

EFFECT OF THE WAVE PROCESSES CAUSED BY EXTERNAL INFLUENCE ON THE FILTERING OF AQUEOUS SOLUTIONS IN THE EXPANDED LOADING LAYER

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Abstract: In this paper, the influence of traveling waves arising from external action on the filtering process of aqueous solutions in the expanded loading layer is investigated. A three-dimensional dynamic generalized model of fluid motion in a porous medium under a nonlinear external influence is used as the primary mathematical model. A model describing traveling waves is obtained. Nine particular cases of this model with three types of nonlinearity process of filtering are examined: power, exponential, and logarithmic. The external influence is also selected power, exponential, and logarithmic. The particular models describe both expansion and contraction of the loading layer on the type of filtration nonlinearity, the type of external influence, and the traveling wave parameters. For filtering with the expanding loading layer we found the time at which maximum its expansion is achieved. When the loading layer contaminates we found the time at which it will be destroyed.

Keywords: filtering of aqueous solutions, expanded loading layer, traveling wave, nonlinear external influence, porous medium

ВЛИЯНИЕ ВОЛНОВЫХ ПРОЦЕССОВ, ВЫЗВАННЫХ ВНЕШНИМ ВОЗДЕЙСТВИЕМ, НА ФИЛЬТРОВАНИЕ ВОДНЫХ РАСТВОРОВ В РАСШИРЕННОМ СЛОЕ ЗАГРУЗКИ

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Аннотация: В настоящей работе исследуется влияние возникающих при внешнем воздействии бегущей волн на процесс фильтрации водных растворов в расширенном слое загрузки. В качестве основной математической модели используется трехмерная динамическая обобщенная модель движения жидкости в пористой среде при наличии нелинейного внешнего воздействия. Получена модель, описывающая бегущие волны. Исследовано 9 частных случаев этой модели с тремя видами нелинейности процесса фильтрации: степенной, экспоненциальной и логарифмической. Внешнее воздействие при этом также выбрано степенным, экспоненциальным и логарифмическим. Эти частные модели описывают как расширение, так и загрязнение слоя загрузки в зависимости от вида нелинейности процесса фильтрации, вида внешнего воздействия и параметров бегущей волны. Для фильтрации с расширяющимся слоем загрузки мы нашли время, при котором достигается максимальное его расширение. При загрязнении слоя загрузки мы нашли время, за которое он разрушится.

Ключевые слова: фильтрация водных растворов, расширенный слой загрузки, бегущая волна, нелинейное внешнее воздействие, пористая среда

INTRODUCTION

The science of the movement of fluids, gases, and their mixtures in porous media is called subsurface

hydromechanics. The object of study in subsurface hydromechanics is filtration flow - the flow of a fluid (gas, gas- fluid mixture) in a porous medium. Subsurface hydromechanics has

extensive applications in other sciences, including hydrogeology, engineering geology, soil studies during the initial stages of building and structure construction, hydraulic engineering, and others. The first experiments to study water filtration in saturated soils were conducted by the French scientist A. Darcy, who in 1856 formulated an experimental law expressing the dependence of filtration rate on pressure gradient. During this same period, another French scientist, J. Dupuis, published a monograph outlining the theory of groundwater filtration, deriving formulas for well flow rates, and solving other filtration problems. American scientists C. Slichter and M. Musket made significant contributions to the development of subsurface hydromechanics. The founders of the Russian school of filtration theory were N.E. Zhukovsky and N.N. Pavlovsky. Underground hydromechanics is one of the constituent theories of oil and gas field development and oil and gas production technology. The founder of Russian oil and gas underground hydromechanics is L.S. Leibenzon. He became the founder of underground hydraulics, which played a major role in creating the scientific foundations for oil field development. Scientists S.A. Khristianovich, B.B. Lapuk, I.A. Charny, V.N. Shchelkachev and others, made outstanding contributions to the development of the theory of fluid and gas filtration in oil-gas-water-bearing formations. All these studies are based on the classical mathematical model of a porous medium. The classical model of a porous medium and some of its simple generalizations have been studied in many works [1–7]. In these studies, boundary value problems related to the motion of fluids and gases in porous media were investigated using analytical and numerical methods.

