_\chi - P_{\varepsilon\chi}) dz. \quad (4.29)$$

Multiply no J_1 by $-2P_{\varepsilon\chi}$ and J_2 on P_χ and add the results, then we get:

$$-J_1 P_{\varepsilon\chi} + J_2 P_\chi = \frac{3}{4} \int \sigma_i de_i. \quad (4.30)$$

We find the expression z^2 by e_i , for this it is necessary to solve the quadratic equation (4.19)

$$z^2 P_\chi - 2z P_{\varepsilon\chi} + P_\varepsilon = \frac{3}{4} e_i^2,$$

Which root which is not contradicting a relation (4.29), is

$$z = \frac{P_{\varepsilon\chi}}{P_\chi} + \frac{\sqrt{3}}{2\sqrt{P_\chi}} \sqrt{e_i^2 - \frac{4}{3} \frac{P_\varepsilon P_\chi - P_{\varepsilon\chi}^2}{P_\chi}} \cdot \text{sign}(z P_\chi - P_{\varepsilon\chi}), \quad (4.31)$$

And it is always necessary to take a positive value of the square root. Differentiating (4.31), we obtain:

$$dz = \frac{\sqrt{3}}{2\sqrt{P_\chi}} \frac{e_i de_i \cdot \text{sign} de_i}{\sqrt{e_i^2 - \frac{4}{3} \frac{P_\varepsilon P_\chi - P_{\varepsilon\chi}^2}{P_\chi}}}. \quad (4.32)$$

The value $\text{sign } z P_\chi - P_{\varepsilon\chi}$, according to (4.29), coincides with a sign $\frac{de_i}{dz}$ and as in intervals interesting us dz always it is positive at change z

$$\text{from } -\frac{h}{2} \text{ to } +\frac{h}{2}$$

integration on de_i should be executed so that de_i too increased, i.e. it is necessary to integrate on $de_i \cdot \text{sign} de_i$.

Let's consider values of intensity of deformations in three points on an axis z :

$$z = -\frac{h}{2}, \quad z = +\frac{h}{2}, \quad z = z_0, \quad z_0 = \frac{P_{\varepsilon\chi}}{P_\chi}. \quad (4.33)$$

Let's designate them accordingly:

$$e_{i1} = \frac{2}{\sqrt{3}} \sqrt{P_\varepsilon + h P_{\varepsilon\chi} + \frac{h^2}{4} P_\chi} \quad \left(z = -\frac{h}{2} \right),$$

$$e_{i2} = \frac{2}{\sqrt{3}} \sqrt{P_\varepsilon - h P_{\varepsilon\chi} + \frac{h^2}{4} P_\chi} \quad \left(z = +\frac{h}{2} \right),$$

$$e_{i0} = \frac{2}{\sqrt{3}\sqrt{P_\chi}} \sqrt{P_\varepsilon P_\chi - P_{\varepsilon\chi}^2} \quad (z = z_0).$$

(4.34)

Apparently from (4.29), the point $z = z_0$ is a minimum point e_i . Hence, inequalities always take place

$$e_{i1} \geq e_{i0}, \quad e_{i2} \geq e_{i0}. \quad (4.34')$$

We shall say that the deformations of the stretching and the shift of the middle surface $\varepsilon_1, \varepsilon_2, \varepsilon_{12}$ are commensurable or small in comparison with deformations of the bending of the shell

$$\pm \frac{h}{2} \chi_1, \quad \pm \frac{h}{2} \chi_2, \quad \pm \frac{h}{2} \chi_{12}$$

or that the latter are dominant if the point z_0 does not exceed the thickness of the shell

$$-\frac{h}{2} \leq z_0 = \frac{P_{\varepsilon\chi}}{P_\chi} \leq \frac{h}{2}. \quad (4.35)$$

Deformations of the middle surface will be called large or dominant as compared with deformation of the bend if the point is located outside the thickness of the shell, that is, if one of the inequalities holds

$$z_0 = \frac{P_{\varepsilon\chi}}{P_\chi} > \frac{h}{2}, \quad z_0 = \frac{P_{\varepsilon\chi}}{P_\chi} < -\frac{h}{2}. \quad (4.36)$$

In case of commensurable tensile deformations and a bending, the integral from any positive

value R on a thickness of a shell is necessary for calculating under the formula:

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} R dz = \frac{\sqrt{3}}{2\sqrt{P_\chi}} \left[\int_{e_{i0}}^{e_{i1}} \frac{Re_i de_i}{\sqrt{e_i^2 - e_{i0}^2}} + \int_{e_{i0}}^{e_{i2}} \frac{Re_i de_i}{\sqrt{e_i^2 - e_{i0}^2}} \right]. \quad (4.35')$$

In case of incommensurable or large tensile deformations such integral should be calculated under the formula:

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} R dz = \frac{\sqrt{3} \cdot \text{sign}(e_{i2} - e_{i1})}{2\sqrt{P_\chi}} \int_{e_{i1}}^{e_{i2}} \frac{Re_i de_i}{\sqrt{e_i^2 - e_{i0}^2}}. \quad (4.36')$$

We introduce the notation of the principal quantities in the theory of shells:

$$\begin{aligned} A = A_0, \quad B = B_0, \quad C = C_0 \quad \left(-\frac{h}{2} \leq z_0 \leq \frac{h}{2} \right), \\ A = A_1, \quad B = B_1, \quad C = C_1 \quad \left(|z_0| > \frac{h}{2} \right), \end{aligned} \quad (4.37)$$

Where the values A_0, B_0, C_0 refer to the case of the dominant deformation of the bending and are equal to:

$$\begin{aligned} A_0 &= - \int_{e_{i0}}^{e_{i1}} \sigma_i de_i + \int_{e_{i0}}^{e_{i2}} \sigma_i de_i = \int_{e_{i1}}^{e_{i2}} \sigma_i de_i, \\ B_0 &= \int_{e_{i0}}^{e_{i1}} \frac{\sigma_i de_i}{\sqrt{e_i^2 - e_{i0}^2}} + \int_{e_{i0}}^{e_{i2}} \frac{\sigma_i de_i}{\sqrt{e_i^2 - e_{i0}^2}}, \\ C_0 &= \int_{e_{i0}}^{e_{i1}} \sigma_i \sqrt{e_i^2 - e_{i0}^2} de_i + \int_{e_{i0}}^{e_{i2}} \sigma_i \sqrt{e_i^2 - e_{i0}^2} de_i. \end{aligned} \quad (4.37')$$

And A_1, B_1, C_1 concern to a case of a dominating stretching of a median surface and are equal:

$$\begin{aligned} A_0 = A_1 &= \int_{e_{i1}}^{e_{i2}} \sigma_i de_i, \quad B_1 = \int_{e_{i1}}^{e_{i2}} \frac{\sigma_i de_i}{\sqrt{e_i^2 - e_{i0}^2}} \cdot \text{sign}(e_{i2} - e_{i1}), \\ C_1 &= \int_{e_{i1}}^{e_{i2}} \sigma_i \sqrt{e_i^2 - e_{i0}^2} de_i \cdot \text{sign}(e_{i2} - e_{i1}). \end{aligned} \quad (4.37'')$$

