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THE MODIFIED METHOD OF SEQUENTIAL LOADS FOR THE ANALYSIS OF SLENDER SHALLOW SHELLS

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Abstract: This study discusses specific features of analyzing shallow shells using a modified sequential load method and a collocation method. This combination shows a rapid rate of convergence when the golden section principle is used to select the collocation node system.

Keywords: geometric nonlinearity, zero approximation selection, collocation method, collocation node selection, golden ratio, solution residual, critical load

Особенности применения модифицированного метода последовательных нагружений при расчете гибких пологих оболочек

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Аннотация: В статье рассматриваются особенности расчета пологих оболочек модифицированным методом последовательных нагружений с применением метода коллокаций. Показано, что наблюдается быстрая сходимость модифицированного метода последовательных нагружений с применением метода коллокаций, если использовать принцип «золотого сечения» для выбора системы узлов коллокаций.

Ключевые слова: геометрическая нелинейность, выбор нулевого приближения, метод коллокаций, выбор узлов коллокаций, золотое сечение, невязка решения, критическая нагрузка

Slender shells can be described by nonlinear differential equations. There is extensive literature on methods for solving geometrically nonlinear problems in structural mechanics. An overview of some of these methods can be found in references [1-4]. Often, there is a practical need to analyze the stress-strain state of slender structures over the entire range of load variations. However, analyzing the relationship between variable lateral load parameters and the corresponding stress-strain state of the structure, while considering the loading history, is labor-intensive. Many iterative methods are sensitive to the initial zero approximation, especially near upper and lower critical loads or bifurcation points. In these regions, the convergence region of the solution tends to a single point.

The modified method of sequential loads (MMSL), which combines the capabilities of the method of sequential loads (MSL) — convenient for studying similar and other nonlinear problems — and the Newton-Kantorovich method (NKM), which converges well when solving nonlinear problems, is sensitive to the choice of zero approximation, especially near limit and bifurcation points. Combining them expands the possibilities for implementing various calculation schemes.

The essence of the MMSL method is that, initially, a force-displacement curve should be constructed for the entire load range using the method of sequential loads with the leading parameter of load increment. This curve will then serve as a carrier of zero approximations while refining the MMSL solution at each selected loading stage. The refined results are connected by a smooth line that characterizes the structure's behavior throughout the entire range of load changes. To reduce the labor intensity of solving the problem, strategies for coordinating the alternation of MSL and MNK can be employed.

Slender shells have areas of multi-valued solutions limited by upper and lower critical loads. If a zero approximation is in an area with a small radius of convergence, it will be difficult to find a solution due to solution instability, even with a large number of iterations. Changing the load parameter helps overcome this. The displacement increment at the center of the shell is taken as the leading parameter, and the load increment becomes the desired parameter alongside the other deflection parameter increments. This change in parameters was proposed in [7]. Many authors have employed it in solving similar problems, and it has proven effective.

Some specific features of applying the modified sequential load method to the analysis of geometrically nonlinear shallow shells are discussed below. For example, consider the flexural problem of a slender, shallow shell that is hinged along its contour and loaded with a uniformly distributed load of intensity q_0, as shown in Figure 1. A rectangular system of dimensionless coordinates $\xi = x/a$, $\eta = y/b$ has been chosen, where a and b represent the shell's dimensions in plan view. The system of equations for the flexure of slender, shallow shells in the dimensionless, incremental form of the sequential load method [1] is as follows:

$$\left.\begin{aligned} &\nabla^4 \Delta\psi_n + \nabla_k^2 \Delta u_n + L\left(u_{n-1}, \Delta u_n\right) = 0 \\ &\nabla^4 \Delta u_n - 12(1-\mu^2)\left[L\left(\psi_{n-1}, \Delta u_n\right) + \nabla_k^2 \Delta\psi_n + L\left(u_{n-1}, \Delta\psi_n\right) + \Delta p_n(\xi,\eta)\right] = 0 \end{aligned}\right\} \qquad (1)$$

$$\left(n = 1, 2, 3, \ldots\right)$$

and the system of incremental equations of the Newton-Kantorovich method in dimensionless form has the form [1]

