DOI:10.22337/2587-9618-2026-22-2-138-148

MODEL OF DEEP BED FILTRATION IN A POROUS MEDIUM WITH HETEROGENEOUS POROSITY

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Abstract: Modeling the transport and sedimentation of small particles of suspensions and colloids in porous rocks is an important problem in subsurface hydromechanics. Particles entrained in fluid are transported and retained in the rock pores. The filtration process is determined by the number and size of pores and is characterized by porosity—the ratio of the void volume to the total soil volume. In homogeneous materials, porosity can be considered constant. However, in practical problems describing filtration of suspended particles in multilayered and heterogeneous soils, the rock porosity is variable, and a constant-porosity model is inapplicable. A one-dimensional model of suspension and colloid filtration in a porous medium with nonuniform porosity is considered. The problem includes a mass balance equation accounting for variable porosity and a kinetic equation for sediment growth. The model describes the injection of a suspension or colloid of constant concentration into a porous medium containing pure water without suspended or sedimented particles. Unlike the case of uniform porosity, the transport velocity of suspended particles in pores is variable, and the boundary between the suspension and pure water, called the concentration front, is curved. Previously, such problems were solved only numerically. In this article, a system of filtration equations in a medium with variable porosity is solved analytically using the method of characteristics. An explicit formula is obtained for the curved front of suspended and sedimented particle concentrations, and exact analytical closed-form solutions are constructed ahead of and behind the front. An explicit solution is found for a model with a linear filtration function.

Key words: deep bed filtration, porous medium, porosity, concentration front, exact solution

МОДЕЛЬ ГЛУБИННОЙ ФИЛЬТРАЦИИ СУСПЕНЗИИ В ПОРИСТОЙ СРЕДЕ С НЕОДНОРОДНОЙ ПОРИСТОСТЬЮ

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Аннотация: Моделирование переноса и осаждения мелких частиц суспензий и коллоидов в пористых породах является важной задачей подземной гидромеханики. Частицы, увлекаемые жидкостью, перемещаются и задерживаются в порах породы. Процесс фильтрации определяется количеством пор и их размерами и характеризуется пористостью – отношением объема пустот к общему объему грунта. В однородных материалах пористость можно считать постоянной. Однако в практических задачах, описывающих фильтрацию суспензии в многослойных и неоднородных грунтах, пористость породы непостоянна и модель с постоянной пористостью неприменима. Рассматривается одномерная модель фильтрации суспензий и коллоидов в пористой среде с неоднородной пористостью. Задача включает уравнение баланса масс, учитывающее переменную пористость, и кинетическое уравнение роста осадка. Модель описывает впрыск суспензии или коллоида постоянной концентрации в пористую среду, содержащую чистую воду без взвешенных и осажденных частиц. В отличие от случая однородной пористости, скорость переноса взвешенных частиц в порах непостоянна и граница раздела суспензии и чистой воды криволинейная. Ранее такие задачи решались только численно. В статье система уравнений фильтрации в среде с переменной пористостью решена аналитически методом характеристик. Получена явная формула для криволинейного фронта концентраций взвешенных и осажденных частиц, перед фронтом и за фронтом построены аналитические точные решения в замкнутой форме. Для модели с линейной функцией фильтрации найдено решение в явном виде.

Ключевые слова: глубинная фильтрация, пористая среда, пористость, фронт концентраций, точные решения

1. INTRODUCTION

Flows of fluid with small particles in porous media are widespread in nature and engineering. Modeling the transport of tiny particles by fluid in rock is used in the strengthening of loose soil and in the design of underground storage facilities for hazardous waste, for tunnels and for hydraulic structures [1-4].

Moving particles of suspensions and colloids in porous media are affected by various forces: mechanical, electrostatic, hydrodynamic, gravitational [5-8]. Under the influence of these forces, some particles settle on the framework of the porous medium and form a stationary deposit. The standard macroscopic model of long-term deep bed filtration of a suspension in a porous medium includes mass conservation for both mobile and settled particulate matter, alongside the formula governing how quickly sediment accumulates. When the number of suspended particles is minimal, the rate at which they deposit is directly linked to their concentration. The factor that quantifies this relationship between deposition speed and suspended particle density is known as the filtration function. Over a short period of time, the filtration function can be considered constant [11-13]. During long-term deep bed filtration, the concentration of sedimented particles increases and the number of vacant sites for retained particles decreases. Consequently, the filtration function depends on the deposit size and decreases as it increases. If the filtration function vanishes upon reaching a certain limiting value, it is called blocking. In many models, a linear decreasing function, called the Langmuir coefficient, is used as the filtration function [14-16].

