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THE GENERALIZED BIFRACTIONAL BROWNIAN MOTION

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Abstract: To extend several known centered Gaussian processes, we introduce a new centered Gaussian process, named the generalized bifractional Brownian motion. This process depends on several parameters, namely $\alpha > 0$, $\beta > 0$, $0 < H < 1$ and $0 < K \le 1$. When $K = 1$, we investigate its convexity properties. Then, when $2HK \le 1$, we prove that this process is an element of the QHASI class, a class of centered Gaussian processes, which was introduced in 2015.

Keywords: convexity, quasi-helix, approximately stationary increments

ОБОБЩЕННОЕ БИФРАКТАЛЬНОЕ БРОУНОВСКОЕ ДВИЖЕНИЕ

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Аннотация: Расширение нескольких центрированных гауссовский процессов требует введения нового процесса, названного бифрактальным броуновским движением. Этот процесс зависит от нескольких параметров, а именно: $\alpha > 0$, $\beta > 0$, $0 < H < 1$ и $0 < K \le 1$. Для случая, когда параметр $K = 1$, исследуется свойство выпуклости. Для случая, когда $2HK \le 1$, доказывается принадлежность этого процесса к квази-классу (обладанием квази-канонической кривой постоянного склона), и к классу центральных гауссовских процессов.

Ключевые слова: выпуклость, квази-каноническая кривая постоянного склона, приближенно стационарные прирощении

1. INTRODUCTION

Let $\{B_{H,K}(t), t \in \mathrm{R}\}$ be a bifractional Brownian motion (bBm) with indices $0 < H < 1$ and $0 < K \le 1$, i.e. a centered Gaussian process such that $B_{H,K}(0) = 0$, with probability 1, and

$$\mathrm{E}\left(B_{H,K}(t)B_{H,K}(s)\right) = \frac{1}{2^K}\left(\left(|t|^{2H} + |s|^{2H}\right)^K - |t-s|^{2HK}\right). \quad (1.1)$$

We can verify that

$$\mathrm{Var} B_{H,K}(t) = |t|^{2HK}$$

and that the bBm is self-similar with index HK. Note also that the process $B_{H,1}$ is the fractional Brownian motion (fBm) and therefore the process $B_{1/2,1}$ is the ordinary Wiener process. Straightforward computations show $B_{H,K}$ has no stationary increment. However, the bBm is a HK-quasi-helix in the sense of Kahane ([1], p. 137) and its increments are approximately stationary for small increments. Houdré and Villa [2] introduced the bBm and established the previous results.

Consider the following centered Gaussian process $Y := Y_{\alpha,\beta,H,K}$ defined as follows:

$$Y(t) := Y_{\alpha,\beta,H,K}(t) = \alpha\, B_{H,K}(t) + \beta\, B_{H,K}(-t) \quad (1.2)$$

with $t \geq 0, \alpha > 0, \beta > 0$. Set

$$\alpha(K) = 1/(2^{(2-K)/2}),\ 0 < K \leq 1 .$$

The introduction of the process Y is motivated by the fact that this process was already introduced for specific values of α, β and K. Indeed, the process $Y_{\alpha(1),\alpha(1),H,1}$ was introduced in [3] and was named the sub-fractional Brownian motion. El-Nouty and Journé [4] extended the former process by introducing the process $Y_{\alpha(K),\alpha(K),H,K}$, which was named the sub-bifractional Brownian process (sbBm). Finally, Zili [5] introduced the process $Y_{\alpha,\beta,H,1}$, which was named the generalized fractional Brownian motion (gfBm). This is why we will name Y the generalized fractional Brownian motion (gbBm). Set for $s,t \geq 0$

$$\sigma^2(s,t) := \sigma^2_{\alpha,\beta,H,K}(s,t) = \mathrm{E}\left(\left(Y_{\alpha,\beta,H,K}(t) - Y_{\alpha,\beta,H,K}(s)\right)^2\right). \quad (1.3)$$

Let us study the convexity properties of

$$\sigma^2(s,t) := \sigma_{\alpha,\beta,H,1}{}^2(s,t)$$

on the set

$$T = \{(s,t) \in [0,1]^2 \ :\ s \leq t\} .$$

Our first result is stated in the following proposition.

Proposition 1. *I. If* $H \geq 1/2$ *, then the function* $\sigma^2_{\alpha,\beta,H,1}(s,t)$, $(s,t) \in T$ *is convex and has a unique maximum at the point* $(0,1)$.
II. If $H < 1/2$ *, then the function* $\sigma^2_{\alpha,\beta,H,1}(s,t)$, $(s,t) \in T$ *is concave and has a unique maximum at the point* $(0,1)$.

