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NUMERICAL ANALYSIS OF NON-LINEAR VIBRATIONS OF A FRACTIONALLY DAMPED CYLINDRICAL SHELL UNDER THE ADDITIVE COMBINATIONAL INTERNAL RESONANCE

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Abstract: Non-linear damped vibrations of a cylindrical shell subjected to the additive type combinational internal resonance are investigated numerically using two different numerical methods. The damping features of the surrounding medium are described by the fractional derivative Kelvin-Voigt model involving the Riemann-Liouville fractional derivatives. Within the first method, the generalized displacements of a coupled set of nonlinear ordinary differential are estimated using numerical solution of nonlinear multi-term fractional differential equations by the procedure based on the reduction of the problem to a system of fractional differential equations. According to the second method, the amplitudes and phases of nonlinear vibrations are estimated from the governing nonlinear differential equations describing amplitude-and-phase modulations for the case of the additive combinational internal resonance. A good agreement in results is declared.

Keywords: cylindrical shell, free nonlinear damped vibrations, additive combinational internal resonance, method of multiple time scales, multi-term fractional differential equations

ЧИСЛЕННЫЙ АНАЛИЗ НЕЛИНЕЙНЫХ КОЛЕБАНИЙ ЦИЛИНДРИЧЕСКОЙ ОБОЛОЧКИ С ДРОБНЫМ ДЕМПФИРОВАНИЕМ ПРИ АДДИТИВНОМ КОМБИНАЦИОННОМ ВНУТРЕННЕМ РЕЗОНАНСЕ

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Аннотация: Рассматриваются нелинейные затухающие колебания цилиндрической оболочки при аддитивном комбинационном внутреннем резонансе. Для решения соответствующих задач применяются два различных численных метода. Демпфирующие особенности окружающей среды описываются с помощью дробной производной модели Кельвина-Фойгта, включающей дробные производные Римана-Лиувилля. В рамках первого метода обобщенные смещения связного набора нелинейных обыкновенных дифференциалов оцениваются на основе численного решения нелинейных многочленных уравнений с дробными производными по методике, предусматривающей сведение исходной задачи к системе уравнений с дробными производными. Согласно второму методу, амплитуды и фазы нелинейных колебаний оцениваются из определяющих нелинейных дифференциальных уравнений, описывающих амплитудно-фазовые модуляции для случая аддитивного комбинационного внутреннего резонанса. Отмечена хорошая согласованность полученных результатов.

Ключевые слова: цилиндрическая оболочка, свободные нелинейно-затухающие колебания, аддитивный комбинационный внутренний резонанс, метод кратных временных шкал, многочленные уравнения с дробными производными

1. INTRODUCTION

In mechanical nonlinear vibrations, the phenomena of internal resonance and energy exchange are quite often what requires the thorough studies, since in the case of low damping it could result in long-time vibrations accompanied by the two-sided or one-sided energy interchange between coupled modes [1]. It will suffice to mention the state-of-the-art articles [1,2] and the monograph [3] involving the extensive review of literature in the field of internal resonances in different mechanical systems. Different types of the internal resonance: one-to-one, two-to-one, three-to-one, as well as a variety of combinational resonances, when three and more natural modes interact, have been discussed. The enumerated internal resonances were investigated in various mechanical systems with multiple degree-of-freedom, as well as in strings, beams, plates, and shells.

It has been emphasized by many researchers [4-13] that the phenomenon of internal resonances can be very critical especially for circular cylindrical shells. Thus, the nonlinear vibrations of infinitely long circular cylindrical shells under the conditions of the two-to-one internal resonance were studied in [6] via the method of multiple time scales using the simple plane strain theory of shells. Parametrically excited vibrations of infinitely long cylindrical shells and nonlinear forced vibrations of a simply supported, circular cylindrical shell filled with an incompressible, inviscid, quiescent and dense fluid were investigated in [4,5,7] using Donnell's nonlinear shallow-shell theory. The flexural deformation is usually expanded by using the linear shell eigenmodes, in so doing the flexural response involves several nodal diameters and one or two longitudinal half-waves. Internal resonances of different types have been analyzed in [8-13].

The extensive review of studies on shallow shells nonlinear vibrations could be found in the state-of-the-art articles [14-16]. In spite of the fact that many studies have been carried out on large amplitude vibrations of circular cylindrical shells and many different approaches to the problem have been used, we agree with Breslavsky and Amabili [10] that this research area is still far from being well understood.

In recent years much attention is given to damping features of mechanical systems subjected to the conditions of different internal resonances. Damping properties of nonlinear systems are described mainly by the first-order time-derivative of a generalized displacement [3]. However, as it has been shown by Rossikhin and Shitikova [17], who analyzed free damped vibrations of suspension combined system under the conditions of the one-to-one internal resonance, for good fit of the theoretical investigations with the experimental results it is better to describe the damping features of nonlinear mechanical systems in terms of fractional time-derivatives of the generalized displacements [18].

During the last decade, fractional calculus entered the mainstream of engineering analysis. And it has been widely applied to structural dynamics problems both in discrete and continuous equations. The history of the fractional calculus applications in mechanics could be found in the retrospective paper by Rossikhin [19], while a comprehensive review of the fractional calculus models in different dynamic problems of solids and structures is presented in the state-of-the-art article [18], wherein the results obtained in the field critically estimated in the light of the present view of the place and role of the fractional calculus in engineering problems and practice.

It has been suggested in 2011 to examine the nonlinear dynamic response of a thin cylindrical shell vibrating in a fractionally damped medium [20], when the dynamic behavior of the shell is described by a set of three coupled nonlinear differential equations with due account for the fact that the shell is being under the conditions of the internal resonance resulting in the interaction of modes corresponding to the mutually orthogonal displacements. The displacement functions are determined in terms of eigenfunctions of linear vibrations.