The capacity of a rock as a reservoir for liquid and gas is characterized by porosity - the ratio of the pore volume to the total volume of the rock. In a homogeneous rock, porosity can be considered constant. To model the filtration of suspensions and colloids in multilayered and heterogeneous soils, one must acknowledge that porosity varies with spatial position within the porous medium. This incorporation of variable porosity enables the filtration challenge to be addressed within a non-uniform porous medium. A similar model was studied in [17], but no analytical solutions were obtained for the problem with variable porosity.

This paper fills the gap. The study examines a large-scale model of deep bed filtration within a sample exhibiting non-uniform porosity, focusing on the process of injecting a suspension or colloidal solution into a porous medium with clean water. The transport of particles by an incompressible carrier fluid in a plane-parallel channel is described by a one-dimensional model, including a quasi-linear system of first-order differential equations, initial conditions and boundary conditions at the inlet of the porous medium [18, 19]. For the model in dimensionless form, analytical and numerical solutions are constructed, for the linear filtration function, an explicit solution is obtained.

2. MATHEMATICAL MODEL

In the domain $\Omega = \{0 \le x \le 1,\ t \ge 0\}$ consider a quasilinear system

$$\varphi(x)\frac{\partial C}{\partial t} + \frac{\partial C}{\partial x} + \frac{\partial S}{\partial t} = 0\,, \tag{1}$$

$$\frac{\partial S}{\partial t} = \Lambda(S)C\,. \tag{2}$$

Here the porosity function $\varphi(x)$ and the filtration function $\Lambda(x,S)$ are continuous and non-negative.

For system (1), (2), the boundary conditions are established at the filter inlet $x = 0$ and at the initial time $t = 0$:

$$C(x,t)\big|_{x=0} = 1\,; \tag{3}$$

$$C(x,t)\big|_{t=0} = 0;\quad S(x,t)\big|_{t=0} = 0\,. \tag{4}$$

According to condition (3), equation (2) at the inlet $x = 0$ has the form

$$\frac{\partial S}{\partial t} = \Lambda(S)\,. \quad (5)$$

Divide both parts of equation (5) by $\Lambda(0,S)$ and integrate with respect to the variable *t*:

$$\int_0^t \frac{\partial S/\partial t}{\Lambda(S)} dt = t\,. \quad (6)$$

Using condition (4), transform the integral on the left side of (6)

$$\int_0^{S_0(t)} \frac{dS}{\Lambda(S)} = t\,. \quad (7)$$

The concentration of particles entering the porous medium $S_0(t) = S(0,t)$ is given by Formula (7).

The interface between the suspension and pure water is marked by a concentration front. This front migrates from the inlet to the outlet as the injected suspension displaces the water. Because conditions (3) and (4) are not met at zero, the concentration solution *C* is discontinuous at the front, while the solution *S* is continuous. The domain $\Omega_S = \{0 \le x \le 1,\, t \ge x\}$ behind the front, containing suspended and settled particles, has a positive solution, whereas the pure water domain $\Omega_0 = \{0 \le x \le 1,\, 0 \le t \le x\}$ ahead of the front has a zero solution.

A non-uniform porosity $\varphi(x)$ results in a curvilinear concentration front. To find the front line $t_\Gamma(x)$, we pass to the characteristic variables in equation (1) [20]. Let

$$t' = \varphi(x), \quad x' = 1\,, \quad (8)$$

where the apostrophe (') denotes the derivative with respect to the "internal time" - the characteristic variable τ.

Substitute (2) into (1) and pass to the characteristic variables in equation (1):

$$C' = \Lambda(S)C\,. \quad (9)$$

In the suspension domain Ω_S, the initial conditions for equations (8) and (9) are

$$t(0) = t_0, \quad x(0) = 0, \quad C(0) = 1\,. \quad (10)$$

Here $t_0 \ge 0$ is the starting point of the characteristic on the *OT* axis.