Note the difference between the case $0 < H < 1/2$ and the case $1/2 < H < 1$, i.e. between short-range dependence and long-range dependence. This phenomenon was already observed by several authors in the fBm case (Beran ([6], p52), Samorodnitsky and Taqqu ([7], p. 123)). Proposition 1 establishes that the fBm and the gfBm are similar from the convexity point of view. However, when one compares Proposition 1 with Proposition 1.1 in [8], he can observe the difference between the gfBm and the bBm. This implies that there is a significant difference between the processes $Y_{\alpha,\beta,H,1}$ and $Y_{\alpha,\beta,H,K}$, with $K < 1$.

The quasi-helix with approximately stationary increments (QHASI) class of centered Gaussian processes was introduced by El-Nouty [9] and was defined as follows. A centered Gaussian process $\{X(t), t \in I \subset R\}$ belongs to the QHASI class if it fulfills the five following assumptions:

- A1: $X(0) = 0$ with probability 1,
- A2: there exists $\lambda > 0$ such that X is self-similar with index λ ,
- A3: there exists $0 < C_1 \leq C_2 < +\infty$ such that $\forall (s,t) \in I^2$

$$C_1 |t-s|^{2\lambda} \leq \mathrm{E}\left(X(t) - X(s)\right)^2 \leq C_2 |t-s|^{2\lambda} ,$$

- A4: there exists $C_3 \in [C_1, C_2]$ such that

$$\forall (s,t) \in I^2,\ t \geq s,\ st \neq 0, \text{ when } t-s \to 0, \quad \mathrm{E}\left(X(t) - X(s)\right)^2 \sim C_3 (t-s)^{2\lambda}$$

- A5: there exists $C_4 \in [C_1, C_2]$ such that

$$\forall t \in I,\ \mathrm{E}X(t)^2 = C_4 |t|^{2\lambda} .$$

Let us state three known results. The first one is due Houdré and Villa [2], the second one to [10] and the last one to El-Nouty [9].

Theorem 2. *The bBM is an element of the QHASI class.*

Theorem 3. *The sfBm is an element of the QHASI class.*

Theorem 4. *The sbBm is an element of the QHASI class.*

We insist on the fact that the values of λ, C_1, C_2, C_3 and C_4 for the bBm, the sfBm and the sbBm can be found in El-Nouty [9]. Using some results of Zili [5] and introducing some additional computations, we get the following result.

Theorem 5. *The gfBm is an element of the QHASI class, with*

- $\lambda = H$,
- $C_1 = \min\left(\alpha^2+\beta^2, (\alpha+\beta)^2 - 2^{2H}\alpha\beta\right)$,
- $C_2 = \max\left(\alpha^2+\beta^2, \left((\alpha+\beta)^2 - 2^{2H}\alpha\beta\right)\right)$,
- $C_3 = \alpha^2+\beta^2$,
- $C_4 = \alpha^2 + 2\left(1-2^{2H-1}\right)\alpha\beta + \beta^2$.

Our main result is stated in the following theorem.

Theorem 6. *Assume that* $2HK \le 1$. *Then the gbBm is an element of the QHASI class, with*

- $\lambda = HK$,
- $C_1 = (\alpha+\beta)^2 - 2^{2-K}\alpha\beta$,
- $C_2 = 2^{1-K}\left((\alpha+\beta)^2 - 2^{2HK}\alpha\beta\right)$,
- $C_3 = 2^{1-K}\left(\alpha^2+\beta^2\right)$,
- $C_4 = \alpha^2 + 2\left(1-2^{2HK-K}\right)\alpha\beta + \beta^2$.

Let us make some comments on the above theorem. When

$$H \le 1/2 \text{ and } K = 1,$$

theorem TH is similar to theorem tutu. When

$$2HK \le 1 \text{ and } \alpha = \beta = \alpha(K),$$

theorem TH is similar to theorem JLJ. Finally, as expected, note the importance of the hyperbola

$$2HK = 1.$$

This phenomenon was already observed in El-Nouty [11], El-Nouty and Journé [12] and Russo and Tudor [13].

Let us investigate the case

$$1 < 2HK < 2 .$$

There is no difficulty to determine λ, C_3 and C_4. Indeed they have the same values as those found in the case

$$2HK \le 1.$$

A careful reading of the proof of theorem TH enables to state the following lemma.

Lemma 7. *Assume that* $2HK > 1$. *Then* $\forall\,(s,t) \in I^2$

$$\sigma^2(s,t) \le 2^{1-K}\left(\alpha^2+\beta^2\right)|t-s|^{2HK} .$$

The question of the existence of a constant C_1 is still an open one.