Studying the motion of fluids or gases in porous media using classical models does not always adequately describe real processes. This is due to the fact that these models do not take into account the presence of an external influence. For more adequately describe real processes, it is necessary to develop and study new, more complex models with an external influence. The study of a general three-dimensional nonlinear

dynamic model of fluid or gas motion in a porous medium with a non-stationary source was initiated in the works of the authors [8–11]. In our study, we investigate the influence of a traveling wave on the filtering of aqueous solutions in an expanded loading layer. As a model for describing this influence, we used a general three-dimensional nonlinear dynamic model of fluid motion in a porous medium with a nonlinear an external influence. This model is defined by the following equation

$$p_t = \Delta\Phi(p) - f(p), \quad (1)$$

where $p = p(t, \mathbf{x})$ is a pressure in the loading layer, $\mathbf{x} = (x, y, z) \in R^3$, $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$,

t is a time, $\Phi(p)$ and $f(p)$ are any functions, which are determined empirically. $\Phi(p) > 0$ defines a nonlinearity of the process, $f(p)$ defines nonlinear nature of the influence of a traveling wave on the filtering of aqueous solutions in an expanded loading layer. A case $f(p) < 0$ corresponds, for example, to the presence of a source. A case $f(p) > 0$ corresponds, for example, to the presence of an absorption. The functions $\Phi(p)$ and $f(p)$ satisfy to the condition

$$\Phi''(p) f'(p) \neq 0.$$

This condition means that the process is nonlinear and the influence is not a constant value.

METHODS

The primary research method in our work is the modeling of physically significant problems using group analysis of differential equations,

which is one of the most effective methods for obtaining maximum information about solutions to differential equations.

The concepts and algorithms of modern group analysis of differential equations can be found, for example, in [12–16] and the references cited therein.

In the specific examples considered, for each obtained nonlinear model, the equation defining this model is reduced to an equivalent system of first-order differential equations. The resulting Cauchy problem for each such system is solved numerically using the Runge-Kutta-Felberg method, the concepts and algorithms of which can be found, for example, in [17] and the references cited therein.

RESULTS AND DISCUSSION

The study conducted in this paper is new and has not been previously reported in the literature.

A general traveling wave for a porous medium has the form

$$p = \phi(\xi), \quad \xi = t + \alpha x + \beta y + \gamma z, \quad (2)$$

Where α, β, γ are arbitrary real numbers, satisfying the condition $\alpha^2 + \beta^2 + \gamma^2 \neq 0$.

Substituting (2) into (1) yields the following differential equation

$$\left(\alpha^2 + \beta^2 + \gamma^2 \right) \left(\frac{d}{d\xi} \left(\frac{d\Phi(\phi)}{d\phi} \frac{d\phi(\xi)}{d\xi} \right) \right) - \frac{d\phi(\xi)}{d\xi} + f(\phi) = 0. \quad (3)$$

The functions $\Phi(\phi)$, $f(\phi)$ and constants α, β, γ specifies each particular model of a traveling wave.

We will be study at some particular models and indicate what influence in these models have on the filtering in the expanded loading layer.

1. First particular models

At $\Phi(\phi) = \mu\phi^\lambda$ where λ, μ are arbitrary real numbers, satisfying the condition $\lambda\mu \neq 0$, equation (3) takes a form

$$\lambda\mu(\alpha^2 + \beta^2 + \gamma^2)\phi^{\lambda-2} \left(\phi \frac{d^2\phi}{d\xi^2} + (\lambda-1) \left(\frac{d\phi}{d\xi} \right)^2 \right) - \frac{d\phi}{d\xi} + f(\phi) = 0. \quad (4)$$

This equation defined model depending on $\alpha, \beta, \gamma, \lambda, \mu$ and $f(\phi)$.

Equation (4) is equivalent to the following system

$$\frac{d\phi}{d\xi} = \phi^{1-\lambda}\psi, \quad \frac{d\psi}{d\xi} = \frac{\phi^{1-\lambda}\psi - f(\phi)}{\lambda\mu(\alpha^2 + \beta^2 + \gamma^2)}, \quad (5)$$

where $\psi = \psi(\xi)$ is new unknown function.