J_1, J_2, J_3 (4.23'), (4.24'), (4.26) and (4.27), it is possible to express the integrals J_1, J_2, J_3 entering in the formulas through the basic values A, B, C depending on the basic quadratic forms $P_\varepsilon, P_\chi, P_{\varepsilon\chi}$, according to formulas (4.37). For this purpose we notice that the integral J_1 on the basis of formulas (4.25) and (4.35')-(4.36') is directly expressed through function B then from (4.30) it is found J_2 through A and B , after from (4.28) is received J_3 through A, B, C . Thus we find following formulas:

$$\begin{aligned} J_1 &= \frac{\sqrt{3}}{2P_\chi^{\frac{1}{2}}} B, \quad J_2 = \frac{\sqrt{3}P_{\varepsilon\chi}}{2P_\chi^{\frac{3}{2}}} B + \frac{3}{4P_\chi} A, \\ J_3 &= \frac{3\sqrt{3}}{8P_\chi^{\frac{3}{2}}} C + \frac{\sqrt{3}P_{\varepsilon\chi}^2}{2P_\chi^{\frac{5}{2}}} B + \frac{3P_{\varepsilon\chi}}{2P_\chi^2} A. \end{aligned} \quad (4.38)$$

Values A, B, C need to attribute an index «0» and to calculate them under formulas (4.37') if bending strain dominates or to attribute an index «1» and to calculate according to (4.37'') if the stretching-compression of a middle surface dominates.

The formula (4.32) and all subsequent calculations lose their meaning when the momentless state is stressful, when the quantities e_i and σ_i are constant in thickness. In this case

$$P_\chi = P_{\varepsilon\chi} = 0, \quad e_i = \frac{2}{\sqrt{3}} \sqrt{P_\varepsilon}, \quad (4.39)$$

And the integrals J_1, J_2, J_3 can be calculated directly. From the formulas (4.25) we have:

$$J_1 = h \frac{\sigma_i}{e_i}, \quad J_2 = 0, \quad J_3 = \frac{h^3}{12} \frac{\sigma_i}{e_i}, \quad (4.40)$$

As equality

$$P_\chi = 0$$

is possible only at

$$\chi_1 = \chi_2 = \chi_{12} = 0$$

all bending moments are equal to zero.

The relations (4.23'), (4.24') or (4.26), (4.27) give the expressions for the forces and moments acting on the shell element through three quadratic forms $P_\varepsilon, P_\chi, P_{\varepsilon\chi}$:

$$\begin{aligned} P_\varepsilon &= \varepsilon_1^2 + \varepsilon_1 \varepsilon_2 + \varepsilon_2^2 + \varepsilon_{12}^2, \quad P_\chi = \chi_1^2 + \chi_1 \chi_2 + \chi_2^2 + \chi_{12}^2, \\ P_{\varepsilon\chi} &= \varepsilon_1 \chi_1 + \varepsilon_2 \chi_2 + \frac{1}{2} \varepsilon_1 \chi_2 + \frac{1}{2} \varepsilon_2 \chi_1 + \varepsilon_{12} \chi_{12} \end{aligned} \quad (4.43)$$

And six components of deformations and distortions $\varepsilon_1, \varepsilon_2, \varepsilon_{12}, \chi_1, \chi_2, \chi_{12}$, hence through the three components of the displacement vector of the point of the middle surface, since deformations and curvatures have differential expressions through u, v, w .

We show that all deformations and curvatures can be expressed in terms of forces and moments. To do this, we find the expressions for the quadratic forms (4.43) in terms of analogous quadratic forms of forces and moments. According to the expressions S, H , through T, M (4.23)-(4.24) we have the identities:

$$\begin{aligned} P_S &= S_1^2 + S_1 S_2 + S_2^2 + 3S_{12}^2 = \\ &= \frac{3}{4} (N_1^2 - N_1 N_2 + N_2^2 + 3N_{12}^2), \\ P_H &= H_1^2 + H_1 H_2 + H_2^2 + 3H_{12}^2 = \\ &= \frac{3}{4} (M_1^2 - M_1 M_2 + M_2^2 + 3M_{12}^2), \\ P_{SH} &= S_1 H_1 + S_2 H_2 + \frac{1}{2} S_1 H_2 + \frac{1}{2} S_2 H_1 + \\ &+ 3S_{12} H_{12} = \\ &= \frac{3}{4} \left(N_1 M_1 + N_2 M_2 - \frac{1}{2} N_1 M_2 - \frac{1}{2} N_2 M_1 + \right. \\ &\quad \left. + 3N_{12} M_{12} \right). \end{aligned} \quad (4.44)$$

We form the quadratic forms P_S, P_H, P_{SH} according to relations (4.23') and (4.24'), replacing the integrals entering them by the notation (4.25) by J_1, J_2, J_3 .

From the group of equations (4.23') we have:

$$P_S = J_1^2 P_\varepsilon - 2J_1 J_2 P_{\varepsilon\chi} + J_2^2 P_\chi. \quad (4.45')$$

Similarly, from the group of equations (4.24') we find:

$$P_H = J_2^2 P_\varepsilon - 2J_2 J_3 P_{\varepsilon\chi} + J_3^2 P_\chi. \quad (4.45'')$$

Constructing from both groups of equations (4.23'), (4.24') a bilinear form P_{SH} and collecting the coefficients of the products J_1, J_2 and J_2, J_3 , we obtain:

$$P_{SH} = J_1 J_2 P_\varepsilon - (J_1 J_3 + J_2^2) P_{\varepsilon\chi} + J_2 J_3 P_\chi. \quad (4.45''')$$

As the left parts of relation (4.45) are known functions (4.44) forces and the moments, and right depend only from $P_\varepsilon, P_\chi, P_{\varepsilon\chi}$ as J_1, J_2, J_3 are expressed under formulas (4.38), (4.37), (4.34) relation (4.45) represent three algebraic

equations from which it is possible to express forms $P_\varepsilon, P_\chi, P_{\varepsilon\chi}$ through P_S, P_H, P_{SH} :

$$\begin{aligned} P_\varepsilon &= f_1(P_S, P_H, P_{SH}), \quad P_\chi = f_2(P_S, P_H, P_{SH}), \\ P_{\varepsilon\chi} &= f_3(P_S, P_H, P_{SH}) \end{aligned} \quad (4.46)$$

Actually it can be executed after the particular characteristic of a material of a shell is given, i.e. the function kind is set $\sigma_i = \Phi(e_i)$.