$$\left.\begin{aligned} &\nabla^4 \Delta\psi_r + \nabla_k^2 \Delta u_r + L\left(u_{r-1}, \Delta u_r\right) = -\nabla^4 \psi_{r-1} - \nabla_k^2 u_{r-1} - \frac{1}{2} L\left(u_{r-1}, u_{r-1}\right) \\ &\nabla^4 \Delta u_r - 12(1-\mu^2)\left[L\left(\psi_{r-1}, \Delta u_r\right) + L\left(u_{r-1}, \Delta\psi_r\right) + \nabla_k^2 \Delta\psi_r\right] = \\ &= -\nabla^4 u_{r-1} + 12(1-\mu^2)\left[L\left(\psi_{r-1}, u_{r-1}\right) + \nabla_k^2 \psi_{r-1} + p(\xi,\eta)\right] \end{aligned}\right\} \qquad (2)$$

$$\left(r = 1, 2, 3, \ldots\right)$$

where n is the number of sequential loading stages when using MSL, r is the iteration number when refining the MNK solution, $\Delta u_n\left(\xi,\eta\right) = \Delta W_n(x,y)/h$ is the increment of dimensionless deflection when using MSL at the n^{th} loading stage, $u_{n-1}\left(\xi,\eta\right) = W(x,y)/h$ is the total dimensionless deflection determined at previous loading stages, $\Delta\psi_n\left(\xi,\eta\right) = \Delta\varphi(x,y)/Eh^3$ is the increment of the dimensionless MSL force function, $\psi_{n-1}(\xi,\eta) = \varphi(x,y)/Eh^3$ is the dimensionless total force function determined at previous loading stages, u_{r-1}, ψ_{r-1} are the dimensionless deflection and force function obtained at previous iterations, $\Delta u_r, \Delta\psi_r$ are the increments of dimensionless deflection and force function at the n^{th} loading stage, $\Delta p_n\left(\xi,\eta\right) = \Delta q\left(x,y\right) a^2 b^2 / Eh^4$ is the increment of dimensionless lateral load $p\left(\xi,\eta\right)$, $\nabla_k^2 = K_2 \dfrac{\partial^2}{\partial\xi^2} + K_1 \dfrac{\partial^2}{\partial\eta^2}$ is the Laplace operator depending on dimensionless curvatures $K_1 = k_1 a^2/h$, $K_2 = k_2 b^2/h$, where k_1, k_2 are the principal dimensional curvatures of the shell, L is the differential operator:

$$L = \frac{\partial^2}{\partial\xi^2}\frac{\partial^2}{\partial\eta^2} + \frac{\partial^2}{\partial\eta^2}\frac{\partial^2}{\partial\xi^2} - 2\frac{\partial^2}{\partial\xi\partial\eta}\frac{\partial^2}{\partial\xi\partial\eta} \qquad (3)$$

The left sides of equations (1) and (2) are the same. The calculation begins by solving system (1) using the sequential load method (MSL). The lateral load is divided into thin layers in our minds and applied sequentially to the shell, and the calculation results are summed up. After several loading stages, the solution is refined using the Newton-Kantorovich method. In this method, the total solution obtained by MSL is assumed to be the zero approximation. By sequentially solving system (2), one can analyze the convergence of the method and make appropriate corrections.

Various approximate methods for solving linear differential equations can be used to solve the linearized systems of equations (1) and (2). Two problems need to be solved beforehand for this purpose: constructing systems of coordinate functions and setting the coordinates of collocation nodes optimally.

The coordinate functions are represented as the product of the main parts of the solution $\omega(\xi,\eta)$ and $\varphi(\xi,\eta)$, satisfying the given boundary conditions, on a system of correction functions of variables ξ and η, containing unknown parameters (generalized coordinates). In the case of shell deflection symmetrical about the coordinate axes ξ and η, we will choose the coordinate functions and their increments in the form of an incomplete polynomial with even degrees. As a result, we obtain:

$$\begin{aligned}
u_{n-1}^*(\xi,\eta) &= \omega(\xi,\eta)\sum_N K_N \xi^{2N}\eta^{2N}, \quad & \Delta u_n^*(\xi,\eta) &= \omega(\xi,\eta)\sum_N \Delta K_N \xi^{2N}\eta^{2N},\\
\psi_{n-1}^*(\xi,\eta) &= \varphi(\xi,\eta)\sum_N T_N \xi^{2N}\eta^{2N}, \quad & \Delta \psi_n^*(\xi,\eta) &= \varphi(\xi,\eta)\sum_N \Delta T_N \xi^{2N}\eta^{2N},\\
u_{r-1}^*(\xi,\eta) &= \omega(\xi,\eta)\sum_N M_N \xi^{2N}\eta^{2N}, \quad & \Delta u_r^*(\xi,\eta) &= \omega(\xi,\eta)\sum_N \Delta M_N \xi^{2N}\eta^{2N},\\
\psi_{r-1}^*(\xi,\eta) &= \varphi(\xi,\eta)\sum_N P_N \xi^{2N}\eta^{2N}, \quad & \Delta \psi_r^*(\xi,\eta) &= \varphi(\xi,\eta)\sum_N \Delta P_N \xi^{2N}\eta^{2N},
\end{aligned} \tag{4}$$