Let at initial time $t = t_0 \geq 0$ at a fixed point

$\mathbf{x}_0 = (x_0, y_0, z_0)$ the pressure and the rate of its change are given

$$p(t_0, \mathbf{x}_0) = p_0 > 0, \quad \frac{\partial p}{\partial t}(t_0, \mathbf{x}_0) = p_1. \quad (6)$$

We will find a function $\phi(\xi)$ satisfying system (5) to solve this problem.

The initial data for system (5) has a form

$$\phi(\xi_0) = p_0, \quad \psi(\xi_0) = (p_0)^{\lambda-1} p_1, \quad (7)$$

$$\xi_0 = t_0 + \alpha x_0 + \beta y_0 + \gamma z_0.$$

Due to the smoothness of the right-hand sides of system (5), the solution of the Cauchy problem (5), (7) exists and is unique in the neighborhood of the point ξ_0 .

1.1. In the first example, we obtain this solution at $f(\phi) = a\phi^b$ (a, b are arbitrary real numbers). At $\alpha = 1, \beta = 1, \gamma = 1, \lambda = 2, \mu = 1, t_0 = 1, x_0 = 1, y_0 = 1, z_0 = 1, p_0 = 1, p_1 = -5, a = 1, b = 1$ we solved numerically the Cauchy problem (5), (7) by the Runge-Kutta-Fehlberg method [17] (with order of accuracy 4). The graph of the function $\phi = \phi(\xi)$ is shown in the Fig. 1.

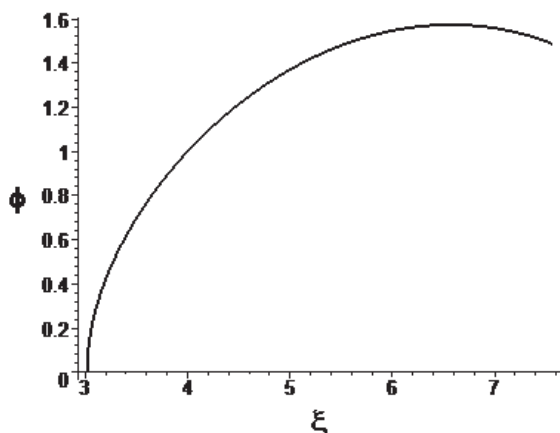


Figure 1. Pressure distributions for the model 1.1

This graph shows that the pressure in the loading layer L increases at $\xi \in [3; 6,6]$. This means that the loading layer gets clogged. At $\xi = 6,6$ the pressure reaches its maximum value and begins to decrease at $\xi > 6,6$. This means that at $\xi = 6,6$, filtering ceases and the loading layer is destroyed. The onset of destruction time is:

$$t = \sup_{x \in L} (6,6 - x - y - z).$$

1.2. In the second example, we obtain this solution at $f(\phi) = a \exp(b\phi)$ (a, b are arbitrary real numbers). At $\alpha = 1, \beta = 1, \gamma = 1, \lambda = 5, \mu = 1, t_0 = 1, x_0 = 1, y_0 = 1, z_0 = 1, p_0 = 1, p_1 = -5,5, a = 1, b = 0,5$ we solved numerically the Cauchy problem (5), (7) by the Runge-Kutta-Fehlberg method [17] (with order

of accuracy 4). The graph of the function $\phi = \phi(\xi)$ is shown in the Fig. 2.

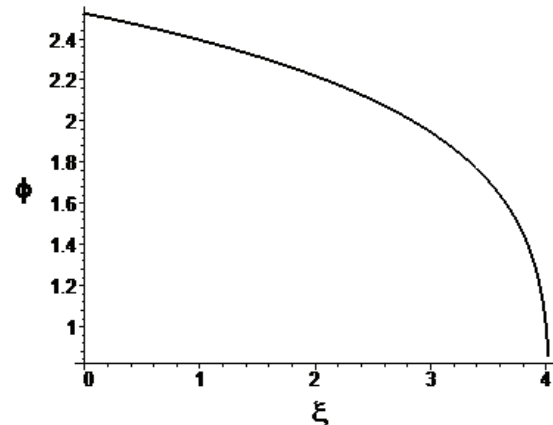


Figure 2. Pressure distributions for the model 1.2

This graph shows that the pressure in the loading layer decreases. This means that the loading layer expands. Let p_* is the pressure at which the maximum expansion of the loading layer L is achieved. And this pressure is reached at the point $x = (x, y, z)$ in the loading layer at $\xi = \xi_*$ ($0 < \xi_* < 4$). This solution has a physical meaning at $\xi \in (0; \xi_*]$. The time to reach maximum expansion of the loading layer is determined by the formula