Assuming that expressions (4.46) are found, we can find expressions of deformations ε, χ through forces T, M or S, H . For this purpose it is necessary to substitute (4.46) in (4.38), to express J_1, J_2, J_3 through P_S, P_H, P_{SH} and to decide the equations (4.23'), (4.24') rather ε, χ . Thus, we receive definitive formulas:

$$\begin{aligned} \varepsilon_1 &= \frac{1}{\Delta}(S_1 J_3 - H_1 J_2), \quad \chi_1 = \frac{1}{\Delta}(S_1 J_2 - H_1 J_1), \\ \varepsilon_2 &= \frac{1}{\Delta}(S_2 J_3 - H_2 J_2), \quad \chi_2 = \frac{1}{\Delta}(S_2 J_2 - H_2 J_1), \\ \varepsilon_{12} &= \frac{1}{\Delta}(S_{12} J_3 - H_{12} J_2), \quad \chi_{12} = \frac{1}{\Delta}(S_{12} J_2 - H_{12} J_1), \\ \Delta &= (J_1 J_3 - J_2^2). \end{aligned} \quad (4.47)$$

1.2. The final relationship between forces and moments and the formulation of the problem of the load-carrying capacity of shells

If intensity of deformations e_i (4.19) any layers of a shell is great enough in comparison with yield strength e_s , i.e.

$$\frac{2}{\sqrt{3}} \sqrt{P_\varepsilon - 2zP_{\varepsilon\chi} + z^2 P_\chi} = e_i \gg e_s, \quad (4.56)$$

and its material does not possess hardening the law $\sigma_i = \Phi(e_i)$ coincides with a condition of

plasticity of Mizes:

$$\sigma_i = \sigma_s = const., \quad (4.57)$$

Or can be approximately replaced by a condition of plasticity of Sen-Venan-Kulon:

$$\tau_{\max} = \frac{\sigma_s}{\sqrt{3}} = const. \quad (4.58)$$

We show that in this case there exists a finite (not differential) relation between the forces and the moments. Using formulas (4.37), taking the integral sign as a constant σ_i , we can calculate the values of the functions A, B, C .

In the case of dominant bending deformations, the formulas (4.37') take the form:

$$\begin{aligned} A_0 &= \sigma_s (e_{i2} - e_{i1}), \\ B_0 &= \sigma_s \ln \frac{(e_{i1} + \sqrt{e_{i1}^2 - e_{i0}^2})(e_{i2} + \sqrt{e_{i2}^2 - e_{i0}^2})}{e_{i0}^2}, \\ C_0 &= \frac{\sigma_s}{2} (e_{i1} \sqrt{e_{i1}^2 - e_{i0}^2} + e_{i2} \sqrt{e_{i2}^2 - e_{i0}^2}) - \frac{1}{2} e_{i0}^2 B_0, \end{aligned} \quad (4.59')$$

and in case of dominating lengthening of a middle surface from formulas (4.37 ") it is found:

$$\begin{aligned} A_1 &= \sigma_s (e_{i2} - e_{i1}), \quad B_1 = \sigma_s \left| \ln \frac{(e_{i2} + \sqrt{e_{i2}^2 - e_{i0}^2})}{(e_{i1} + \sqrt{e_{i1}^2 - e_{i0}^2})} \right|, \\ C_1 &= \frac{\sigma_s}{2} |e_{i2} \sqrt{e_{i2}^2 - e_{i0}^2} - e_{i1} \sqrt{e_{i1}^2 - e_{i0}^2}| - \frac{e_{i0}^2}{2} B_1. \end{aligned} \quad (4.59'')$$

In both cases of value e_{i1}, e_{i2}, e_{i0} are expressed by formulas (4.34). Considering the last as the equations concerning three quadratic forms $P_\chi, P_{\varepsilon\chi}, P_\varepsilon$, we copy them in a kind:

$$P_{\varepsilon} + hP_{\varepsilon\chi} + \frac{h^2}{4}P_{\chi} = \frac{3}{4}e_{i1}^2, \quad P_{\varepsilon} - hP_{\varepsilon\chi} + \frac{h^2}{4}P_{\chi} = \frac{3}{4}e_{i2}^2,$$

$$P_{\varepsilon}P_{\chi} - P_{\varepsilon\chi}^2 = \frac{3}{4}e_{i0}^2P_{\chi}.$$

Solving them with respect to quadratic forms leads to the following results:

$$\begin{aligned} hP_{\varepsilon\chi} &= \frac{3}{8}(e_{i1}^2 - e_{i2}^2), \quad P_{\varepsilon} = \frac{3}{8}(e_{i1}^2 + e_{i2}^2) - \frac{h^2}{4}P_{\chi}, \\ \frac{h^2}{4}P_{\chi} &= \frac{3}{16}\left(\sqrt{e_{i1}^2 - e_{i0}^2} \pm \sqrt{e_{i2}^2 - e_{i0}^2}\right)^2. \end{aligned} \quad (4.60)$$

To determine the sign in the last formula, we use inequalities (4.35) and (4.36). In the case of the dominant bending strain from (4.35), we have:

$$-2 \cdot \frac{h^2}{4}P_{\chi} \leq hP_{\varepsilon\chi} \leq 2 \cdot \frac{h^2}{4}P_{\chi}.$$

This inequality will take place, if in the formula (4.60) for P_{χ} in brackets to take a sign (+). The inequality (4.36) will take place, if for P_{χ} in brackets to take a sign (-).

Below, in all formulas with two signs, the upper sign will refer to the case of the dominant bending of the shell, and the lower sign to the case of the dominant extension-compression.

We introduce two basic parameters λ and μ :

$$\lambda = \frac{e_{i2}}{e_{i1}}, \quad \mu = \frac{e_{i0}}{e_{i1}}. \quad (4.61)$$

These parameters satisfy the following conditions:

$$0 \leq \lambda \geq \mu \leq 1, \quad (4.61')$$

Since e_{i0} - is the minimum value of the strain

intensity at a given point of the shell. Then the formulas (4.60) can be rewritten in the form:

$$P_{\chi} = \frac{3e_{i1}^2}{4h^2}\Delta_1^2, \quad P_{\varepsilon\chi} = \frac{3e_{i1}^2}{8h}\Delta\Delta_1, \quad P_{\varepsilon} = \frac{3e_{i1}^2}{16}(4\mu^2 + \Delta^2), \quad (4.62)$$

where Δ_1 and Δ designate following functions:

$$\Delta_1 = \left| \sqrt{1-\mu^2} \pm \sqrt{\lambda^2 - \mu^2} \right|, \quad \Delta = \frac{1-\lambda^2}{\Delta_1}. \quad (4.63)$$

The kind of the formula (4.62) for P_{ε} becomes clear if to take into consideration identity:

$$4\mu^2 + \Delta^2 = 1 + \lambda^2 + 2\mu^2 \mp 2\sqrt{(1-\mu^2)(\lambda^2 - \mu^2)}.$$

Using the notation λ , μ and the established rule for applying two-valued formulas, we can rewrite the expressions for the functions A , B , C (4.59) in the form:

$$\begin{aligned} A &= \sigma_s e_{i1} \varphi(\lambda, \mu), \quad B = \sigma_s \psi(\lambda, \mu), \\ C &= \frac{\sigma_s}{2} e_{i1}^2 [\chi(\lambda, \mu) - \mu^2 \psi(\lambda, \mu)], \end{aligned} \quad (4.64)$$

Functions φ , ψ also χ are determined so:

$$\begin{aligned} \varphi &= \lambda - 1, \quad \psi = \left| \ln \frac{1 + \sqrt{1-\mu^2}}{\mu} \pm \ln \frac{\lambda + \sqrt{\lambda^2 - \mu^2}}{\mu} \right|, \\ \chi &= \left| \sqrt{1-\mu^2} \pm \lambda \sqrt{\lambda^2 - \mu^2} \right|. \end{aligned} \quad (4.65)$$

Using formulas (4.62) and (4.64), we can be convinced that quadratic forms P_S , P_H , P_{SH} , according to formulas (4.45) and (4.38), do not depend on value e_{i1} and are functions only parameters λ , μ .