In section 2, we prove Proposition 1. The proof of Theorem 6is postponed to section 3. In the sequel, there is no loss of generality in assuming $K < 1$.

2. PROOF OF PROPOSITION 1

Recall that $K=1$ in this section. We have for any $t \geq s$

$$\begin{aligned}\sigma^2(s,t) &:= \sigma_{\alpha,\beta,H,1}{}^2(s,t)\\ &= s^{2H}\Big((\alpha^2+\beta^2)(u-1)^{2H}\\ &\quad + 2^{2H}\alpha\beta\big(2^{1-2H}(u+1)^{2H}-u^{2H}-1\big)\Big) \qquad (2.1)\\ &:= s^{2H}\ \lambda(u),\end{aligned}$$

where

$$u = t/s \geq 1.$$

There is no difficulty to deal with the case $H=1/2$. When $H \neq 1/2$, the derivative of order 2 of the function λ is

$$\lambda^{(2)}(u) = 2H(2H-1)s^{2H}\ g(u), \qquad (2.2)$$

where

$$\begin{aligned}g(u) &= (\alpha^2+\beta^2)(u-1)^{2H-2}\\ &\quad + 2^{2H}\alpha\beta\big(2^{1-2H}(u+1)^{2H-2}-u^{2H-2}\big).\end{aligned}$$

Let us study the sign of the function g. We have

$$g(u) \geq 2\,\alpha\,\beta\,h(u)\,, \qquad (2.3)$$

where

$$h(u) = (u-1)^{2H-2}+(u+1)^{2H-2}-2^{2H-1}u^{2H-2}.$$

Since

$$2H-2<0\,,$$

the function $u \to u^{2H-2}$, $u \geq 1$ is convex, and therefore

$$(u-1)^{2H-2}+(u+1)^{2H-2} \geq 2u^{2H-2}.$$

Hence,

$$h(u) = (2-2^{2H-1})u^{2H-2} > 0\,. \qquad (2.4)$$

Combining (2.4) and (2.3) with (2.1), we establish that, if $H>1/2$, then the function $\sigma_{\alpha,\beta,H,1}{}^2(s,t)$, $(s,t)\in T$ is convex, else the function is concave.
By using (2.1), we have for any real $s,\ t, s \neq t$, and $a>0$

$$\sigma^2(s,t)>0,\ \sigma^2(s,s)=0$$

and

$$\sigma^2(as,at) = a^{2H}\sigma^2(s,t)\,.$$

Thus, we get

$$s\frac{\partial\sigma^2(s,t)}{\partial s}+t\frac{\partial\sigma^2(s,t)}{\partial t} = 2H\sigma^2(s,t)\,. \qquad (2.5)$$

If

$$\frac{\partial\sigma^2(s,t)}{\partial s} = \frac{\partial\sigma^2(s,t)}{\partial t} = 0\,,$$

then (2.5) yields that

$$\sigma^2(s,t)=0$$

and consequently

$$s=t\,.$$

Thus, there is no maximum of $\sigma^2(s,t)$ in the interior of T.
Let us investigate the existence of a maximum of $\sigma^2(s,t)$ on the border of T. Note that

$$\sigma^2(0,t) = \big(\alpha^2+(2-2^{2H})\alpha\,\beta+\beta^2\big)\ t^{2H}$$

has a unique maximum at the point $t=1$. Thus, we have to study the function

$$\sigma^2(s,1) = (\alpha^2+\beta^2)(1-s)^{2H} + 2^{2H}\alpha\beta\left(2^{1-2H}(1+s)^{2H} - s^{2H} - 1\right). \quad (2.6)$$

We have by differentiation

$$\frac{d\sigma^2(s,1)}{ds} = 2H\left(-\left(\alpha^2+\beta^2\right)(1-s)^{2H-1} + 2^{2H}\alpha\beta\left(2^{1-2H}(1+s)^{2H-1} - s^{2H-1}\right)\right).$$

We must consider the following three cases:

Case 1. $H = 1/2$.

Since

$$\frac{d\sigma^2(s,1)}{ds} = -\left(\alpha^2+\beta^2\right) < 0,$$

the function $\sigma^2(s,t),(s,t)\in T$ has a unique maximum at the point $(0,1)$. Using (2.6), we have

$$\sigma^2(0,1) = \alpha^2+\beta^2.$$

Case 2. $2H > 1$.