$$t_* = \sup_{x \in L} (\xi_* - x - y - z).$$

1.3. In the second example, we obtain this solution at $f(\phi) = a \ln \phi$ (a is arbitrary real number). At $\alpha = 1, \beta = 1, \gamma = 1, \lambda = 1,5, \mu = 1, t_0 = 1, x_0 = 1, y_0 = 1, z_0 = 1, p_0 = 1, p_1 = -0,05, a = 1, b = 1$ we solved numerically the Cauchy problem (5), (7) by the Runge-Kutta-Fehlberg method [17] (with order of accuracy 4). The graph of the function $\phi = \phi(\xi)$ is shown in the Fig.3.

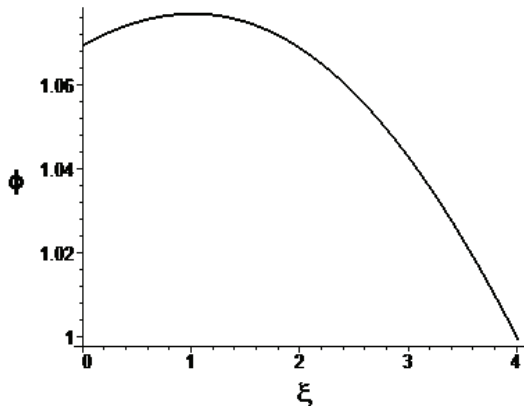


Figure 3. Pressure distributions for the model 1.2.

From this graph, it follows that the pressure increases briefly at $\xi \in (0; 1,2]$ and reaches its maximum value at $\xi = 1,2$, then the pressure at $\xi > 1,2$ decreases. This means that the loading layer L is heavily contaminated and is rapidly beginning to deteriorate. The onset of destruction time is:

$$t = \sup_{\mathbf{x} \in L} (1,2 - x - y - z).$$

2. Second particular models

At $\Phi(\phi) = \mu \exp(\lambda\phi)$ where λ, μ are arbitrary real numbers, satisfying the condition $\lambda\mu \neq 0$, equation (3) takes a form

$$\lambda\mu(\alpha^2 + \beta^2 + \gamma^2) \exp(\lambda\phi) \left(\frac{d^2\phi}{d\xi^2} + \left(\lambda \frac{d\phi}{d\xi} \right)^2 \right) - \frac{d\phi}{d\xi} + f(\phi) = 0. \tag{8}$$

This equation defined model depending on $\alpha, \beta, \gamma, \lambda, \mu$ and $f(\phi)$.

Equation (8) is equivalent to the following system

$$\begin{aligned} \frac{d\phi}{d\xi} &= \exp(-\lambda\phi)\psi, \\ \frac{d\psi}{d\xi} &= \frac{\exp(-\lambda\phi)\psi - f(\phi)}{\lambda\mu(\alpha^2 + \beta^2 + \gamma^2)}, \end{aligned} \tag{9}$$

where $\psi = \psi(\xi)$ is new unknown function.

We use the function $\phi(\xi)$ satisfying this system and conditions (6).

The initial data for system (5) has a form

$$\begin{aligned} \phi(\xi_0) &= p_0, \quad \psi(\xi_0) = p_1 \exp(\lambda p_0), \\ \xi_0 &= t_0 + \alpha x_0 + \beta y_0 + \gamma z_0. \end{aligned} \tag{10}$$

Due to the smoothness of the right-hand sides of system (9), the solution of the Cauchy problem (9), (10) exists and is unique in the neighborhood of the point ξ_0 .

2.1. In the first example, we obtain this solution at $f(\phi) = a\phi^b$ (a, b are arbitrary real numbers

At $\alpha = 1, \beta = 1, \gamma = 1, \lambda = 5, \mu = 1, t_0 = 1, x_0 = 1, y_0 = 1, z_0 = 1, p_0 = 1, p_1 = -1, a = 1,$

$b = 2$ we solved numerically the Cauchy problem (9), (10) by the Runge-Kutta-Fehlberg method [17] (with order of accuracy 4). The graph of the function $\phi = \phi(\xi)$ is shown in the Fig. 4.