Solution of equations (8) with conditions (10) are

$$t = \int_0^\tau \varphi(y)dy + t_0, \quad x = \tau\,.$$

In Cartesian coordinates, the characteristics in the domain Ω_S are given by the relation

$$t = \int_0^x \varphi(y)dy + t_0\,. \quad (11)$$

The concentration front is a characteristic that emerges from the origin of coordinates:

$$t_\Gamma(x) = \int_0^x \varphi(y)dy\,.$$

From the continuity of the deposit concentration follows the condition on the concentration front

$$S\big|_{t=t_\Gamma(x)} = 0\,. \quad (12)$$

According to condition (12), on the concentration front, equation (1) takes the form

$$\varphi(x)\frac{\partial C}{\partial t} + \frac{\partial C}{\partial x} + \Lambda(0)C = 0\,. \quad (13)$$

Equation (13) with the initial condition (3) determines the solution on the concentration front:

$$C = e^{-\Lambda(0)x}\,. \quad (14)$$

3. EXACT SOLUTION TO MODEL WITH NON-UNIFORM POROSITY

Consider the problem (1)-(4) in the suspension domain Ω_S, where the solution is nonzero. Express the unknown C from equation (2):

$$C = \frac{\partial S/\partial t}{\Lambda(S)}, \tag{15}$$

and substitute it into equation (1)

$$\varphi(x)\frac{\partial}{\partial t}\left(\frac{\partial S/\partial t}{\Lambda(S)}\right) + \frac{\partial}{\partial x}\left(\frac{\partial S/\partial t}{\Lambda(S)}\right) + \frac{\partial S}{\partial t} = 0. \tag{16}$$

Using the ratio

$$\frac{\partial}{\partial x}\left(\frac{\partial S/\partial t}{\Lambda(S)}\right) = \frac{\partial^2 S/\partial t\partial x}{\Lambda(S)} - \frac{\partial S}{\partial t}\frac{\Lambda'(S)}{\Lambda^2(S)}\frac{\partial S}{\partial x} = \frac{\partial}{\partial t}\left(\frac{\partial S/\partial x}{\Lambda(S)}\right), \tag{17}$$

equation (16) takes the form

$$\frac{\partial}{\partial t}\left(\varphi(x)\frac{\partial S/\partial t}{\Lambda(S)}\right) + \frac{\partial}{\partial t}\left(\frac{\partial S/\partial x}{\Lambda(S)}\right) + \frac{\partial S}{\partial t} = 0. \tag{18}$$

Integrating equation (18) with respect to the variable t from 0 to t yields

$$\varphi(x)\frac{\partial S/\partial t}{\Lambda(S)} + \frac{\partial S/\partial x}{\Lambda(S)} + S = K(x). \tag{19}$$

Substitute $t = 0$ into (19), and from the initial condition (4) obtain the integration constant: $K(x) = 0$. Equation (19) takes the form

$$\varphi(x)\frac{\partial S/\partial t}{\Lambda(S)} + \frac{\partial S/\partial x}{\Lambda(S)} + S = 0. \tag{20}$$

Pass to the characteristic variables in equation (20):

$$\frac{S'}{\Lambda(S)} + S = 0. \tag{22}$$

The initial condition for equation (22) is set on the OT axis:

$$S(\tau)\big|_{\tau=0} = S_0(t_0), \tag{23}$$

where the function S_0 is given by formula (7).

The solution to equation (22) with condition (23) has an implicit form:

$$\int_{S_0(t_0)}^{S(\tau)} \frac{ds}{s\Lambda(s)} = -\tau. \tag{24}$$

Express the initial characteristic coordinate t_0 from equation (11):

$$t_0 = t - \int_0^x \varphi(y)dy. \tag{25}$$

In Cartesian coordinates, the solution (24) is given by the formula

$$\int_{S_0\left(t-\int_0^x \varphi(y)dy\right)}^{S(x,t)} \frac{ds}{s\Lambda(s)} = -x. \tag{26}$$

If $S(x,t)$ is known, the solution $C(x,t)$ can be obtained from the formula (15). Let us derive another relation that does not use derivatives of the solution and specifies the relationship between the solutions on the characteristics (11). Differentiate the equality (26) with respect to the variable t:

$$\frac{\partial S/\partial t}{S\Lambda(S)} - \frac{\partial S_0/\partial t}{S_0\Lambda(S_0)} = 0. \tag{27}$$

Substituting the formulae (2) and (5) into equation (27) yields

$$\frac{C}{S}-\frac{1}{S_0}=0. \qquad (28)$$

Express the solution C from (28):

$$C=\frac{S(x,t)}{S_0\left(t-\int_0^x \varphi(y)dy\right)}. \qquad (29)$$

Relation (29) is called the Riemann invariant [21].