A new procedure resulting in decoupling linear parts of equations has been proposed in Rossikhin and Shitikova [21] with the further utilization of the method of multiple scales for solving nonlinear governing equations of motion, in so doing the amplitude functions are expanded into power series in terms of the small parameter and depend on different time scales. It is shown that the phenomenon of the internal resonance between vibrational subsystems of the cylindrical shell under consideration can be very critical, since in the circular cylindrical shell of such a type the two-to-one [21], one-to-one, three-to-one [22] internal resonances, as well as combinational internal resonances [23] could occur, which are governed by the order of smallness of viscosity. All possible cases of the internal resonance have been recently revealed in [22], which belong to the resonances of the constructive type, since all of them depend on the geometrical dimensions of the shell under consideration and its mechanical characteristics, that is why such resonances could not be ignored and eliminated for a particularly designed shell. It has been shown that the energy exchange could occur between two or three subsystems at a time: normal vibrations of the shell, its torsional vibrations and shear vibrations along the shell axis. Such an energy exchange, if it takes place for a rather long time, could result in crack formation in the shell, and finally to its failure. The energy exchange has been illustrated pictorially by the phase portraits, wherein the phase trajectories of the phase fluid motion are depicted.

In the present paper, we are going to verify parameter values of the cylindrical shell model [20-23], resulting in the nonlinear vibrations of a fractionally damped cylindrical shell under the conditions of combinational internal resonance, and to study such phenomenon using two different numerical methods [24]. In the first method, the generalized displacements of a coupled set of nonlinear ordinary differential equations of the second order are estimated using numerical solution of nonlinear multi-term fractional differential equations by the procedure based on the reducing of the problem to a system of fractional differential equations [25-28]. According to the second method, the amplitudes and phases of nonlinear vibrations are estimated from the governing nonlinear differential equations describing amplitude-and-phase modulations for the case of the combinational internal resonance [23] using the Runge-Kutta fourth order method.

2. PROBLEM FORMULATION AND GOVERNING EQUATIONS

Let us examine the dynamic response of a free supported non-linear elastic circular cylindrical shell of radius R and length l, vibrations of which in the cylindrical system of coordinates described by the Donnell–Mushtari–Vlasov equations with respect to the three displacements [12] considering that damping features of the surrounding medium are described by the time-differential operator of the fractional order [20]:

$$\begin{aligned}
&\frac{\partial^2 u}{\partial x^2}+\frac{1-\sigma}{2}\frac{1}{R^2}\frac{\partial^2 u}{\partial \varphi^2}+\\
&+\frac{1+\sigma}{2}\frac{1}{R}\frac{\partial^2 v}{\partial x\partial\varphi}-\sigma\frac{1}{R}\frac{\partial w}{\partial x}+\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2}+\\
&+\frac{1+\sigma}{2}\frac{1}{R^2}\frac{\partial w}{\partial \varphi}\frac{\partial^2 w}{\partial x\partial\varphi}+\frac{1-\sigma}{2}\frac{1}{R^2}\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial \varphi^2}=\\
&=\frac{\rho(1-\sigma^2)}{E}\frac{\partial^2 u}{\partial t^2}+æ_1\left(\frac{d}{dt}\right)^{\gamma}u,
\end{aligned}\tag{1}$$

$$\begin{aligned}
&\frac{1}{R^2}\frac{\partial^2 v}{\partial \varphi^2}+\frac{1-\sigma}{2}\frac{\partial^2 v}{\partial x^2}+\frac{1+\sigma}{2}\frac{1}{R}\frac{\partial^2 u}{\partial x\partial\varphi}-\\
&-\frac{1}{R^2}\frac{\partial w}{\partial \varphi}+\frac{1}{R^3}\frac{\partial w}{\partial \varphi}\frac{\partial^2 w}{\partial \varphi^2}+\\
&+\frac{1+\sigma}{2}\frac{1}{R}\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x\partial\varphi}+\frac{1-\sigma}{2}\frac{1}{R}\frac{\partial w}{\partial \varphi}\frac{\partial^2 w}{\partial x^2}=\\
&=\frac{\rho(1-\sigma^2)}{E}\frac{\partial^2 v}{\partial t^2}+æ_2\left(\frac{d}{dt}\right)^{\gamma}v,
\end{aligned}\tag{2}$$

$$\frac{h^2}{12}\nabla^4 w + \frac{1}{R^2}w - \sigma\frac{1}{R}\frac{\partial u}{\partial x} - \frac{1}{R^2}\frac{\partial v}{\partial \varphi} -$$
$$-\frac{1}{2}\frac{\sigma}{R}\left(\frac{\partial w}{\partial x}\right)^2 - \frac{1}{2}\frac{1}{R^3}\left(\frac{\partial w}{\partial \varphi}\right)^2 -$$
$$-\frac{\partial}{\partial x}\left[\frac{\partial w}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\sigma}{R}\frac{\partial v}{\partial \varphi} - \frac{\sigma}{R}w\right) +\right.$$
$$\left. + \frac{1-\sigma}{2}\frac{1}{R}\frac{\partial w}{\partial \varphi}\left(\frac{1}{R}\frac{\partial u}{\partial \varphi} + \frac{\partial v}{\partial x}\right)\right] -$$
$$-\frac{1}{R}\frac{\partial}{\partial \varphi}\left[\frac{1}{R}\frac{\partial w}{\partial \varphi}\left(\sigma\frac{\partial u}{\partial x} + \frac{1}{R}\frac{\partial v}{\partial \varphi} - \frac{1}{R}w\right) +\right.$$
$$\left. + \frac{1-\sigma}{2}\frac{\partial w}{\partial x}\left(\frac{1}{R}\frac{\partial u}{\partial \varphi} + \frac{\partial v}{\partial x}\right)\right] -$$
$$= -\frac{\rho(1-\sigma^2)}{E}\frac{\partial^2 w}{\partial t^2} - æ_3\left(\frac{d}{dt}\right)^{\gamma} w, \tag{3}$$

where x-axis is directed along the axis of the cylinder, φ is the polar angle in the plane perpendicular to the x-axis,