We have

$$\frac{d\sigma^2(s,1)}{ds} \le 0 \Leftrightarrow 2\alpha\beta(1+s)^{2H-1} \le \left(\alpha^2+\beta^2\right)(1-s)^{2H-1} + 2\alpha\beta(2s)^{2H-1}. \quad (2.7)$$

Recall that

$$2\alpha\beta \le \alpha^2+\beta^2.$$

To prove (2.7), it suffices to verify

$$(1+s)^{2H-1} \le (1-s)^{2H-1} + (2s)^{2H-1}. \quad (2.8)$$

Inequality (2.8) is true at the points 0 and 1. Set

$$u = 1/s \ge 1.$$

Thus, inequality (2.8) can be rewritten as follows:

$$(u+1)^{2H-1} \le (u-1)^{2H-1} + 2^{2H-1}. \quad (2.9)$$

Set

$$g(u) = (u+1)^{2H-1} - (u-1)^{2H-1} - 2^{2H-1}.$$

We have

$$g'(u) = (2H-1)\left((u+1)^{2H-2} - (u-1)^{2H-2}\right) \le 0.$$

Since

$$g(1) = 0,$$

we prove that $g \le 0$ and consequently inequality (2.9).

The function $\sigma^2(s,t),(s,t)\in T$ has a unique maximum at the point $(0,1)$. Using (2.6), we have

$$\sigma^2(0,1) = \alpha^2 + \left(2-2^{2H}\right)\alpha\beta + \beta^2.$$

Case 3. $2H < 1$.

To show that

$$\frac{d\sigma^2(s,1)}{ds} \le 0$$

it suffices to establish that

$$2^{1-2H}\left(1+s\right)^{2H-1}-s^{2H-1}\leq 0$$
$$\Leftrightarrow\left(1+s\right)^{2H-1}\leq\left(2s\right)^{2H-1}.$$

Since

$$2H-1<0 \text{ and } s\leq 1,$$

the result is true. Therefore, the function $\sigma^2\left(s,t\right),\left(s,t\right)\in T$ has a unique maximum at the point $\left(0,1\right)$. Using (2.6), we have

$$\sigma^2\left(0,1\right)=\alpha^2+\left(2-2^{2H}\right)\alpha\beta+\beta^2 .$$

The proof of Proposition 1 is complete. ■

3. PROOF OF THEOREM 6

We can easily remark that the process Y is a centered Gaussian process such that $Y(0)=0$ with probability 1 and Y is self-similar with index HK. The covariance function of the process Y is given in the following lemma.

Lemma 8. We have for $t\geq s\geq 0$

$$\mathrm{E}\left(Y(t)Y(s)\right)=\frac{1}{2^K}\Big(\left(\alpha+\beta\right)^2\left(t^{2H}+s^{2H}\right)^K -\left(\alpha^2+\beta^2\right)\left(t-s\right)^{2HK} -2\alpha\beta\left(t+s\right)^{2HK}\Big)$$

and therefore

$$\mathrm{E}\left(Y\left(t\right)^2\right)=\left(\alpha^2+2\left(1-2^{2HK-K}\right)\alpha\beta+\beta^2\right)t^{2HK}.$$

Proof. It suffices to combine (1.1) with (1.2).

Remark 9. Lemma 8 gives the value of the constant C_4 .

Set for $t\geq s\geq 0$

$$F_{H,K}\left(s,t\right)=2\left(\frac{t^{2H}+s^{2H}}{2}\right)^K -t^{2HK}-s^{2HK}\geq 0, \tag{3.1}$$

$$F_{\frac{1}{2},2HK}\left(s,t\right)=2\left(\frac{t+s}{2}\right)^{2HK} -t^{2HK}-s^{2HK}. \tag{3.2}$$

The functions given in (3.1) and (3.2) will play a key role in the proofs of our results. Let us recall the following basic proposition.

Proposition 10. *When* $0<2HK<1$, $F_{1/2,2HK}\geq 0$. *When* $2HK=1$, $F_{1/2,1}=0$. *When* $1<2HK<2$, $F_{1/2,2HK}\leq 0$.

Remark 9. *When* $2HK\leq 1$, *the function* $F_{\frac{1}{2},2HK}$ *can be viewed as* $F_{H,K}$ *with* $H=\frac{1}{2}$.

We can state the second technical lemma.

Lemma 12. *We have for* $t\geq s\geq 0$

$$\sigma^2\left(s,t\right)=2^{1-K}\left(\alpha^2+\beta^2\right)\left(t-s\right)^{2HK} -\left(\alpha+\beta\right)^2 F_{H,K}\left(s,t\right) +2^{1-K+2HK}\alpha\,\beta\,F_{\frac{1}{2},2HK}\left(s,t\right).$$

Proof. It suffices to combine (1.3), (3.1) and (3.2) with lemma 8. ■

The next step consists in determining the value of the constant C_2.