In this connection it is natural to introduce the

notation for the characteristic value of forces N_1, N_2, N_{12} and moments M_1, M_2, M_{12} :

$$N_s = \sigma_s h, \quad M_s = \frac{\sigma_s h^2}{4}. \quad (4.66)$$

The quantities N_s, M_s in the problems of momentless deformations of shells and problems of purely moment deformations play the same role as the yield stress σ_s in the plane stress problem. Therefore, it is useful to introduce the notation for dimensionless forces and moments:

$$\begin{aligned} n_1 &= \frac{N_1}{N_s}, & n_2 &= \frac{N_2}{N_s}, & n_{12} &= \frac{N_{12}}{N_s}, \\ m_1 &= \frac{M_1}{M_s}, & m_2 &= \frac{M_2}{M_s}, & m_{12} &= \frac{M_{12}}{M_s}, \end{aligned} \quad (4.67)$$

and instead of quadratic forms (4.44), consider quadratic forms of dimensionless forces and moments:

$$\begin{aligned} Q_n &= n_1^2 - n_1 n_2 + n_2^2 + 3n_{12}^2, \\ Q_m &= m_1^2 - m_1 m_2 + m_2^2 + 3m_{12}^2, \\ Q_{nm} &= n_1 m_1 + n_2 m_2 - \frac{1}{2} n_1 m_2 - \frac{1}{2} n_2 m_1 + 3n_{12} m_{12}. \end{aligned} \quad (4.68)$$

The last are connected with P_S, P_H, P_{SH} obvious relation:

$$Q_n = \frac{4P_S}{3N_s^2}, \quad Q_m = \frac{4P_H}{3M_s^2}, \quad Q_{nm} = \frac{4P_{SH}}{3N_s M_s}. \quad (4.69)$$

Performing rather cumbersome transformations of the right-hand sides of equations (4.45), namely squaring polynomials and multiplying, and then collecting the coefficients for $\psi^2, \phi^2, \phi\psi, \chi\psi, \phi\chi, \chi^2$, we obtain the follow-

ing equations:

$$\begin{aligned} Q_n &= \frac{1}{\Delta_1^2} (\mu^2 \psi^2 + \phi^2), \\ Q_{nm} &= \frac{2}{\Delta_1^3} (\mu^2 \Delta \psi^2 + \Delta \phi^2 + \mu^2 \phi \psi + \phi \chi), \\ Q_m &= \frac{4}{\Delta_1^4} \left[\mu^2 (\mu^2 + \Delta^2) \psi^2 + (4\mu^2 + \Delta^2) \phi^2 + \right. \\ &\quad \left. + 2\mu^2 \Delta \phi \psi - 2\mu^2 \psi \chi + 2\Delta \phi \chi + \chi^2 \right]. \end{aligned} \quad (4.70')$$

Since the right-hand sides of equations (4.70'), according to (4.63) and (4.65), are functions of two parameters λ, μ , in a three-dimensional space with variables Q_n, Q_m, Q_{nm} they represent a surface

$$F(Q_n, Q_m, Q_{nm}) = 0, \quad (4.70)$$

and (4.70') is the parametric equation of this surface. The relation between the quadratic forms (4.68) obtained in this way is called the final relation between the forces and moments acting in the shells. The final relationship was obtained from the Mizes hypothesis $\sigma_i = \sigma_s$ and therefore it is a generalization of the Mizes condition. The final relation derived from the equations of the theory of small elastic-plastic deformations will have the same form, according to the theory of flow the Sen-Venan- Mizes.

Existence of a final relation between forces N and the moments M in case of ideal plasticity, i.e. under condition of Mizes and at small elastic deformations, follows and is direct from formulas (4.23') and (4.24') as thus they are uniform zero degree concerning six values $\varepsilon_1, \varepsilon_2, \varepsilon_{12}, \chi_1, \chi_2, \chi_{12}$.

The surface (4.70) represents a three-dimensional image of the indicated surface of the six-measurement space.

We pass to more in-depth study of a final relation (4.70'). We note three special cases of a final relation.

1. The momentless state of stress takes place at

$$\chi_1 = \chi_2 = \chi_{12} = 0, \text{ with } P_{\varepsilon\chi} = 0 \quad (4.68).$$

The final relation is obtained from (4.70') if we assume that the deformations of the fibers along the thickness of the shell are the same

$$e_{i1} = e_{i2} = e_{i0}, \quad \lambda = \mu = 1.$$

In formulas (4.63), (4.65), one should take the lower sign and then uncover the uncertainties in formulas (4.70'). Then we find, obviously, the Mizes condition:

$$Q_m = Q_{nm} = 0, \quad Q_n = 1, \quad (4.71')$$

Or in expanded form:

$$N_1^2 - N_1 N_2 + N_2^2 + 3N_{12}^2 = N_s^2. \quad (4.71)$$

2. Purely moments the tension takes place in the absence of lengthening of a middle surface. The quadratic form

$$P_\varepsilon = 0,$$

that is why

$$P_{\varepsilon\chi} = 0.$$

As appears from the formula (4.19), intensity of deformations e_i is even function z and, according to (4.34), we have

$$e_{i1} = e_{i2}, \quad e_{i0} = 0, \quad \lambda = 1, \quad \mu = 0.$$

In formulas (4.63), (4.65) it is necessary to take the upper sign as from (4.33) it is had $z_0 = 0$, thus we receive

$$\Delta_1 = 2, \quad \Delta = 0, \quad \varphi = 0, \quad \mu\psi = 0, \quad \chi = 2.$$

The final relation (4.70') becomes:

$$Q_n = Q_{nm} = 0, \quad Q_m = 1, \quad (4.72')$$

Or in expanded form:

$$M_1^2 - M_1 M_2 + M_2^2 + 3M_{12}^2 = M_s^2. \quad (4.72)$$

3. The elementary difficult tension of shells at

$$P_\chi \neq 0, \quad P_\varepsilon \neq 0$$

takes place, if the bilinear form $P_{\varepsilon\chi}$ addresses in zero:

$$P_{\varepsilon\chi} = \chi_1 \left(\varepsilon_1 + \frac{1}{2} \varepsilon_2 \right) + \chi_2 \left(\varepsilon_2 + \frac{1}{2} \varepsilon_1 \right) + \chi_{12} \varepsilon_{12} = 0. \quad (4.73)$$

It can take place in cases

$$\begin{aligned} a) \quad & \chi_1 \neq 0, \quad \chi_{12} = \chi_2 = 0, \quad \varepsilon_1 + \frac{1}{2} \varepsilon_2 = 0, \\ b) \quad & \varepsilon_1 \neq 0, \quad \varepsilon_{12} = \varepsilon_2 = 0, \quad \chi_1 + \frac{1}{2} \chi_2 = 0 \end{aligned}$$

And many other things. From (4.60) it is thus had $e_{i1} = e_{i2} > e_{i0}$, $\lambda = 1$, $\mu < 1$, i.e. dominating bending strain is available. We find:

$$\begin{aligned} \Delta = \varphi = 0, \quad \Delta_1 = \chi = 2\sqrt{1-\mu^2}, \\ \psi = 2 \ln \frac{1+\sqrt{1-\mu^2}}{\mu}, \end{aligned}$$

and after simple transformations the final relation becomes:

$$Q_n = \frac{\mu^2}{1-\mu^2} \ln^2 \frac{1+\sqrt{1-\mu^2}}{\mu}, \quad Q_{nm} = 0,$$

$$Q_m = \left(\frac{\mu^2}{1-\mu^2} \ln \frac{1+\sqrt{1-\mu^2}}{\mu} - \frac{1}{\sqrt{1-\mu^2}} \right)^2. \quad (4.74)$$