$u = u(x,\varphi,t)$, $v = v(x,\varphi,t)$, and $w = w(x,\varphi,t)$

are the displacements of points located in the shell's middle surface in three mutually orthogonal directions x, φ, r with r as the polar radius, h is the thickness, ρ is the density, E and σ are the elastic modulus and Poisson's ratio, respectively, t is the time, $æ_1$, $æ_2$, $æ_3$ are the damping coefficients, and

$$\nabla^4 = \nabla^2\nabla^2 = \frac{\partial^4}{\partial x^4} + 2\frac{1}{R^2}\frac{\partial^4}{\partial x^2 \partial \varphi^2} + \frac{1}{R^4}\frac{\partial^4}{\partial \varphi^4}$$

The initial conditions

$$u\,|_{t=0} = v\,|_{t=0} = w\,|_{t=0}; \tag{4}$$

$$\dot{u}\,|_{t=0} = \varepsilon V_1^0(x,\varphi),$$
$$\dot{v}\,|_{t=0} = \varepsilon V_2^0(x,\varphi), \tag{5}$$
$$\dot{w}\,|_{t=0} = \varepsilon V_3^0(x,\varphi)$$

where $V_i^0(x,\varphi)$ (i=1,2,3) are the corresponding initial velocities, and ε is a small value, should be added to Eqs. (1)-(3). Hereafter over dots denote time-derivatives.

The boundary conditions for the simply supported shell (the Navier-type conditions for the edges free supported in the x-direction) have the form [12]:

$$w\,|_{x=0} = w\,|_{x=l} = 0, \quad v\,|_{x=0} = v\,|_{x=l} = 0,$$
$$\frac{\partial^2 w}{\partial x^2}\,|_{x=0} = \frac{\partial^2 w}{\partial x^2}\,|_{x=l} = 0,$$
$$\frac{\partial u}{\partial x}\,|_{x=0} = \frac{\partial u}{\partial x}\,|_{x=l} = 0. \tag{6}$$

From relationships (5) it follows that free vibrations are excited by the weak disturbance from the equilibrium position.

It has been proposed in [20] to rewrite Eqs. (1)-(5) in the nondimensioned form in terms of the following dimensionless parameters:

$$u^* = \frac{u}{l}, \quad v^* = \frac{v}{l}, \quad w^* = \frac{w}{l},$$
$$x^* = \frac{x}{l}, \quad t^* = \frac{t}{l}\sqrt{\frac{E}{\rho(1-\sigma^2)}}.$$

Dropping hereafter the asterisks for the ease of presentation, let us admit the solution of the Navier type for Eqs. (1)-(3) in the form

$$u(x,\varphi,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} x_{1mn}(t)\eta_{1mn}(x,\varphi);$$
$$v(x,\varphi,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} x_{2mn}(t)\eta_{2mn}(x,\varphi); \tag{7}$$
$$w(x,\varphi,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} x_{3mn}(t)\eta_{3mn}(x,\varphi),$$

where $x_{jmn}(t)$ and $\eta_{jmn}(x,\varphi)$ (j=1,2,3) are, respectively, the generalized displacements and eigenfunctions satisfying the boundary conditions (6), and m and n are integers.

Distinct to the traditional modeling the viscous resistance forces via first order time-derivatives [16], in the present research we adopt the fractional order time-derivative

$$(d/dt)^{\gamma},$$

what, as it had been shown in [17,18], allows one to obtain the damping coefficients dependent on the natural frequency of vibrations. It has been demonstrated in [29] on the example of the Golden Gate suspension bridge that such an approach for modeling the damped non-linear vibrations provides the good agreement between the theoretical results and the experimental data through the appropriate choice of the fractional parameter (the order of the fractional derivative) and the viscosity coefficient.

It was shown in Samko et al. [30] (see Chapter 2, Paragraph 5, point 7^0) that the fractional order of the operator of differentiation

$$(d/dt)^{\gamma}$$

is equal to the Marsho fractional derivative, which, in its turn, equal to the Riemann–Liouville derivative D_+^{γ}.

It has been noted in [17,18] that a fractional derivative is the immediate extension of an ordinary derivative. In fact, when $\gamma \to 1$ the fractional derivative goes over into the ordinary time-derivative of the first order, and the mathematical model of the viscoelastic shell under consideration transforms into the conventional Kelvin–Voigt model, wherein the elastic element behaves non-lineally, but the viscous element behaves linearly. When

$$\gamma \to 0,$$

the fractional derivative

$$D_+^{\gamma} f \text{ tends to } f(t).$$

To put it otherwise, the introduction of the new fractional parameter along with the parameters $æ_i$ allows one to change not only the magnitude of viscosity at the cost of an increase or decrease in the parameters $æ_i$, but also the character of viscosity at the sacrifice of variations in the fractional parameter γ.

Now substituting the proposed solution (7) in nondimensioned Eqs. (1)-(3), multiplying then each equation by the corresponding function $\eta_{jmn}(x,\varphi)$, integrating over x and φ, and using the orthogonality conditions for linear modes within the domains of

$$0 \le x \le 1 \text{ and } 0 \le \varphi \le 2\pi,$$

we are led to a coupled set of nonlinear ordinary differential equations of the second order in $x_{imn}(t)$. However, a new procedure has been proposed in [21] for decoupling the linear parts of nonlinear differential equations.