$$(N = 1,2,3,....)$$

where K_N, ΔK_N, T_N, ΔT_N, M_N, ΔM_N, P_N, ΔP_N are the generalized coordinates and their increments, N is the number of collocation nodes, and the approximate solution is marked with an asterisk (*). The number of members in the correction functions must be equal to the number of collocation nodes and determines the approximation number.

The main parts of the solution $\omega(\xi,\eta)$ and $\varphi(\xi,\eta)$, satisfying the given boundary conditions, are constructed in various ways. One of them is the first approximation obtained from solving a similar problem using some approximate method. For the hinged support of the shell along the contour, the main parts of the solution of the function $\omega(\xi,\eta)$ and $\varphi(\xi,\eta)$ are selected in the form [4]

$$\begin{aligned}
\omega(\xi,\eta) &= \cos\pi\xi\cdot\cos\pi\eta\\
\varphi(\xi,\eta) &= \cos\pi\xi\cdot\cos\pi\eta
\end{aligned}. \tag{5}$$

Substituting the constructed functions (4) into the systems of equations MMSL (1) and (2), one obtains expressions for the discrepancies in solutions of the incremental equations of systems (1) and (2)

$$\left.\begin{aligned}
&F_1(\xi,\eta,\Delta K_N,\Delta T_N) = \nabla^4\Delta\psi_n^* + \nabla_k^2\Delta u_n^* + L\left(u_{n-1},\Delta u_n^*\right)\\
&F_2(\xi,\eta,\Delta K_N,\Delta T_N) = \nabla^4\Delta u_n^* - 12(1-\mu^2)\left[L\left(\psi_{n-1},\Delta u_n^*\right)+\right.\\
&\left.+\nabla_k^2\Delta\psi_n^* + L\left(u_{n-1},\Delta\psi_n^*\right) + \Delta p(\xi,\eta)\right]
\end{aligned}\right\}$$

$$\left.\begin{aligned}
&F_3(\xi,\eta,\Delta M_N,\Delta P_N) = \nabla^4\Delta\psi_r^* + \nabla_k^2\Delta u_r^* + L\left(u_{r-1},\Delta u_r^*\right)+\\
&+\nabla^4\psi_{r-1} + \nabla_k^2 u_{r-1} + \frac{1}{2}L\left(u_{r-1},u_{r-1}\right)\\
&F_4(\xi,\eta,\Delta M_N,\Delta P_N) = \nabla^4\Delta u_r^* - 12(1-\mu^2)\left[L\left(\psi_{r-1},\Delta u_r^*\right)+\right.\\
&\left.+L\left(u_{r-1},\Delta\psi_r^*\right) + \nabla_k^2\psi_{r-1} + p(\xi,\eta) + L\left(\psi_{r-1},u_{r-1}\right)\right] + \nabla^4 u_{r-1}
\end{aligned}\right\} \tag{6}$$

According to the collocation method, set the solution discrepancies equal to zero at each collocation node. To obtain a solution with the required accuracy, one should specify a certain number of collocation nodes and their coordinates. Next, set the solution discrepancy at these nodes equate to zero, resulting in a system of linear equations with respect to the increments of the parameters of the generalized coordinate correction functions.

The selection of coordinates for the arrangement of collocation nodes faces a certain challenge. To reduce the number of collocation nodes without significant loss of accuracy, a method for selecting a system of collocation nodes has been proposed by the authors based on the use of the "golden ratio." This method has been studied in numerous calculations of linear and physically nonlinear problems [8,9] and has demonstrated rapid convergence of the solution. One variant of this method is shown in Figure 1, using the example of calculating a slender shallow shell under axisymmetric loading and boundary conditions. It is sufficient to consider one quarter of the shell plan, which in dimensionless form has dimensions 1 x 1. To determine the coordinates of the first collocation node A_1, the distance from the center to the edge of the shell is divided in the "golden ratio" proportion, and the length of the short segment (near the center) determines the coordinate η of the collocation node $A_1(0.191;0.191)$. Next, the second (long segment) is similarly divided in the "golden ratio" proportion, and we determine the coordinate of the collocation node $A_2(0.309;0.309)$. In the same way, we find the coordinates of the collocation nodes $A_3(0.382;0.382)$ and $A_4(0.427;0.427)$. If necessary, this process can be continued.