It gives a line of interception of a surface (4.70) with a plane $Q_{nm} = 0$. As Q_n , Q_m are essentially positive, all surface is disposed between planes

$$Q_n = 0 \text{ and } Q_m = 0,$$

and a line (4.74) between positive directions of axes Q_n , Q_m , i.e. in the first quadrant of a plane

$$Q_{nm} = 0.$$

The point

$$Q_m = 0, \quad Q_n = 1$$

corresponding to a non-propulsive condition of a shell, is received from (4.74) at $\mu = 1$, and the point

$$Q_n = 0, \quad Q_m = 1$$

corresponding purely moment to a condition of a shell, is received at

$$\mu = 0, \text{ as } \mu \ln \mu = 0 \text{ at } \mu = 0.$$

The curve Q_n , Q_m can be constructed on the points which coordinates are introduced in table 4 [9] (the expanded version of the table it is resulted in 2 parts of the article). On Figure 53 [9] coordinates (it is resulted in 2 parts of the article) the curve (4.74) and a straight line is represented

$$Q_n + Q_m = 1, \quad (4.75)$$

which well enough approximates it. The maxi-

mum deviation of a straight line makes about 9 %. The surface (4.70) is symmetric concerning a plane

$$Q_{nm} = 0.$$

Thus, it is enough to know about a surface (4.70), only in the first octant of co-ordinate system Q_n , Q_m , Q_{nm} . It is possible to be convinced that on a line $\lambda = 1$ in a value plane (Q_n , Q_m) Q_n , Q_m have a maximum. If to use Schwarz's inequality concerning quadratic forms Q_n , Q_m , Q_{nm} $Q_{nm}^2 \leq Q_n \cdot Q_m$, it is possible to conclude that the value Q_{nm} on the module also is limited.

Table 5 [9] (the expanded version of the table is resulted in 2 parts of the article) gives coordinates of some points of a surface on lines $\lambda = \text{const}$, and against each value λ are given: in the first line - Q_n in the second - Q_m and in the third - Q_{nm} .

The greatest values Q_{nm} will be, when Schwarz's inequality is transformed into equality

$$Q_{nm}^2 = Q_n \cdot Q_m, \quad (4.77)$$

and it is possible only when values n and m are proportional:

$$\frac{n_1}{m_1} = \frac{n_2}{m_2} = \frac{n_{12}}{m_{12}}. \quad (4.77')$$

Let's show that the hyperbolic paraboloid (4.77) is crossed with a surface (4.70) on a line $\mu = 0$.

From (4.65) at $\mu = 0$ it is had:

$$\varphi = \lambda - 1, \quad \chi = 1 \pm \lambda^2, \quad \mu\psi = 0,$$

$$\Delta_1 = 1 \pm \lambda, \quad \Delta = \frac{1 - \lambda^2}{1 \pm \lambda}. \quad (4.77'')$$

Introducing these values to the equations (4.70), we receive:

$$Q_n = \frac{\varphi^2}{\Delta_1^2}, \quad Q_{nm} = \frac{2\varphi}{\Delta_1^3}(\Delta\varphi + \chi), \quad Q_m = \frac{4}{\Delta_1^4}(\Delta\varphi + \chi)^2. \quad (4.78)$$

From here in case of a dominating stretching of a shell at the lower sign in (4.77 ") it is had:

$$Q_n = 1, \quad Q_{nm} = Q_m = 0,$$

I.e. the line $\mu = 0$ degenerates in a point.

In case of a dominating bending of a shell it is received:

$$Q_n = \left(\frac{1-\lambda}{1+\lambda}\right)^2, \quad Q_{nm} = -\frac{4\lambda(1-\lambda)}{(1+\lambda)^3}, \quad Q_m = \frac{16\lambda^2}{(1+\lambda)^4}, \quad (4.79')$$

whence follows (4.77). Besides, from last equations it is found other relation

$$Q_m = (1 - Q_n)^2, \quad (4.79)$$

Combining it with (4.77), we find:

$$|Q_{nm}| = (1 - Q_n)\sqrt{Q_n}. \quad (4.80)$$

From here we conclude that the line $\mu = 0$ determining greatest on the module of value of the bilinear form Q_{nm} , represents a line of interception of two parabolic cylinders from which the cylinder (4.79) passes through points:

$$\begin{aligned} Q_n = 1, \quad Q_m = 0, \quad Q_{nm} = 0, \\ Q_n = 0, \quad Q_m = 1, \quad Q_{nm} = 0, \end{aligned}$$

Having forming, parallel to co-ordinate Q_{nm} , the cylinder (4.80) has forming, parallel to co-ordinate Q_m , and passes through the same

points. The line $\mu = 0$ limiting a piece of a surface (4.70) for dominating bending on which values Q_n , Q_{nm} , Q_m have mechanical sense, is shown on fig. 54 (it is resulted in 2 parts of the article).

The maximum value of ordinate Q_{nm} on the module will be at

$$Q_n = \frac{1}{3}, \quad Q_m = \frac{4}{9}$$

and

$$|Q_{nm}|_{\max} = \frac{2}{3\sqrt{3}}.$$

The final relation between forces and the moments in case of a dominating bending matters

$$|Q_{nm}|_{\max} = \frac{2}{3\sqrt{3}}.$$

can be approximately presented, as pair of the planes passing through a line (4.75) and through points

$$Q_n = \frac{1}{3}, \quad Q_m = \frac{4}{9}, \quad Q_{nm} = \pm \frac{2}{3\sqrt{3}}.$$

They have the equation:

$$Q_n + Q_m + \frac{1}{\sqrt{3}}|Q_{nm}| = 1. \quad (4.81)$$

As six components of deformations and bendings are expressed by means of differential operations on curvilinear coordinates through three components of a displacement vector u , v , w of a middle surface, they should satisfy to the equations of compatibility of deformations.

Generally it is possible to express the compatibility equations through forces N and the moments M , but they will contain one more func-

tion of coordinates e_{i1} . The differential equations of equilibrium and conditions of compatibility of deformations will be insufficiently for definition of forces N_1, N_2, N_{12} , the moments M_1, M_2, M_{12} and unknown function e_{i1} .

The final relation (4.70') between forces and the moments will be the missing equation also. In a kind of that this relation not differential and from it follows that forces and the moments and their quadratic forms Q_n, Q_m, Q_{nm} are limited on value, at any external forces equilibrium of a shell is impossible.

As lift capability of a shell is called limiting value of external forces at which internal forces N and the moments M satisfy to a final relation (4.70'), to the equilibrium equations, conditions of compatibility of deformations and boundary conditions.

In special cases thanks to a final relation the problem about equilibrium becomes statically definable and does not demand conditions of compatibility of deformations. Then the question on lift capability of a shell is decided rather simply.

It more becomes simpler, if forces and the moments can be expressed through external forces only by means of the equilibrium equations that takes place, for example, in the non-propulsive theory of shells, in that case the final relation (4.70') determines lift capability.

Conditions of compatibility of deformations do a problem about definition of lift capability rather difficult and consequently the approximate methods of its solution have great value.