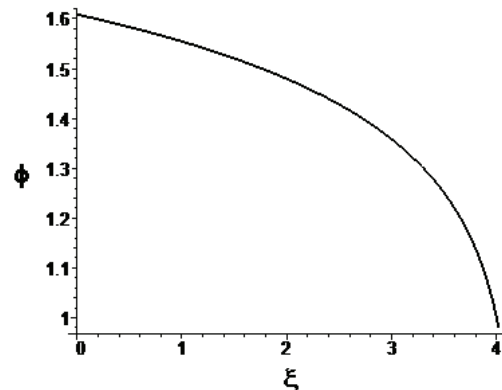


Figure 4. Pressure distributions for the model 2.1.

From this graph, it follows that the pressure in the loading layer decreases. This means that the loading layer expands. Let p_* is the pressure at which the maximum expansion of the loading layer L is achieved. And this pressure is reached at the point $\mathbf{x} = (x, y, z)$ in the loading layer at $\xi = \xi_*$ ($0 < \xi_* < 4$). This solution

has a physical meaning at $\xi \in (0; \xi_*]$. The time to reach maximum expansion of the loading layer is determined by the formula

$$t_* = \sup_{x \in L} (\xi_* - x - y - z).$$

2.2. In the second example, we obtain this solution at $f(\phi) = a \exp(b\phi)$ (a, b are arbitrary real numbers) At $\alpha = 1, \beta = 1, \gamma = 1, \lambda = 2, \mu = 1, t_0 = 1, x_0 = 1, y_0 = 1, z_0 = 1, p_0 = 1, p_1 = -0,01, a = 1, b = -1$ we solved numerically the Cauchy problem (9), (10) by the Runge-Kutta-Fehlberg method [17] (with order of accuracy 4). The graph of the function $\phi = \phi(\xi)$ is shown in the Fig. 5.

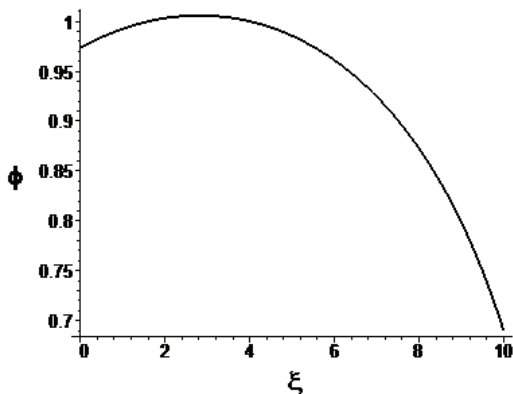


Figure 5. Pressure distributions for the model 2.2.

From this graph, it follows that the pressure increases briefly at $\xi \in (0; 3]$ and reaches its maximum value at $\xi = 3$, then the pressure at $\xi > 3$ decreases. This means that the loading layer L is heavily contaminated and is rapidly beginning to deteriorate. The onset of destruction time is:

$$t = \sup_{x \in L} (3 - x - y - z).$$

2.3. In the second example, we obtain this solution at $f(\phi) = a \ln \phi$ (a is arbitrary real number). At $\alpha = 1, \beta = 1, \gamma = 1, \lambda = 2,$

$t_0 = 1, x_0 = 1, y_0 = 1, z_0 = 1, p_0 = 1, p_1 = 0, a = 1, b = 1$ we solved numerically the Cauchy problem (9), (10) by the Runge-Kutta-Fehlberg method [17] (with order of accuracy 4). The graph of the function $\phi = \phi(\xi)$ is shown in the Fig.6.

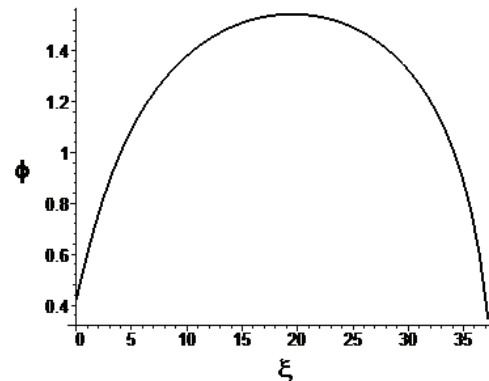


Figure 6. Pressure distributions for the model 2.2.