By equating the discrepancies of solution (6) to zero in the constructed collocation nodes (Figure 1), we obtain systems of linear algebraic equations by MSL

$$\left.\begin{cases} F_1(\xi_s,\eta_s,\Delta K_N,\Delta T_N)=0 \\ F_2(\xi_s,\eta_s,\Delta K_N,\Delta T_N)=0 \end{cases}\right|_{s=A_N}, \quad N=1,2,3,4 \tag{7}$$

and systems of linear algebraic equations for calculating by MNK

$$\left.\begin{cases} F_3(\xi_s,\eta_s,\Delta M_N,\Delta P_N)=0 \\ F_4(\xi_s,\eta_s,\Delta M_N,\Delta P_N)=0 \end{cases}\right|_{s=A_N}, \quad N=1,2,3,4 \tag{8}$$

where ξ_s,η_s are the coordinates of specific collocation nodes: node A_1 - in the first approximation; nodes A_1, A_2 - in the second approximation; nodes A_1, A_2, A_3 - in the third approximation; nodes A_1, A_2, A_3, A_4 - in the fourth approximation by MK. Solving the system of algebraic equations (7), one finds the coefficients $\Delta K_N,\Delta T_N,\Delta M_N,\Delta P_N$ and, substituting them into (4), obtains the desired values of the deflection increment function and the force function of the slender shallow shell. Next, one determines the necessary characteristics of the shell's internal forces.

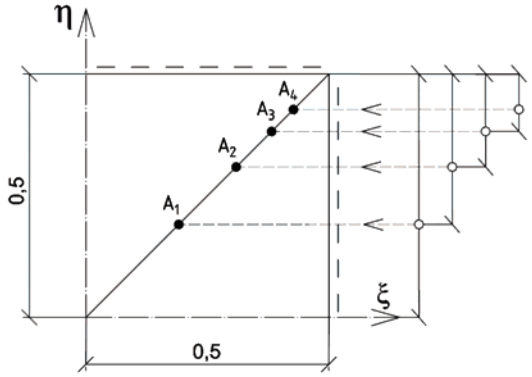

Figure 1. Scheme for selecting collocation points on a quarter of the shell plan

Figure 2 shows the load-displacement dependencies at the center of a shallow shell with dimensionless curvatures K_1=K_2=18, obtained by MMSL using the MK. The curve number corresponds to the number of the approximate solution. Curve 5 was obtained by the MBG in the fourth approximation [7].

Alternating convergence of the collocation method has been observed. The difference in the upper critical loads between the first, second, and fourth approximations of the MK and the fourth approximation of the MBG is less than 10%. The values of the second and fourth approximations of the MK are close to the lower critical load obtained by the MBG. The difference is less than 8%. The curve obtained in the second approximation of MK almost merges with the dependence obtained by MBG in the fourth approximation. This allows us to conclude that the second approximation of MK is sufficient to determine the upper critical load for the shell under consideration.

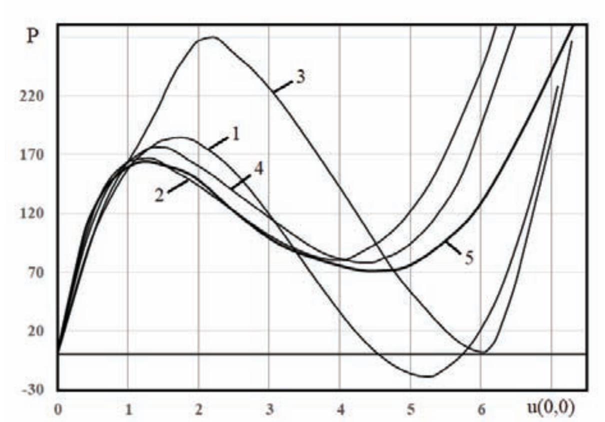

Figure 2. Load-displacement relationships at the center of a shallow shell in different MK approximations

Note that such rapid convergence of the solution is directly dependent on the rational selection of collocation nodes using the "golden ratio" proportion.