Thus, the system is reduced to the following form:

$$\ddot{X}_{1mn} + æ_1 D^{\gamma} X_{1mn} + \Omega_{1mn}^2 X_{1mn} = = -\sum_{i=1}^{3} F_{imn} L_{imn}^{I}; \tag{8}$$

$$\ddot{X}_{2mn} + æ_2 D^{\gamma} X_{2mn} + \Omega_{2mn}^2 X_{2mn} = = -\sum_{i=1}^{3} F_{imn} L_{imn}^{II}; \tag{9}$$

$$\ddot{X}_{3mn} + æ_3 D^{\gamma} X_{3mn} + \Omega_{3mn}^2 X_{2mn} = = -\sum_{i=1}^{3} F_{imn} L_{imn}^{III}, \tag{10}$$

where $D^{\gamma} = (d/dt)^{\gamma}$, and X_i (i=1,2,3) are new generalized displacements which are connected with $x_{imn}(t)$ via eigenvectors L_{imn}^{I}, L_{imn}^{II}, L_{imn}^{III}

$$x_{imn}(t) = X_{1mn} L_{imn}^{I} + X_{2mn} L_{imn}^{II} + X_{3mn} L_{imn}^{III}$$

of the matrix S_{ij}^{mn} with the corresponding eigenvalues Ω_{1mn}, Ω_{2mn}, and Ω_{3mn}, the elements of which are the following:

$$S_{ij}^{mn} = \begin{bmatrix} S_{11}^{mn} & S_{12}^{mn} & S_{13}^{mn} \\ S_{21}^{mn} & S_{22}^{mn} & S_{23}^{mn} \\ S_{31}^{mn} & S_{32}^{mn} & S_{33}^{mn} \end{bmatrix} = \begin{bmatrix} \left(\pi^2 m^2 + \frac{1-\sigma}{2}\beta_1^2 n^2\right) & \frac{1+\sigma}{2}\beta_1 \pi mn & \sigma\beta_1 \pi m \\ \frac{1+\sigma}{2}\beta_1 \pi mn & \left(\frac{1-\sigma}{2}\pi^2 m^2 + \beta_1^2 n^2\right) & \beta_1^2 n \\ \sigma\beta_1 \pi m & \beta_1^2 n & \frac{\beta_2^2}{12}(\pi^2 m^2 + \beta_1^2 n^2)^2 + \beta_1 \end{bmatrix}, \quad (11)$$

where

$$\beta_1 = l / R \text{ and } \beta_2 = h / l$$

are the parameters defining the dimensions of the shell.

From Eqs. (8)-(10) it is seen that their left-hand side parts are linear and independent of each other, while they are coupled only by non-linear terms F_{imn} in their right-hand sides.

It is known [3, 31] that during nonstationary excitation of thin bodies not all possible modes of vibration would be excited. Moreover, the modes which are strongly coupled by any of the so-called internal resonance conditions are initiated and dominate in the process of vibration, resulting in the energy transfer from one subsystem to another between the coupled modes, in so doing the types of modes to be excited are dependent of the character of the external excitation. It was emphasized in [31] that in the presence of damping, all modes that are not directly or indirectly excited by an internal resonance decay with time.

Assume hereafter that the vibration process occurs in such a way that only three natural modes corresponding to the complex generalized displacements

$$X_{1s_1s_2}, \; X_{2l_1l_2}, \text{ and } X_{3k_1k_2}$$

are excited and dominate over other natural modes. In this case, the right parts of Eqs. (8)-(10) are significantly simplified.

According to [20], the approximate solution of these three nonlinear equations (wherein the low indices s_1s_2, l_1l_2 and k_1k_2 are omitted for the ease of presentation) for small but finite amplitudes weakly varying with time could be represented by a uniform expansion in terms of different time scales:

$$X_i = \varepsilon X_{i1}(T_0, T_1, ...) + \varepsilon^2 X_{i2}(T_0, T_1, ...) + ... \quad (12)$$

where $i = 1, 2, 3$, ε is a small dimensionless parameter of the same order of magnitude as the amplitudes,

$$T_n = \varepsilon^2 t \;\; (n = 0, 1, 2, \ldots)$$

are new independent variables, among them:

$$T_0 = t$$

is a fast scale characterizing motions with the natural frequencies, and

$$T_1 = \varepsilon t$$

is a slow scale characterizing the modulation of the amplitudes and phases of the modes with nonlinearity.

Applying the method of multiple scales directly to the governing partial-differential equations by substituting (12) in them and considering that the first and second time-derivatives, as well as the fractional order time-derivative are defined in terms of new time scales, respectively, as follows:

$$\frac{d}{dt} = D_0 + \varepsilon D_1 + ..., \quad \frac{d^2}{dt^2} = D_0^2 + 2\varepsilon D_0 D_1 ... \quad (13)$$

$$\left(\frac{d}{dt}\right)^{\gamma}=D^{\gamma}=(D_0+\varepsilon D_1+...)^{\gamma}=$$
$$=D_0^{\gamma}+\varepsilon\gamma D_0^{\gamma-1}D_1+\frac{1}{2}\varepsilon^2\gamma(\gamma-1)D_0^{\gamma-2}D_1^2+... \quad (14)$$

where $D_n=\partial/\partial T_n$, and D_0^{γ}, $D_0^{\gamma-1}$, $D_0^{\gamma-2}$... are the Riemann-Liouville fractional derivatives in time t [17,18], after separating the terms at the same powers of ε, we could obtain the equations corresponding to different orders of ε [21].

It should be noted that the expansion of the fractional order operator of differentiation (14) for the first time was suggested in 1998 by Rossikhin and Shitikova [17], and nowadays it is used by the researchers worldwide when solving the nonlinear dynamic problems with fractional order damping.

The case of the order of ε has been considered in detail in [21], wherein all types of the internal resonance, which could occur on this step, have been detected and classified: (1) the two-to-one internal resonance, when one natural frequency is twice the other natural frequency, (2) the one-to-one-to-two or one-to-two-to-two internal resonance, and (3) the combinational resonances of the additive-difference type of the first order, among them, the case of $\Omega_2=\Omega_1+\Omega_3$, which we are going to study below numerically.