From this graph, it follows that the pressure increases at $\xi \in (0; 29]$ and reaches its maximum value at $\xi = 29$, then the pressure at $\xi > 29$ decreases. This means that the loading layer L is heavily contaminated and is rapidly beginning to deteriorate. The onset of destruction time is:

$$t = \sup_{x \in L} (29 - x - y - z).$$

3. Third particular models

At $\Phi(\phi) = \lambda \ln \phi$ where λ is arbitrary real nonzero number, , equation (3) takes a form

$$\frac{\lambda}{\phi^2} (\alpha^2 + \beta^2 + \gamma^2) \left(\phi \frac{d^2\phi}{d\xi^2} - \left(\frac{d\phi}{d\xi} \right)^2 \right) - \left(-\frac{d\phi}{d\xi} + f(\phi) \right) = 0. \quad (11)$$

This equation defined model depending on $\alpha, \beta, \gamma, \lambda$ and $f(\phi)$.

Equation (11) is equivalent to the following system

$$\begin{aligned} \frac{d\phi}{d\xi} &= \phi\psi, \\ \frac{d\psi}{d\xi} &= \frac{\phi\psi - f(\phi)}{\lambda(\alpha^2 + \beta^2 + \gamma^2)}, \end{aligned} \quad (12)$$

where $\psi = \psi(\xi)$ is new unknown function.

We use the function $\phi(\xi)$ satisfying this system and conditions (6).

The initial data for system (12) has a form

$$\begin{aligned} \phi(\xi_0) &= p_0, \quad \psi(\xi_0) = \frac{p_1}{p_0}, \\ \xi_0 &= t_0 + \alpha x_0 + \beta y_0 + \gamma z_0. \end{aligned} \quad (13)$$

Due to the smoothness of the right-hand sides of system (12), the solution of the Cauchy problem (12), (13) exists and is unique in the neighborhood of the point ξ_0 .

3.1. In the first example, we obtain this solution at $f(\phi) = a\phi^b$ (a, b are arbitrary real numbers). At $\alpha = 1, \beta = 1, \gamma = 1, \lambda = 5, \mu = 1, t_0 = 1, x_0 = 1, y_0 = 1, z_0 = 1, p_0 = 1, p_1 = -1, a = 1, b = -1$ we solved numerically the Cauchy problem (12), (13) by the Runge-Kutta-Fehlberg method [17] (with order of accuracy 4). The graph of the function $\phi = \phi(\xi)$ is shown in the Fig. 7.

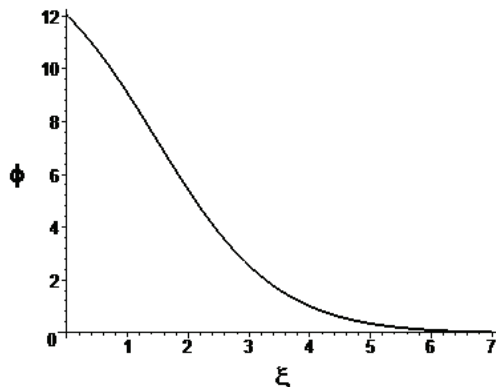


Figure 7. Pressure distributions for the model 3.1.

From this graph, it follows that the pressure corresponding to maximum expansion of the loading layer L is reached at $\xi = \xi_*$ ($0 < \xi_* < 6$). This solution has a physical meaning at $\xi \in (0; \xi_*]$. The time to reach maximum expansion of the loading layer is determined by the formula

$$t_* = \sup_{\mathbf{x} \in L} (\xi_* - x - y - z).$$

3.2. In the second example, we obtain this solution at $f(\phi) = a \exp(b\phi)$ (a, b are arbitrary real numbers). At $\alpha = 1, \beta = 1, \gamma = 1, \lambda = 5, \mu = 1, t_0 = 1, x_0 = 1, y_0 = 1, z_0 = 1, p_0 = 1, p_1 = 0.1, a = 1, b = -1$ we solved numerically the Cauchy problem (12), (13) by the Runge-Kutta-Fehlberg method [17] (with order of accuracy 4). The graph of the function $\phi = \phi(\xi)$ is shown in the Fig.8.