The formulae (24), (26) and (29) generalize the known solution of the filtration model with constant porosity [22].

In the case where the filtration function is equal to the Langmuir coefficient

$$\Lambda(S)=\lambda\left(1-\frac{S}{S_m}\right),\ \lambda>0,\ S_m>0, \qquad (30)$$

the solution of (26) and (29) can be obtained in explicit form. Calculate the integral in the formula (7):

$$\int_0^{S_0}\frac{dS}{\Lambda(S)}=\int_0^{S_0}\frac{dS}{\lambda\left(1-\frac{S}{S_m}\right)}=-\frac{S_m}{\lambda}\ln\left(1-\frac{S_0}{S_m}\right).$$

Now we can find the solution at the porous medium inlet:

$$S_0(t)=S_m\left(1-e^{-\frac{\lambda t}{S_m}}\right). \qquad (31)$$

Calculation of the integral on the left-hand side of (26) gives

$$\int_{S_0\left(t-\int_0^x \varphi(y)dy\right)}^{S(x,t)}\frac{ds}{s\Lambda(s)}=\int_{S_0\left(t-\int_0^x \varphi(y)dy\right)}^{S(x,t)}\frac{ds}{s\lambda\left(1-\frac{s}{S_m}\right)}=\frac{1}{\lambda}\left(\ln\frac{S}{S_m-S}-\ln\frac{S_0}{S_m-S_0}\right). \qquad (32)$$

Substitute the formula (32) into (26) and obtain the solution:

$$S(x,t)=\frac{S_m\left(e^{\frac{\lambda}{S_m}\left(t-\int_0^x \varphi(y)dy\right)}-1\right)}{e^{\frac{\lambda}{S_m}\left(t-\int_0^x \varphi(y)dy\right)}+e^{\lambda x}-1}. \qquad (33)$$

Substitute the solutions (31) and (33) into the Riemann invariant (29):

$$C=\frac{S_m\left(e^{\frac{\lambda}{S_m}\left(t-\int_0^x \varphi(y)dy\right)}-1\right)}{e^{\frac{\lambda}{S_m}\left(t-\int_0^x \varphi(y)dy\right)}+e^{\lambda x}-1}\Bigg/ S_m\left(1-e^{-\frac{\lambda}{S_m}\left(t-\int_0^x \varphi(y)dy\right)}\right)=\frac{e^{\frac{\lambda}{S_m}\left(t-\int_0^x \varphi(y)dy\right)}}{e^{\frac{\lambda}{S_m}\left(t-\int_0^x \varphi(y)dy\right)}+e^{\lambda x}-1}. \qquad (34)$$

The formulae (31), (33) and (34) define an explicit solution to problem (1)-(4) in the suspension domain Ω_S for the filtration function (30). At constant porosity $\varphi(x)=1$, the solution to the filtration problem is known [23]. The case of linear porosity was studied in [24].

4. NUMERICAL SIMULATION

The numerical calculation of the model was performed at $\lambda = 1,\ S_m = 1,\ \varphi(x) = 1.1 - 0.2x$.

The solution profiles (how the solution changes with position at a specific moment) are shown in Fig. 1.

a) b)