Utilizing the procedure described in [22] and considering that the fractional order damping coefficients have the form of

$$æ_i=\varepsilon\mu_i\tau_i^{\gamma},$$

where τ_i is the relaxation time of the i-th generalized displacement and μ_i is a finite value, the following six first-order nonlinear ordinary-differential equations governing the modulation of the amplitudes and phases of the three interacting modes in case of combinational additive internal resonance $\Omega_2=\Omega_1+\Omega_3$ have been obtained:

$$(a_1^2)^{\cdot}+s_1a_1^2=-\Omega_1^{-1}a_{23}^{I}\,a_1a_2a_3\sin\delta\,; \quad (15)$$
$$(a_2^2)^{\cdot}+s_2a_2^2=\Omega_2^{-1}a_{13}^{II}\,a_1a_2a_3\sin\delta\,; \quad (16)$$
$$(a_3^2)^{\cdot}+s_3a_3^2=-\Omega_3^{-1}a_{12}^{III}\,a_1a_2a_3\sin\delta\,; \quad (17)$$
$$\dot{\varphi}_1-\frac{1}{2}\sigma_1-\frac{1}{2}\frac{a_{23}^{I}}{\Omega_1}\frac{a_2a_3}{a_1}\cos\delta=0\,; \quad (18)$$
$$\dot{\varphi}_2-\frac{1}{2}\sigma_2-\frac{1}{2}\frac{a_{13}^{II}}{\Omega_2}\frac{a_1a_3}{a_2}\cos\delta=0\,; \quad (19)$$
$$\dot{\varphi}_3-\frac{1}{2}\sigma_3-\frac{1}{2}\frac{a_{12}^{III}}{\Omega_3}\frac{a_1a_2}{a_3}\cos\delta=0\,; \quad (20)$$

where a_i and φ_i (i=1,2,3) are the amplitudes and phases, respectively,

$$\delta=\varphi_2-(\varphi_1+\varphi_3)$$

is the phase difference, an over dot denotes the differentiation with respect to T_1,

$$s_i=\mu_i\tau_i^{\gamma}\Omega_i^{\gamma-1}\sin\psi\,,\quad \sigma_i=\mu_i\tau_i^{\gamma}\Omega_i^{\gamma-1}\cos\psi\ \left(i=1,2,3\right),$$
$$\psi=\frac{1}{2}\pi\gamma\,,$$

and a_{23}^{I}, a_{13}^{II}, a_{12}^{III} are constant coefficients defined by the coupled modes of vibrations [22].

3. NUMERICAL METHOD OF SOLUTION

3.1. Defining the shell parameters that satisfy the condition of the combinational internal resonance $\Omega_2=\Omega_1+\Omega_3$

Before proceeding to numerical investigations, let us find the shell parameters which could satisfy the condition of the additive combinational internal resonance

$$\Omega_2=\Omega_1+\Omega_3\,.$$

Table 1. Part of shell parameters which satisfy the resonance condition $\Omega_2 = \Omega_1 + \Omega_3$.

Ω_1	m_1	n_1	Ω_2	m_2	n_2	Ω_3	m_3	n_3	σ	β_1	β_2
30.4137	5	3	44.412	4	5	13.9983	3	1	0.33	8.37	0.004
27.0251	5	2	43.1784	3	4	16.1531	3	1	0.33	10.23	0.004
19.8875	5	1	44.6532	4	4	24.7656	3	2	0.33	10.42	$5\ 10^{-5}$
18.9932	5	1	48.9931	4	5	29.9997	2	3	0.33	9.30	0.005
17.1999	5	1	33.883	3	5	16.6832	3	2	0.33	6.40	0.002
16.4713	3	3	23.0467	2	5	6.57529	1	1	0.33	4.36	0.004
15.7683	4	1	41.7693	1	5	26.0007	1	3	0.33	8.17	0.005

For this purpose we should use the properties of the symmetric matrix s_{ij}^{mn} (11) possessing three real eigenvalues $\Omega_{i\,mn}$ (i =1,2,3) which are in the correspondence with three mutually orthogonal eigenvectors $L_{i\,mn}$.

We search for values $\Omega_{1m_1n_1}$, $\Omega_{2m_2n_2}$, and $\Omega_{3m_3n_3}$ corresponding to the fixed shell's parameters σ, β_1 and β_2, which could satisfy the additive combinational resonance $\Omega_2 = \Omega_1 + \Omega_3$ (here subindices m_in_i are omitted for the ease of presentation), resulting in coupling of these particular three modes of vibration. Some results are shown in Table 1, from which it is evident that the situation of such a combinational resonance could be realized rather often in real shells used as parts of different civil engineering structures.

3.2. Numerical solution of general multi-term linear equations

Using the numerical method proposed in [25]-[28], the procedure based on the reduction of the problem to a set of fractional differential equations to estimate numerically the solution of Eqs. (8-10) is as follows:

let

$$Y_1 = X_1, \tag{21}$$
$$Y_2 = D^{\gamma} X_1 = D^{\gamma} Y_1,$$
$$Y_3 = DX_1 = DY_1,$$
$$\ddot{X}_1 = DDX_1 = DY_3.$$

First substitute these equalities in equation (8), resulting in

$$DY_3 = -\sum_{i=1}^{3} F_{1\,mn} L_{i\,mn}^{I} - æ_1 Y_2 - \Omega_1^2 Y_1, \tag{22}$$
$$Y_4 = X_2,$$
$$Y_5 = D^{\gamma} X_2 = D^{\gamma} Y_4,$$
$$Y_6 = DX_2 = DY_4,$$
$$\ddot{X}_2 = DDX_2 = DY_6;$$

then in equation (9), resulting in

$$DY_6 = -\sum_{i=1}^{3} F_{2\,mn} L_{i\,mn}^{II} - æ_2 Y_5 - \Omega_2^2 Y_4, \tag{23}$$
$$Y_7 = X_3,$$
$$Y_8 = D^{\gamma} X_3 = D^{\gamma} Y_7,$$
$$Y_9 = DX_3 = DY_7,$$

$$\ddot{X}_3 = DDX_3 = DY_9,$$

and finally in equation (10), resulting in

$$DY_9 = -\sum_{i=1}^{3} F_{3\,mn} L^{II}_{i\,mn} - æ_3 Y_8 - \Omega_3^2 Y_7 . \quad (24)$$