The energy method of the solution consists in the following: are set by the suitable form of the deformed surface of shells and, making expressions of a variation of activity of internal forces and activity of external forces on variations of movings, compare them. Approximate limiting value of external forces can be received, if material hardening to put equal to zero, and deformations beyond all bounds to increase or saving constants yield strength

$$\sigma_s = 3Ge_s,$$

G to aim to infinity, and e_s - to zero.

On Figures 2.1-2.4 the fluidity surface

$$F(Q_n, Q_m, Q_{nm}) = 0$$

in three-dimensional space with variables is presented Q_n, Q_m, Q_{nm} . A black line – section of a surface a plane

$$Q_{nm} = 0,$$

formulas (4.74), a red line - a line of a maximum $|Q_{nm}|$ (4.79)-4.80).

1.3. The relationship between internal forces, moments and deformations of the shell on the basis of flow theory for an ideal plastic material

We show that the relations (4.26-4.27) remain valid also in the framework of the flow theory. Specific power dissipation of energy per unit volume:

$$D = \sigma_x \dot{\epsilon}_x + \sigma_y \dot{\epsilon}_y + \sigma_z \dot{\epsilon}_z + \tau_{xy} \dot{\gamma}_{xy} + \tau_{zx} \dot{\gamma}_{zx} + \tau_{zy} \dot{\gamma}_{zy}. \quad (1.3.1)$$

The plasticity condition of R. Mizes:

$$F = (\sigma_x - \sigma_y)^2 + (\sigma_x - \sigma_z)^2 + (\sigma_y - \sigma_z)^2 + 6(\tau_{xy}^2 + \tau_{zx}^2 + \tau_{zy}^2) - 2\sigma_s^2 \quad (1.3.2)$$

On the basis of the associate law of flow and a postulate of Druker for true fields of speeds of movings power of a dissipation of energy receives the maximum value, speeds of deformations are determined from a condition of a maximum of function

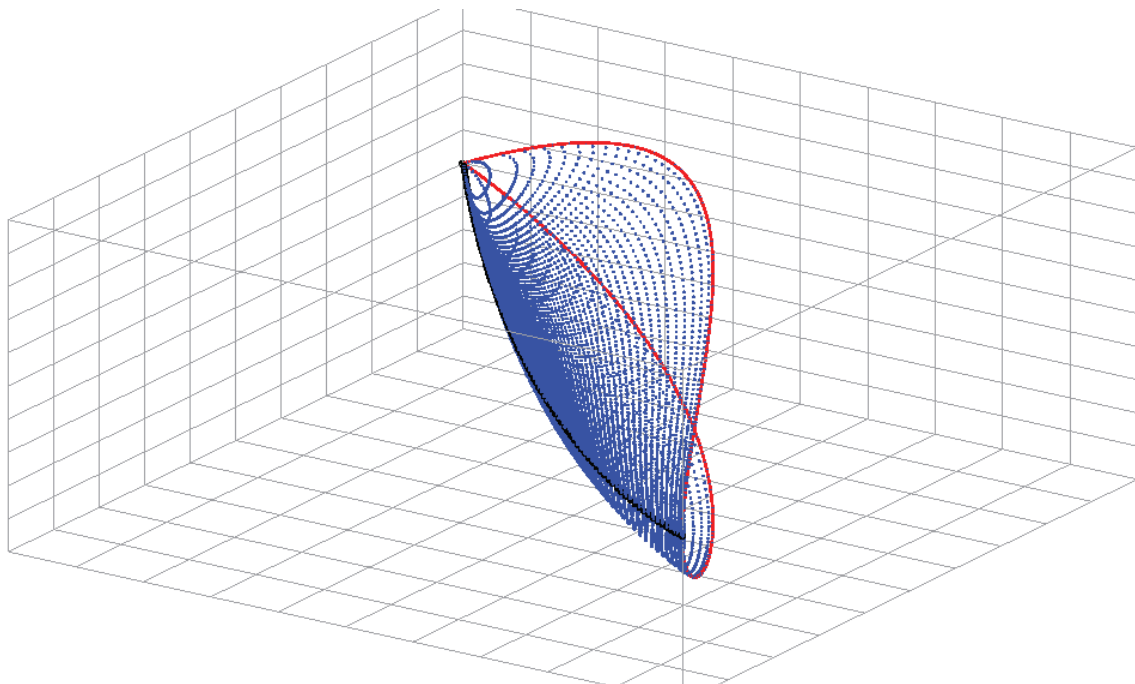


Figure 2.1. A fluidity surface $F(Q_n, Q_m, Q_{nm}) = 0$ in three-dimensional space with variables Q_n, Q_m, Q_{nm} .

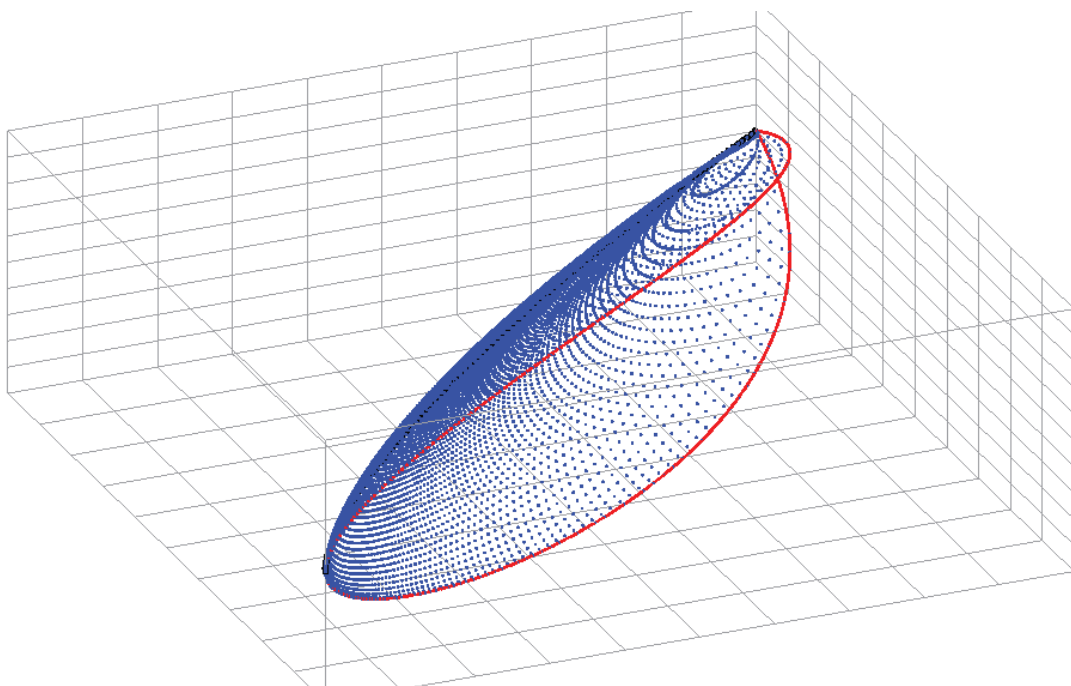


Figure 2.2. A fluidity surface $F(Q_n, Q_m, Q_{nm}) = 0$ in three-dimensional space with variables Q_n, Q_m, Q_{nm} .