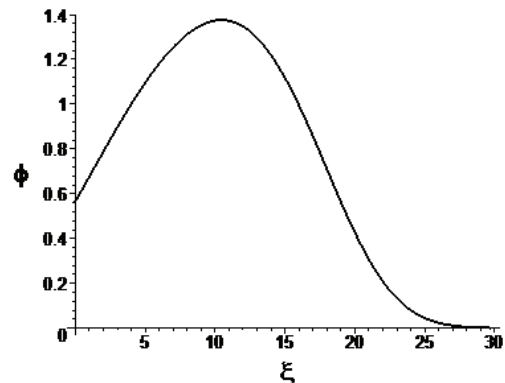


Figure 8. Pressure distributions for the model 3.2.

From this graph, it follows that the pressure increases at $\xi \in (0; 11]$ and reaches its maximum value at $\xi = 11$, then the pressure at $\xi > 11$ decreases. This means that the loading layer L is heavily contaminated and is rapidly beginning to deteriorate. The onset of destruction time is:

$$t = \sup_{\mathbf{x} \in L} (29 - x - y - z).$$

3.3. In the second example, we obtain this solution at $f(\phi) = a \ln \phi$ (a is arbitrary real number). At $\alpha = 1, \beta = 1, \gamma = 1, \lambda = 6, \mu = 1, t_0 = 1, x_0 = 1, y_0 = 1, z_0 = 1, p_0 = 1, p_1 = -0,1, a = 1, b = -1$ we solved numerically the Cauchy problem (12), (13) by the Runge-Kutta-Fehlberg method [17] (with order of accuracy 4). The graph of the function $\phi = \phi(\xi)$ is shown in the Fig. 9.

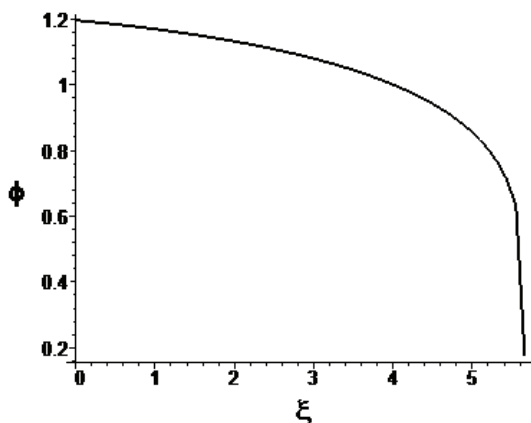


Figure 9. Pressure distributions for the model 2.2.

From this graph, it follows that the pressure corresponding to experimentally determined maximum expansion of the loading layer L is reached at $\xi = \xi_*$ ($0 < \xi_* < 5,5$). This solution has a physical meaning at $\xi \in (0; \xi_*]$. The time to reach maximum expansion of the loading layer is determined by the formula

$$t_* = \sup_{x \in L} (\xi_* - x - y - z).$$

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CONCLUSION

In this paper, we investigate the influence of externally applied traveling waves on the filtering of aqueous solutions in an expanded loading layer. To describe this process, we used a three-dimensional nonlinear dynamic generalized mathematical model of fluid motion in a porous medium with a nonlinear external influence. A general model describing traveling waves, defined by equation (3), was obtained.

Specific cases of this model with three types of filtering process nonlinearity were studied: power, exponential, and logarithmic. These nonlinear models are defined by equations (4), (8), and (11), respectively. For each of these models, three special cases were examined, for which the external influence was chosen as power, exponential, and logarithmic. The resulting nine models depend on seven arbitrary constants, which are determined empirically depending on the filtering process.

For each of these nine models, we investigated the problem of determining the pressure of an aqueous solution in a loading layer assuming that the pressure and its rate of change at a fixed point in the bed are known at the initial time. For specific values of the constants on which these models depend, this problem was solved numerically. The results are shown in Figures 1, 2, 3, 4, 5, 6, 7, 8, and 9. These solutions describe both the expansion and fouling of the bed, depending on the type of nonlinearity in the filtering process, the type of external influence, and the parameters of the traveling wave. For filtering with an expanding loading layer, we found the time at which maximum expansion is achieved. For fouled loading layers, we found the time it takes for the loading layer to collapse.

The presence of seven arbitrary constants on which the nine models studied depend allows these models to be used to solve other physically significant problems.

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