The "golden ratio" is a rule of proportion: "the smaller part relates to the larger part as the larger part relates to the whole." It is also called "divine harmony." This concept was first used by Pythagoras and has accompanied human civilization throughout its development. It is a marker of beauty and harmony. Works of art have been created according to the canons of this proportion. This proportion is present everywhere: in art, in nature, in spiral galaxies, and in humans themselves. The golden ratio is a universal numerical constant, perfect for the development of all objects, systems, and processes, that is, with minimum potential energy. According to Euclid of Alexandria, at the points of the "golden ratio," there is a balance between the whole and its parts, as well as between the parts themselves, and the increase in potential energy is zero. The "golden ratio" ensures the development of living beings and all processes in the most energy-efficient way. It is believed that in nature it is the basis for development. Extensive literature is devoted to the study of these issues. We recommend that those interested in these issues refer, for example, to monograph [10], which contains an extensive bibliography.

Figure 3 (to the left of the axis line) shows the graphs of the solution discrepancies in the section $\eta = 0$, obtained by the MMSL at a load of P=120 and in the upper critical load region (to the right of the axis line). The curve numbers coincide with the approximation numbers obtained by the MK.

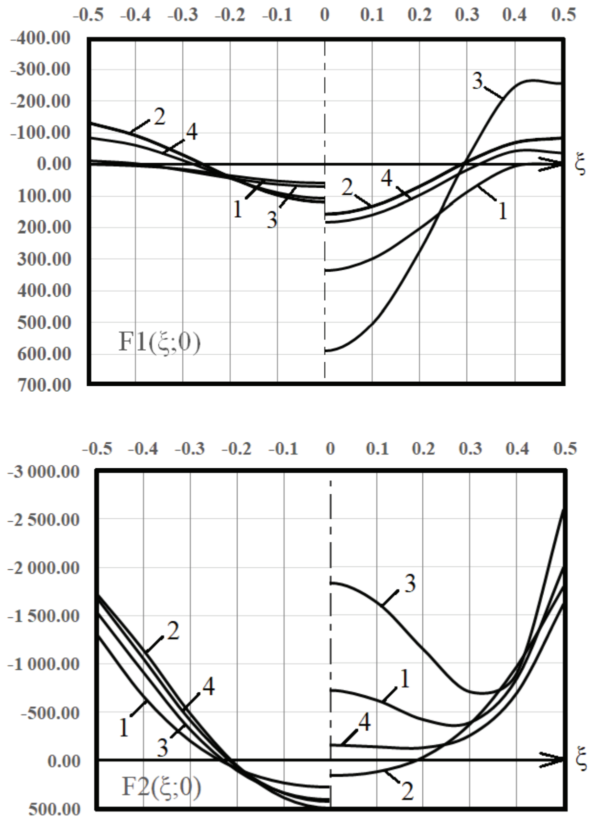

Figure 3. Changes in the discrepancies of solutions F_1 and F_2 with increasing load

Analysis of the results shown in Figure 3 demonstrates the convergence of solution discrepancies at the center of the shell is alternating in nature. The solution discrepancies obtained in the vicinity of the upper critical load are significantly smaller than the solution discrepancies at a load of P=120.

Figures 4-7 show the change in the surfaces of the solution discrepancies of the continuity equation F_1 and the equilibrium equation F_2 at maximum load in the first and fourth approximations of the collocation method.

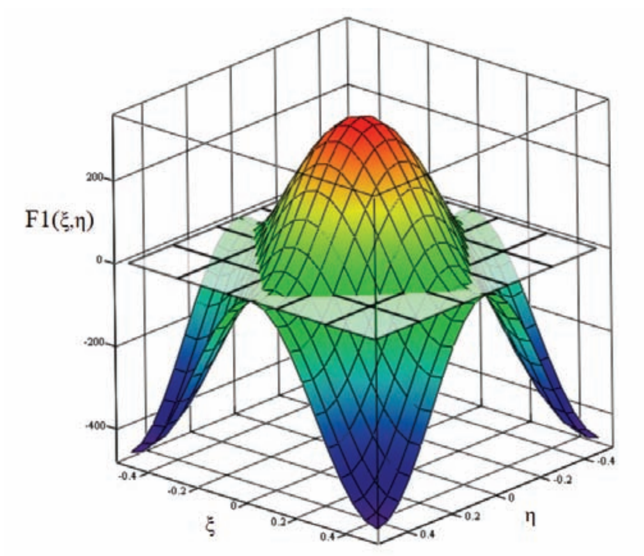

Figure 4. Surface of the solution F_1 discrepancy (1st approximation by MK)