Combining Lemma 12 with Proposition 10, we have

if $0<2HK<1$, then

$$\sigma^2\left(s,t\right)\leq 2^{1-K}\left(\alpha^2+\beta^2\right)\left(t-s\right)^{2HK} +2^{1-K+2HK}\alpha\beta F_{\frac{1}{2},2HK}\left(s,t\right),$$

if $2HK = 1$, then

$$\sigma^2(s,t) \le 2^{1-K}\left(\alpha^2+\beta^2\right)(t-s),$$

and

if $1 < 2HK < 2$, then

$$\sigma^2(s,t) \le 2^{1-K}\left(\alpha^2+\beta^2\right)(t-s)^{2HK}.$$

El-Nouty and Journé [12] showed that we have for $0 < 2HK < 1$,

$$(t-s)^{2HK} + 2^{2HK-1}F_{\frac{1}{2},2HK}(s,t) \le \left(2-2^{2HK-1}\right)(t-s)^{2HK}.$$

Then,

$$\begin{aligned}\sigma^2(s,t) &\le 2^{1-K}\left(\alpha^2+\beta^2\right)(t-s)^{2HK} \\ &+2^{1-K+2HK}\alpha\beta 2^{1-2HK}\left(1-2^{2HK-1}\right)(t-s)^{2HK} \\ &= 2^{1-K}\left((\alpha+\beta)^2-2^{2HK}\alpha\beta\right)(t-s)^{2HK} \\ &:= C_2\,(t-s)^{2HK}.\end{aligned}$$

Let us determine the value of the constant C_1.

Combining Lemma 12 with Proposition 10, we get

$$\sigma^2(s,t) \ge 2^{1-K}\left(\alpha^2+\beta^2\right)(t-s)^{2HK} - (\alpha+\beta)^2 F_{H,K}(s,t).$$

It was proved by El-Nouty and Journé [12] that, when $2HK \le 1$,

$$(t-s)^{2HK} + 2^K\left(t^{2HK}+s^{2HK}\right) - 2\left(t^{2H}+s^{2H}\right) \ge \left(2^K-1\right)(t-s)^{2HK},$$

that is

$$\left(2-2^K\right)(t-s)^{2HK} \ge 2^K F_{H,K}(s,t).$$

Then,

$$\begin{aligned}\sigma^2(s,t) &\ge \left((\alpha+\beta)^2 - 2^{2-K}\,\alpha\,\beta\right)(t-s)^{2HK} \\ &:= C_1(t-s)^{2HK}.\end{aligned}$$

Finally, we determine the value of the constant C_3. Recall that $s > 0$. Set

$$t-s=h.$$

When $h \to 0$, the Taylor expansions of the functions $F_{H,K}$ and $F_{1/2,2HK}$, given in (3.1) and (3.2), are

$$F_{H,K}(s,t) = s^{2HK}\left(H^2K(1-K)\frac{h^2}{s^2} + o\left(\frac{h^2}{s^2}\right)\right) \tag{3.3}$$

and for $2HK \ne 1$

$$F_{\frac{1}{2},2HK}(s,t) = -s^{2HK}\left(\frac{HK(2HK-1)}{2}\frac{h^2}{s^2} + o\left(\frac{h^2}{s^2}\right)\right). \tag{3.4}$$

Combining Lemma 12 with (3.3) and (3.4) (or Proposition 10 if $2HK = 1$), we obtain the Taylor expansion of $\sigma(s,t)$. Hence, since $2HK < 2$, we get the value of C_3.

To complete the proof of Theorem 6, we have to verify that

$$C_1 \le C_3 \le C_2$$

and

$$C_1 \le C_4 \le C_2 .$$

Assume

$$\alpha \ge \beta > 0$$

and set

$$x = \frac{\alpha}{\beta},\ x \ge 1 .$$

The inequalities $C_1 \le C_3 \le C_2$ are equivalent to

$$\begin{aligned} x^2 + \left(2 - 2^{2-K}\right)x + 1 \\ &\le 2^{1-K}\left(x^2+1\right) \\ &\le 2^{1-K}\left(x^2 + \left(2-2^{2HK}\right)x + 1\right), \end{aligned}$$

that is

$$\left(x+1\right)^2 \ge 0$$

and

$$\left(2 - 2^{2HK}\right)x \ge 0 .$$

Since $2HK \le 1$, it proves the result. Similarly, we can establish that

$$C_1 \le C_4 \le C_2 .$$

The proof of Theorem 6 is complete. ■

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