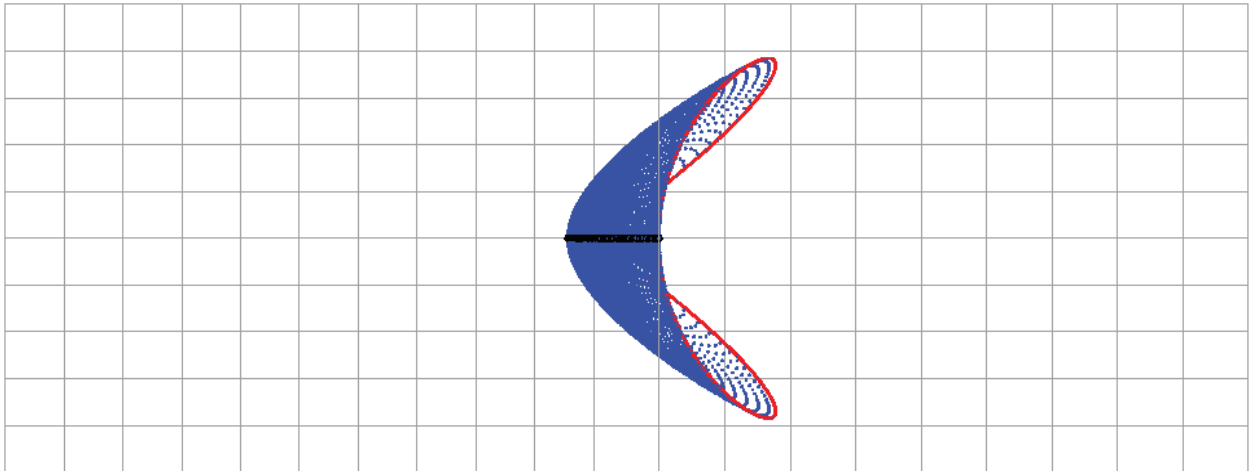


Figure 2.3. A fluidity surface $F(Q_n, Q_m, Q_{nm}) = 0$ in three-dimensional space with variables Q_n, Q_m, Q_{nm} .

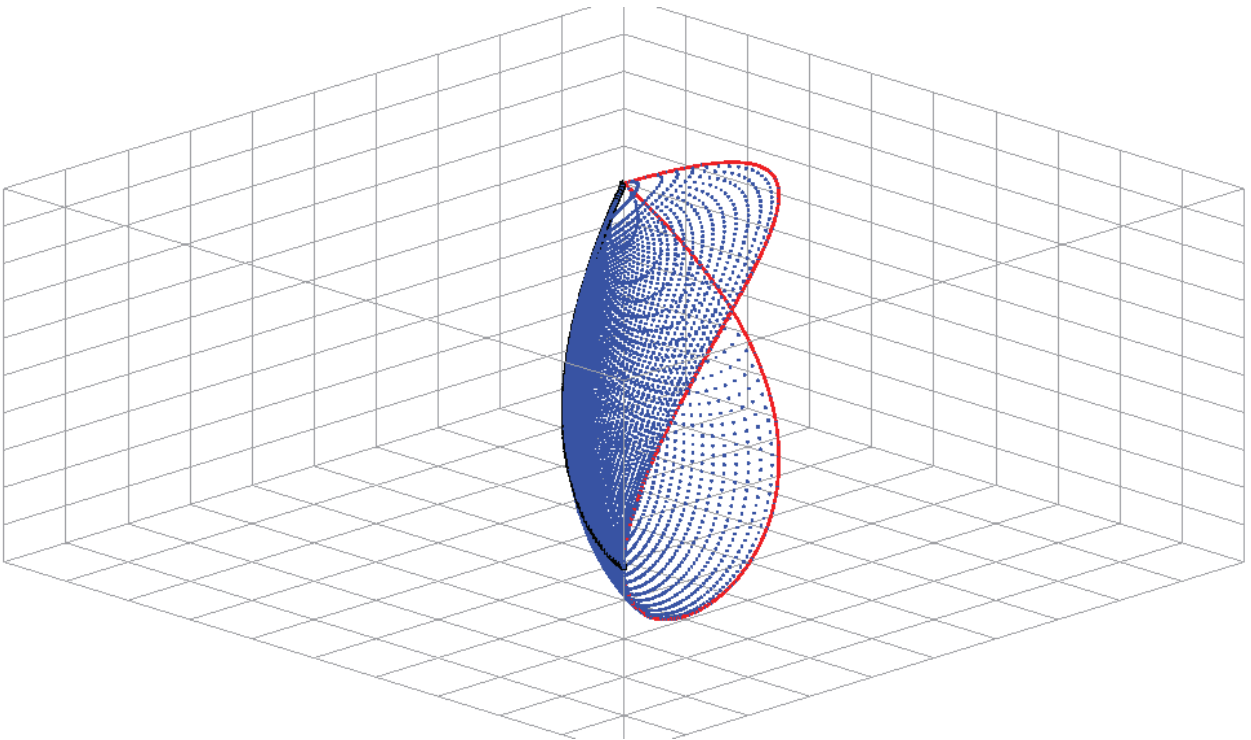


Figure 2.4. A fluidity surface $F(Q_n, Q_m, Q_{nm}) = 0$ in three-dimensional space with variables Q_n, Q_m, Q_{nm} .

$$\Phi = D - \lambda F,$$

where D and F according to (1.3.1)-(1.3.2):

$$\begin{aligned}\dot{\varepsilon}_x &= 6\lambda(\sigma_x - \sigma_0), \quad \dot{\varepsilon}_y = 6\lambda(\sigma_y - \sigma_0), \\ \dot{\varepsilon}_z &= 6\lambda(\sigma_z - \sigma_0), \quad \frac{1}{2}\dot{\gamma}_{xy} = 6\lambda\tau_{xy}, \quad \frac{1}{2}\dot{\gamma}_{xz} = 6\lambda\tau_{xz}, \\ \frac{1}{2}\dot{\gamma}_{zy} &= 6\lambda\tau_{zy}, \quad \sigma_0 = \frac{\sigma_x + \sigma_y + \sigma_z}{3}.\end{aligned}\quad (1.3.3)$$

Excluding λ by means of (1.3.2), we receive relation of flow of Sen-Venan-Mizes-Levi-Ishlinsky

$$\begin{aligned}\sigma_x - \sigma_0 &= \frac{2\sigma_s}{3\dot{\varepsilon}_i} \cdot \dot{\varepsilon}_x, \quad \sigma_y - \sigma_0 = \frac{2\sigma_s}{3\dot{\varepsilon}_i} \cdot \dot{\varepsilon}_y, \\ \sigma_z - \sigma_0 &= \frac{2\sigma_s}{3\dot{\varepsilon}_i} \cdot \dot{\varepsilon}_z, \quad \tau_{xy} = \frac{2\sigma_s}{3\dot{\varepsilon}_i} \cdot \frac{1}{2}\dot{\gamma}_{xy}, \\ \tau_{zx} &= \frac{2\sigma_s}{3\dot{\varepsilon}_i} \cdot \frac{1}{2}\dot{\gamma}_{zx}, \quad \tau_{zy} = \frac{2\sigma_s}{3\dot{\varepsilon}_i} \cdot \frac{1}{2}\dot{\gamma}_{zy},\end{aligned}\quad (1.3.4)$$

where intensity of speeds of deformations

$$\dot{\varepsilon}_i = \frac{\sqrt{2}}{3} \sqrt{(\dot{\varepsilon}_x - \dot{\varepsilon}_y)^2 + (\dot{\varepsilon}_x - \dot{\varepsilon}_z)^2 + (\dot{\varepsilon}_y - \dot{\varepsilon}_z)^2 + \frac{6}{4}(\dot{\gamma}_{xy}^2 + \dot{\gamma}_{zx}^2 + \dot{\gamma}_{zy}^2)}.\quad (1.3.5)$$