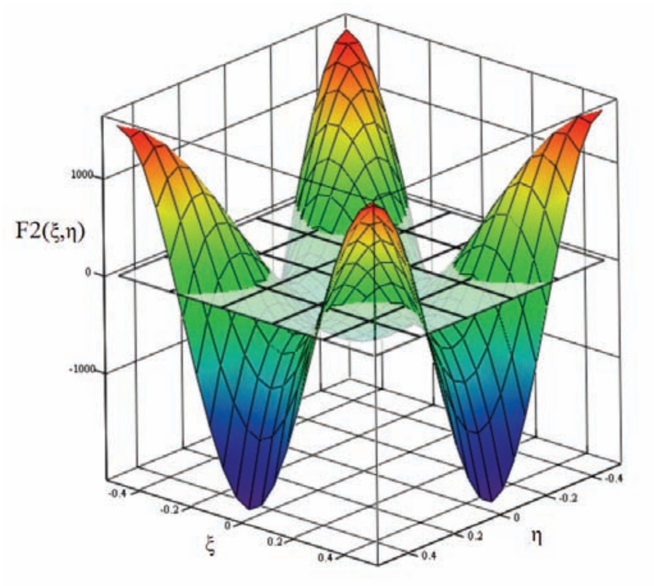

Figure 5. Surface of the solution F_2 discrepancy (1st approximation by MK)

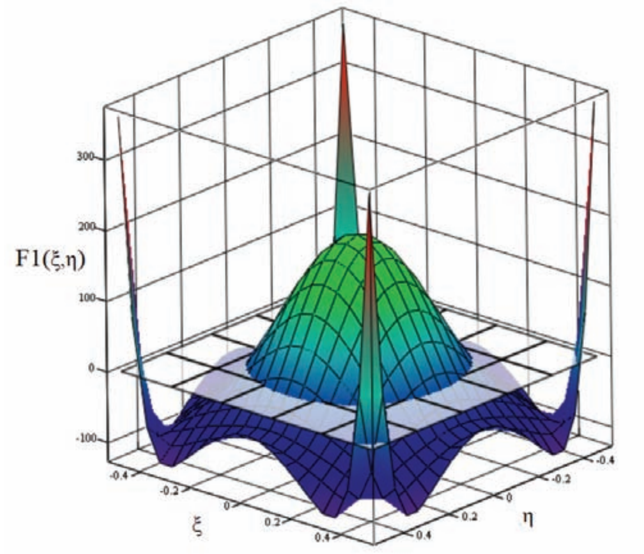

Figure 6. Surface of the solution F_1 discrepancy (4th approximation by MK)

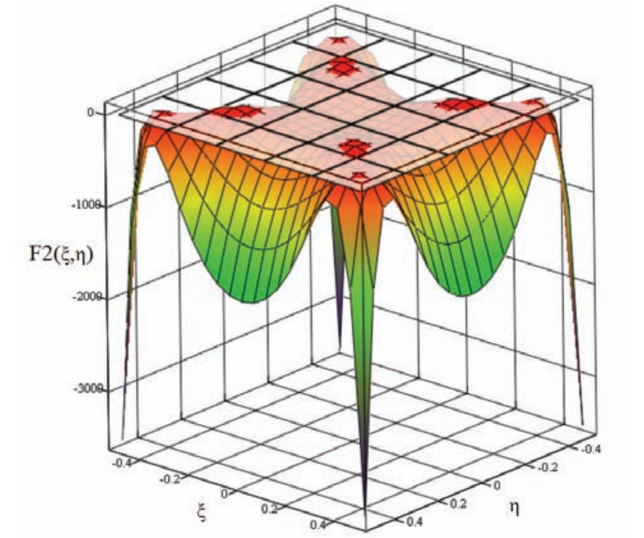

Figure 7. Surface of the solution F_2 discrepancy (4th approximation by MK)

These figures show that maximum changes in solution discrepancy values occur near the corner points of the contour and in the central region of the shell. Increasing the number of collocation nodes significantly decreases the values of solution discrepancies in these regions. The corner points of the shell are "special points" where a volumetric stress state arises that is not described by the equations of the shallow shell model adopted in the analysis. Therefore, these areas should be excluded from the analysis.

The following conclusion can be drawn: The modified method of sequential loads combined with the collocation method can obtain an engineering-sufficient solution for geometrically nonlinear shallow shells in the second approximation if the collocation node selection scheme is based on the "golden ratio." This allows the final solution to be presented in a form that facilitates calculations.

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