Thus, the governing set of nine equations in nine unknown values Y_i in the matrix form could be written as

$$\begin{bmatrix} D^{\gamma} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ D & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & D & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & D^{\gamma} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & D & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & D & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & D^{\gamma} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & D & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & D \end{bmatrix} * \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \\ Y_5 \\ Y_6 \\ Y_7 \\ Y_8 \\ Y_9 \end{bmatrix} =$$

$$= \begin{bmatrix} Y_2 \\ Y_3 \\ -\sum_{i=1}^{3} F_{1\,mn} L^{I}_{i\,mn} - æ_1 Y_2 - \Omega_1^2 Y_1 \\ Y_5 \\ Y_6 \\ -\sum_{i=1}^{3} F_{2\,mn} L^{II}_{i\,mn} - æ_2 Y_5 - \Omega_2^2 Y_4 \\ Y_8 \\ Y_9 \\ -\sum_{i=1}^{3} F_{3\,mn} L^{III}_{i\,mn} - æ_3 Y_8 - \Omega_3^2 Y_7 \end{bmatrix} \quad (25)$$

Two different types of discretization of derivatives in (25) could be utilized [25-28]. For the first order of differentiation, the trapezoidal rule is usually used:

$$DY = f \xrightarrow{yields} Y_i = Y_{i-1} + \frac{1}{2} h(f_i + f_{i-1}) . \quad (26)$$

So in our problem, the discrete derivatives (D) will take the form

$$Y_i - \frac{1}{2} hf_i = Y_{i-1} + \frac{1}{2} h(f_{i-1}) . \quad (27)$$

To discretize the fractional derivative, the Diethelm's method could be used [25]:

$$D^{\gamma} Y = \frac{1}{{}^{\gamma}\chi_i} \left(\sum_{k=0}^{i} {}^{\gamma}\omega_{k,i} Y_{i-k} + \frac{Y_0}{\alpha} \right) \quad (28)$$

where

$${}^{\gamma}\chi_i = (ih)^{\alpha}\, \Gamma(-\gamma)$$

and ${}^{\gamma}\omega_{k,0}, \ldots, {}^{\gamma}\omega_{k,i}$ [25, 26] are the convolution weights derived from the fact that the fractional operator defined in terms of a convolution integral. We will use the weights of the quadrature formula [25]

$$f(\gamma) = \gamma(1-\gamma)\, j^{-\gamma}\, {}^{\gamma}\omega_{k,0} = $$
$$= \begin{cases} -1 & \text{for } k = 0 \\ 2k^{1-\gamma} - (k-1)^{1-\gamma} - (k+1)^{1-\gamma} & \text{for } k = 1, 2, ., j-1 \\ (\gamma-1)k^{-\gamma} - (k-1)^{1-\gamma} + k^{1-\gamma} & \text{for } k = j \end{cases} \quad (29)$$

Discretization of the equation

$$D^{\gamma} Y_1 = Y_2 \quad (30)$$

results in the following relationships (note $\alpha = \gamma$):

$$\frac{1}{{}^{\gamma}\chi_i} \left(\sum_{k=0}^{i} {}^{\gamma}\omega_{k,i} Y_{1\,i-k} + \frac{Y_{10}}{\gamma} \right) = Y_2 ; \quad (31)$$

$$\left({}^{\gamma}\omega_{0,i} Y_{1i} + \sum_{k=1}^{i} {}^{\gamma}\omega_{k,i} Y_{1\,i-k} + \frac{Y_{10}}{\alpha} \right) = {}^{\gamma}\chi_i Y_{2\,i} ; \quad (32)$$

Let

$$\sum_{k=1}^{i} {}^{\gamma}\omega_{k,i} Y_{1\,i-k} + \frac{Y_{10}}{\gamma} = s_{1\,i-1},$$

so we have

$$({}^{\gamma}\omega_{0,i} Y_{1i} + s_{1i-1}) = {}^{\gamma}\chi_i Y_{2i}\,, \tag{33}$$

$$s_{1i} = -{}^{\gamma}\omega_{0,i} Y_{1i} + {}^{\gamma}\chi_i Y_{2i}\,. \tag{34}$$

By the trapezoidal rule we can represent

$$Y_3 = DY_1$$

in a discrete form as

$$Y_{1i} = Y_{1i-1} + \frac{h}{2}(Y_{3i} + Y_{3i-1})\,. \tag{35}$$

So rearranging the terms

$$Y_{1i} - \frac{h}{2}Y_{3i} = Y_{1i-1} + \frac{h}{2}Y_{3i-1} = s_{2i-1}\,, \tag{36}$$

and utilizing the trapezoidal rule, we can discretize

$$DY_3 = -\sum_{j=1}^{3} F_{1mn} L^I_{j\,mn} - æ_1 Y_2 - \Omega_1^2 Y_1\,; \tag{37}$$

$$Y_{3i} = Y_{3i-1} + \frac{h}{2}\Big[-\sum_{j=1}^{3}\left(F_{1\,mn\,i} + F_{1\,mn\,i-1}\right)L^I_{j\,mn} - æ_1(Y_{2i} + Y_{2i-1}) - \Omega_1^2(Y_{1i} + Y_{1i-1})\Big]. \tag{38}$$

Rearranging the terms, we have

$$\left(Y_{3i} + \frac{h}{2}(æ_1 Y_{2i} + \Omega_1^2 Y_{1i})\right) = Y_{3i-1} + \frac{h}{2}\left(-\sum_{j=1}^{3}(F_{1\,mn\,i} + F_{1\,mn\,i-1})L^I_{j\,mn} - æ_1 Y_{2i-1} - \Omega_1^2 Y_{1i-1}\right) = s_{3i-1} \tag{39}$$