For a flat tension and problems of a bending of plates and shells it agree hypotheses of Kirhgof-fa-Ljava

$$\sigma_z = 0, \quad \tau_{zx} = \tau_{zy} = 0$$

and a condition of an incompressibility of a material

$$\dot{\varepsilon}_x + \dot{\varepsilon}_y + \dot{\varepsilon}_z = 0:$$

$$\begin{aligned}\dot{\varepsilon}_x &= \dot{\varepsilon}_1 - \dot{\chi}_1 z, \quad \dot{\varepsilon}_y = \dot{\varepsilon}_2 - \dot{\chi}_2 z, \quad \frac{1}{2}\dot{\gamma}_{xy} = \dot{\varepsilon}_{12} - \dot{\chi}_{12} z, \\ \dot{\varepsilon}_1 &= \dot{\varepsilon}_x|_{z=0}, \quad \dot{\varepsilon}_2 = \dot{\varepsilon}_y|_{z=0}, \quad \dot{\varepsilon}_{12} = \frac{1}{2}\dot{\gamma}_{xy}|_{z=0},\end{aligned}\quad (1.3.6)$$

The equations (1.3.4) and (1.3.5) taking into account (1.3.5) become

$$\begin{aligned}\sigma_x &= \frac{4\sigma_s}{3\dot{\varepsilon}_i} \left(\dot{\varepsilon}_x + \frac{1}{2}\dot{\varepsilon}_y \right) = \\ &= \frac{4\sigma_s}{3\dot{\varepsilon}_i} \left[\left(\dot{\varepsilon}_1 + \frac{1}{2}\dot{\varepsilon}_2 \right) - \left(\dot{\chi}_1 + \frac{1}{2}\dot{\chi}_2 \right) z \right], \\ \sigma_y &= \sigma_x = \frac{4\sigma_s}{3\dot{\varepsilon}_i} \left(\dot{\varepsilon}_y + \frac{1}{2}\dot{\varepsilon}_x \right) = \\ &= \frac{4\sigma_s}{3\dot{\varepsilon}_i} \left[\left(\dot{\varepsilon}_2 + \frac{1}{2}\dot{\varepsilon}_1 \right) - \left(\dot{\chi}_2 + \frac{1}{2}\dot{\chi}_1 \right) z \right], \\ \tau_{xy} &= \frac{2\sigma_s}{3\dot{\varepsilon}_i} \cdot \frac{1}{2}\dot{\gamma}_{xy} = (\dot{\varepsilon}_{12} - \dot{\chi}_{12} z), \\ \dot{\varepsilon}_i &= \frac{2}{\sqrt{3}} \sqrt{\dot{\varepsilon}_x^2 + \dot{\varepsilon}_x \dot{\varepsilon}_y + \dot{\varepsilon}_y^2 + \frac{1}{4}\dot{\gamma}_{xy}^2}.\end{aligned}\quad (1.3.7)$$

Longitudinal and shearing forces, bending and twisting moments according to (4.21)-(4.22)

$$\begin{aligned}\frac{3}{4}N_1 &= \left(\dot{\varepsilon}_1 + \frac{1}{2}\dot{\varepsilon}_2 \right) J_1 - \left(\dot{\chi}_1 + \frac{1}{2}\dot{\chi}_2 \right) J_2, \\ \frac{3}{4}N_2 &= \left(\dot{\varepsilon}_2 + \frac{1}{2}\dot{\varepsilon}_1 \right) J_1 - \left(\dot{\chi}_2 + \frac{1}{2}\dot{\chi}_1 \right) J_2, \quad (1.3.9) \\ \frac{3}{2}N_{12} &= \dot{\varepsilon}_{12} J_1 - \dot{\chi}_{12} J_2, \\ \frac{3}{4}M_1 &= \left(\dot{\varepsilon}_1 + \frac{1}{2}\dot{\varepsilon}_2 \right) J_2 - \left(\dot{\chi}_1 + \frac{1}{2}\dot{\chi}_2 \right) J_3, \\ \frac{3}{4}M_2 &= \left(\dot{\varepsilon}_2 + \frac{1}{2}\dot{\varepsilon}_1 \right) J_2 - \left(\dot{\chi}_2 + \frac{1}{2}\dot{\chi}_1 \right) J_3, \quad (1.3.10) \\ \frac{3}{2}M_{12} &= \dot{\varepsilon}_{12} J_2 - \dot{\chi}_{12} J_3,\end{aligned}$$

where integrals J_1, J_2, J_3 :

$$J_1 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\sigma_s}{\dot{\epsilon}_i} dz, \quad J_2 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\sigma_s}{\dot{\epsilon}_i} z dz, \quad J_3 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\sigma_s}{\dot{\epsilon}_i} z^2 dz, \quad (1.3.11)$$

and intensity of speeds of deformations:

$$\begin{aligned} \dot{\epsilon}_i &= \frac{2}{\sqrt{3}} \sqrt{\dot{P}_\epsilon - 2z\dot{P}_{\epsilon\chi} + z^2\dot{P}_\chi}, \\ \dot{P}_\epsilon &= \dot{\epsilon}_1^2 + \dot{\epsilon}_1\dot{\epsilon}_2 + \dot{\epsilon}_2^2 + \dot{\epsilon}_{12}^2, \quad \dot{P}_\chi = \dot{\chi}_1^2 + \dot{\chi}_1\dot{\chi}_2 + \dot{\chi}_2^2 + \dot{\chi}_{12}^2, \\ \dot{P}_{\epsilon\chi} &= \dot{\epsilon}_1\dot{\chi}_1 + \dot{\epsilon}_2\dot{\chi}_2 + \frac{1}{2}\dot{\epsilon}_1\dot{\chi}_2 + \frac{1}{2}\dot{\epsilon}_2\dot{\chi}_1 + \dot{\epsilon}_{11}\dot{\chi}_{12}. \end{aligned} \quad (1.3.12)$$

Thus the final relation remains fair and within the limits of the flow theory if in all formulas of sections 1.1-1.2 to replace deformations $\epsilon_1, \epsilon_2, \epsilon_{12}$ and changes of curvature of a median surface $\chi_1, \chi_2, \chi_{12}$ with speeds of deformations $\dot{\epsilon}_1, \dot{\epsilon}_2, \dot{\epsilon}_{12}$ and speed of change of curvature of a median surface $\dot{\chi}_1, \dot{\chi}_2, \dot{\chi}_{12}$. For the hardening account in formulas it is necessary to consider (1.3.11) yield strength as function of intensity of deformations and intensity of speeds of deformations $\sigma_s = \sigma_s(\epsilon_i, \dot{\epsilon}_i)$.

CONCLUSIONS

The geometrical image of an exact surface of fluidity in space of the generalised pressure which A.A. Ilyushin in the works and in references is absent that allows to execute its approximation for the solution of practical problems is received. It is shown that a final relation remain fair and within the limits of the theory of flow for ideally plastic material.

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