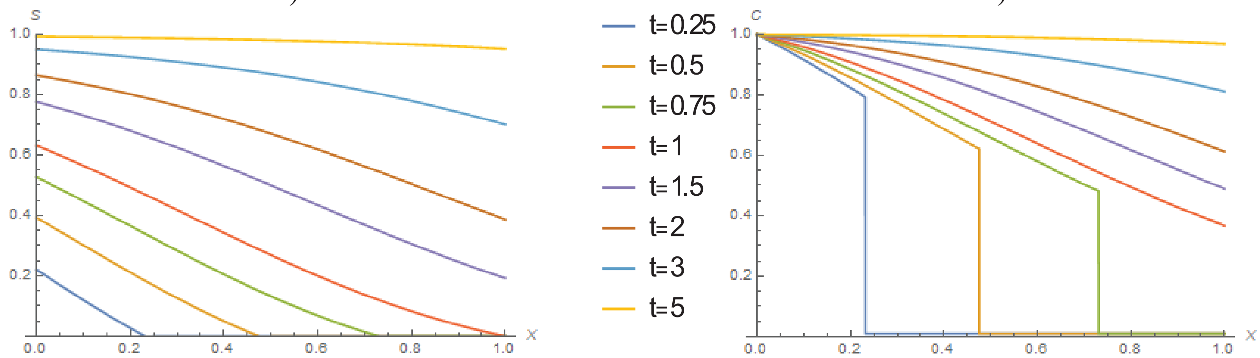

Figure 1. Profiles a) retained particles concentration S b) suspended particles concentration C

Since the suspension spreads in the porous medium from the inlet $x = 0$ to the outlet $x = 1$, the profiles decrease monotonically. At $t < 1$ the profiles vanish before the concentration front.

Figure 2 shows the evolution of the solution (the dependence of the solution on time for a fixed spatial coordinate).

a) b)

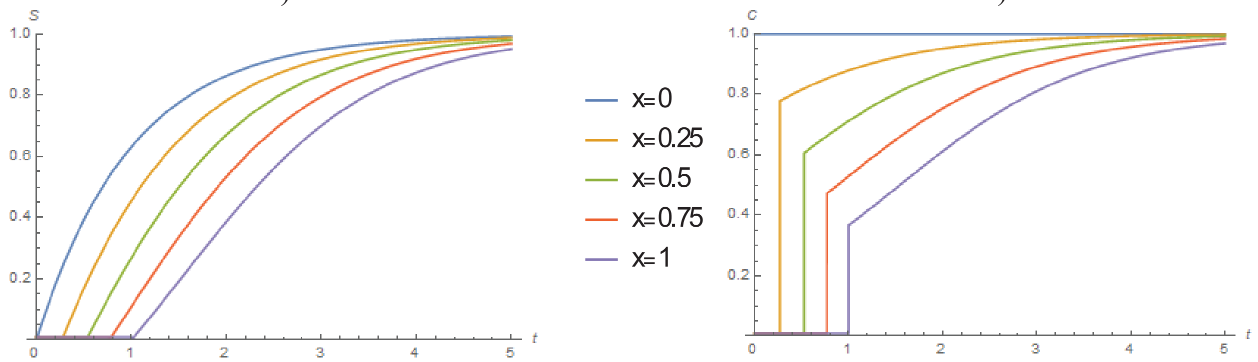

Figure 2. Evolution a) retained particles concentration S b) suspended particles concentration C

Before the front, the concentrations of suspended and retained particles are zero, behind the front the evolution increases and tends to the limiting values: $S \to S_m = 1,\ C \to C(0,t) = 1$. The discontinuity in the concentration of suspended particles at the front is indicated by a vertical line.

5. DISCUSSION

The analytical solutions found for the problem of filtration with variable porosity allow fine-tuning of laboratory experiments and predicting the behavior of particles carried by water in rock [25-27].

Comparison of solutions of models with variable and constant porosity makes it possible to estimate the averaging errors when replacing variable porosity with constant porosity.

Exact solutions allow us to correctly solve the inverse filtration problem – finding the porosity and filtration function from a known solution [28, 29].

The general model of filtration in a heterogeneous porous medium assumes the dependence of porosity and filtration function on the spatial variable. In the general case, the analytical solution of the problem in closed form is unknown; numerical solutions are used to analyze the model [30-32].

The problem of particle migration in a porous medium is of interest [33, 34]. The displacement of a suspension by pure water from a sample with variable porosity will be considered separately.

6. CONCLUSIONS

The study of particle transport in a porous medium with variable porosity leads to the following conclusions.

- A model of filtration with variable porosity was studied.
- A new analytical solution to the model is constructed, generalizing the solution of the standard filtration problem with constant porosity.
- For the linear filtration function (Langmuir coefficient) a solution in explicit form is obtained.

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