Repeating these steps (as we have done in Eqs. (30)-(39)) for all other values (Y_4-Y_9), and arranging them in a matrix form, we obtain (40). Then it is quite straightforward to solve (40) (Figure 1).

$$\begin{bmatrix} -{}^{\gamma}\omega_{0,i} & {}^{\gamma}\chi_i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -\frac{h}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{h}{2}\Omega_1^2 & \frac{h}{2}æ_1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -{}^{\gamma}\omega_{0,i} & {}^{\gamma}\chi_i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -\frac{h}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{h}{2}\Omega_2^2 & \frac{h}{2}æ_2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -{}^{\gamma}\omega_{0,i} & {}^{\gamma}\chi_i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -\frac{h}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{h}{2}\Omega_3^2 & \frac{h}{2}æ_3 & 1 \end{bmatrix} * \begin{bmatrix} Y_{1i} \\ Y_{2i} \\ Y_{3i} \\ Y_{4i} \\ Y_{5i} \\ Y_{6i} \\ Y_{7i} \\ Y_{8i} \\ Y_{9i} \end{bmatrix} = \begin{bmatrix} s_{1i-1} \\ s_{2i-1} \\ s_{3i-1} \\ s_{4i-1} \\ s_{5i-1} \\ s_{6i-1} \\ s_{7i-1} \\ s_{8i-1} \\ s_{9i-1} \end{bmatrix} \tag{40}$$

Figure 1. Formula (40).

3.3. Numerical solution of the governing equations for the combinational additive internal resonance using the Runge-Kutta fourth order method

To utilize the Runge-Kutta fourth order method to estimate numerically the solution of equations (15)-(20), we first rewrite these equations as follows:

$$\dot{a}_1 = \frac{1}{2}(-\Omega_1^{-1} a_{23}^{I} a_2 a_3 \sin\delta - s_1 a_1)\,; \qquad (41)$$

$$\dot{a}_2 = \frac{1}{2}(\Omega_2^{-1} a_{13}^{II} a_1 a_3 \sin\delta - s_2 a_2)\,; \qquad (42)$$

$$\dot{a}_3 = \frac{1}{2}(-\Omega_3^{-1} a_{12}^{III} a_1 a_2 \sin\delta - s_3 a_3)\,; \qquad (43)$$

$$\dot{\varphi}_1 = \frac{1}{2}\sigma_1 + \frac{1}{2}\frac{a_{23}^{I}}{\Omega_1}\frac{a_2 a_3}{a_1}\cos\delta\,; \qquad (44)$$

$$\dot{\varphi}_2 = \frac{1}{2}\sigma_2 + \frac{1}{2}\frac{a_{13}^{II}}{\Omega_2}\frac{a_1 a_3}{a_2}\cos\delta\,; \qquad (45)$$

$$\dot{\varphi}_3 = \frac{1}{2}\sigma_3 + \frac{1}{2}\frac{a_{12}^{III}}{\Omega_3}\frac{a_1 a_2}{a_3}\cos\delta\,. \qquad (46)$$

4. NUMERICAL RESULTS

4.1. Method 1: multi-step fractional differential equations.

The numerical solution using the multi step method of equation (40) has been carried out at the dimensionless parameters presented in Table 1 (for the case presented in the first line), and the results are presented in Fig. 1 for different magnitudes of the fractional parameter.

4.2. Method 2: the analysis of the amplitudes and phases using multiple time scales

Variation of the fractional parameter γ from 0 to 1 allows one to investigate vibrations of cylindrical shells in surrounding media with different viscous properties, including the pure elastic case at $\gamma = 0$ and conventional Kelvin-Voigt model at $\gamma \to 1$.

The dynamic behavior of a cylindrical shell in a viscous medium at γ = 0.02, 025, 0.5, and 0.98, which is found by using the first and second methods, is shown in Figures 2 and 3, respectively, for the parameters taken from Table 1, which correspond to the combinational internal resonance at $\Omega_2 = \Omega_1 + \Omega_3$.

The behavior of amplitudes of vibrations reveals the exchange of energy between the generalized displacements of the system under the considered case of the combinational internal resonance.

CONCLUSION

Free damped vibrations of a shallow nonlinear thin cylindrical shell in a fractional derivative viscoelastic medium are investigated numerically by two different methods based on the new approach proposed in [20-23].

The numerical solutions of the damped vibrations of the nonlinear cylindrical shell subjected to the conditions of the internal resonance have been estimated, and good agreement between the two methods has been achieved. Within the first method, the generalized displacements of a coupled set of nonlinear ordinary differential equations of the second order are calculated using the numerical solution of nonlinear multi-term fractional differential equations by the procedure based on the reducing the problem to a system of fractional differential equations. According to the second method, the amplitudes and phases of nonlinear vibrations are estimated from the governing nonlinear differential equations describing amplitude-and-phase modulations for the case of the combinational internal resonance.

It has been shown that, as in [22], the phenomenon of the internal resonance could be very critical, since in a circular cylindrical shell the internal additive and difference combinational resonances are always present. The effect of viscosity on the energy exchange mechanism is analyzed.

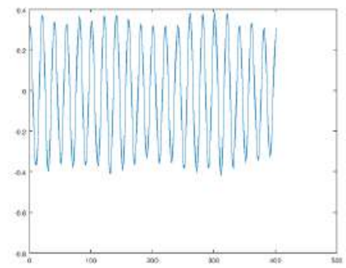
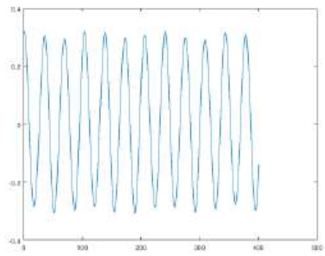
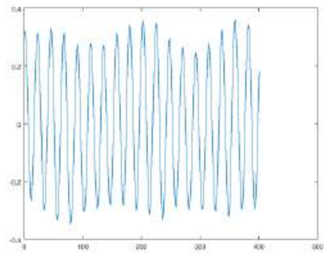
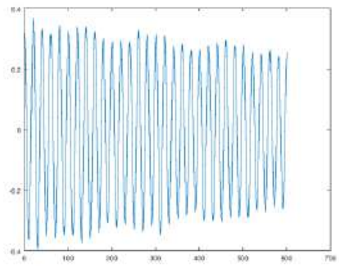
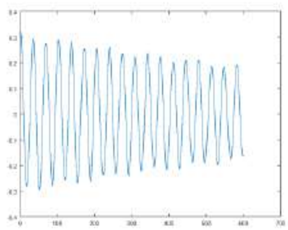
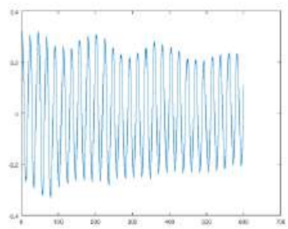
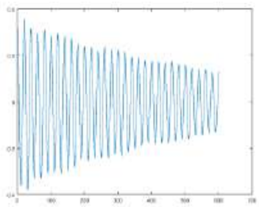
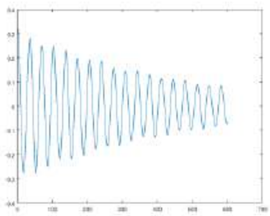
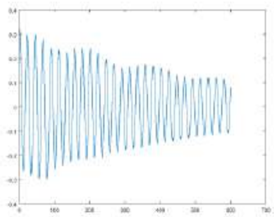
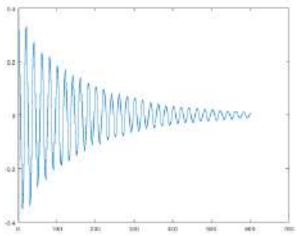
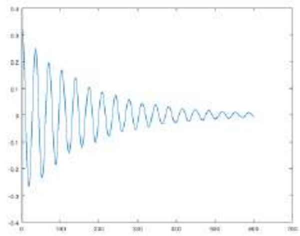
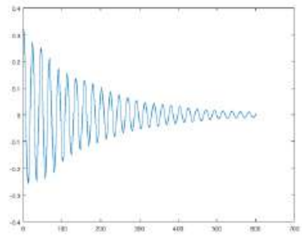
γ =0.02		
X_1	X_2	X_3
		
γ =0.25		
X_1	X_2	X_3
		
γ =0.5		
X_1	X_2	X_3
		
γ =0.98 ≈ 1		
X_1	X_2	X_3
		

Figure 2. The time-dependence of the generalized displacements at different magnitudes of the fractional parameter.

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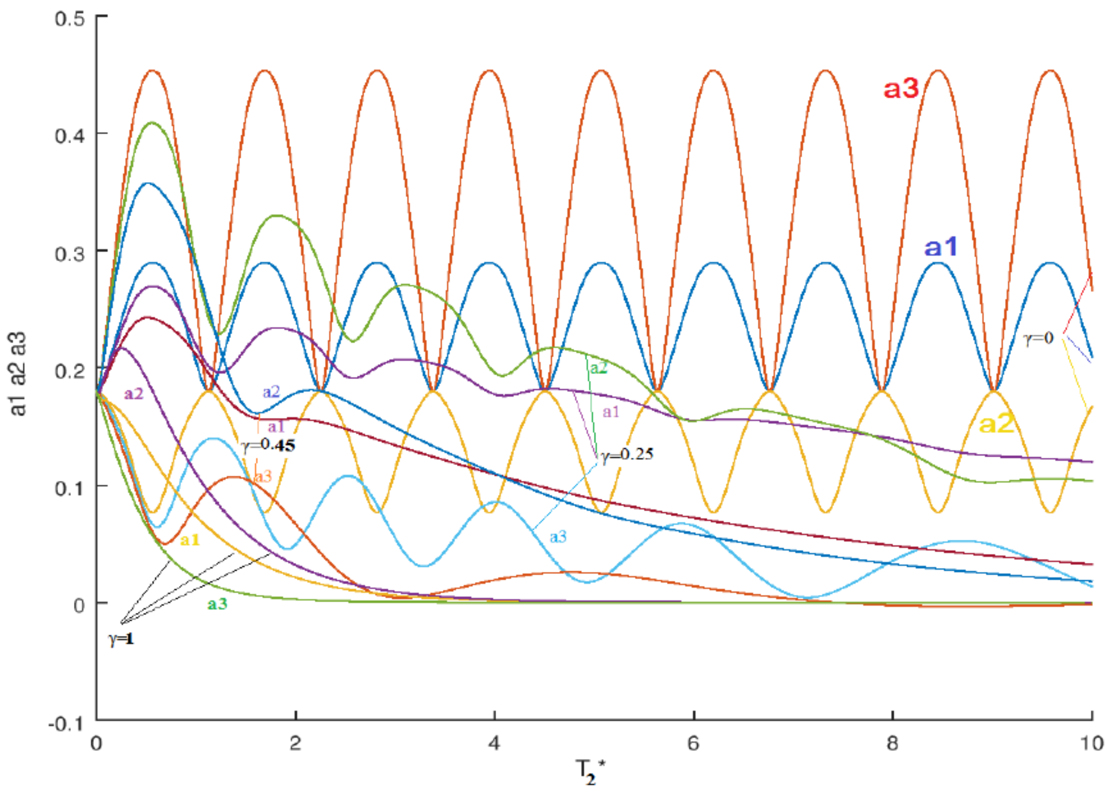

Figure 3. Dimensionless amplitude vs. dimensionless time as the solution of equations (41)-(46).

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