

AXISIMMETRIC FLUID MOTION IN A POROUS MEDIUM IN THE PRESENCE OF A NON-STATIONARY EXTERNAL SOURCE OR ABSORPTION

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Abstract: The generalized axisymmetric model of fluid motion in a porous medium in the presence of a non-stationary external source or absorption is studied by methods of group (symmetry) analysis of differential equations. All its invariant submodels of rank 1 are studied. They are specified by invariant solutions of rank 1 of the equation of the original model. These solutions are obtained either explicitly, or their search is reduced to solving systems of ordinary differential equations of the first order. For explicit solutions at specific values of the parameters included in their expressions, graphs of the pressure distribution in the porous medium are constructed. The remaining solutions are used to study physically meaningful boundary value problems for which, at the initial moment of time, the pressure and either the rate of its change along the axis of symmetry or the radial rate of its change are specified at a fixed point of the medium. These boundary value problems are solved numerically for some specific values of the parameters included in them. Graphs of the functions determining these solutions are obtained. The conducted research is relevant in many areas of applied science and technology: filtration, soil mechanics, rock mechanics, oil field engineering, construction engineering, petroleum geology, biology and biophysics, materials science.

Keywords: axisymmetric model of fluid motion, porous medium, non-stationary external source, non-stationary external absorption, symmetry analysis, invariant solutions, invariant submodels

ОСЕСИММЕТРИЧНОЕ ДВИЖЕНИЕ ЖИДКОСТИ В ПОРИСТОЙ СРЕДЕ ПРИ НАЛИЧИИ НЕСТАЦИОНАРНОГО ВНЕШНЕГО ИСТОЧНИКА ИЛИ ПОГЛОЩЕНИЯ

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Аннотация: Методами группового (симметричного) анализа дифференциальных уравнений изучается обобщенная осесимметричная модель движения жидкости в пористой среде при наличии нестационарного внешнего источника или поглощения. Изучаются все ее инвариантные подмодели ранга 1. Они задаются инвариантными решениями ранга 1 уравнения исходной модели. Эти решения либо получаются в явном виде, либо их поиск сводится к решению систем обыкновенных дифференциальных уравнений первого порядка. Для явных решений при конкретных значениях входящих в их выражения параметров строятся графики распределения давления в пористой среде. Оставшиеся решения используются для исследования физически содержательных краевых задач, для которых в начальный момент времени в фиксированной точке среды задаются давление и либо скорость его изменения вдоль оси симметрии, либо радиальная скорость его изменения. Эти краевые задачи решаются численно при некоторых конкретных значениях входящих в них параметров. Получены графики функций, определяющих эти решения. Проведенные исследования актуальны во многих областях прикладной науки и техники: фильтрация, механика грунтов, механика горных пород, нефтепромысловое дело, строительство, нефтяная геология, биология и биофизика, материаловедение.

Ключевые слова: осесимметричная модель движения жидкости, пористая среда, нестационарный внешний источник, нестационарное внешнее поглощение, симметричный анализ, инвариантные решения, инвариантные подмодели

INTRODUCTION

The classical model of fluid motion in a porous medium has been studied in many works (see, for example, [1 – 7] and the large bibliography provided there). However, the study of this fluid motion within the framework of the classical model does not always adequately describe real processes. First of all, this applies to processes with an external non-stationary source or absorption.

The study of rank 0 invariant submodels for a general three-dimensional model of fluid motion in a porous medium in the presence of an external non-stationary source or absorption was started in [8, 9].

In [10], rank 1 invariant submodels were studied for a general two-dimensional model of fluid motion in a porous medium in the presence of an external non-stationary source or absorption.

In our work we will study an axisymmetric model describing the motion of a fluid in a porous medium in the presence of an external non-stationary source or absorption. This model is obtained from the classical model by adding a term describing the presence of an external source or absorption. It is specified by the following partial differential equation

$$\partial_t q = \left(\partial_r^2 + \frac{1}{r} \partial_r + \partial_z^2 \right) (q^\lambda) + g(t)q, \quad (1)$$

where $q = q(t, \mathbf{x})$ is a pressure, t is a time, $\mathbf{x} = (x, y, z) \in R^3$; $r = \sqrt{x^2 + y^2}$; λ ($\lambda(\lambda - 1) \neq 0$) is arbitrary real number, characterizing the nonlinearity of the process, a function $g(t)$ defines a non-stationary external source or absorption. For each real process, this function is specified empirically. The case

$g(t) > 0$ corresponds to the presence of an external source. The case $g(t) < 0$ corresponds to the presence of external absorption. Let

$$g(t) = \frac{(\ln f'(t))'}{\lambda - 1},$$

where $f(t) \neq k_1 \exp t + k_2$ (k_1, k_2 are arbitrary real numbers). The prime denotes, as everywhere below, the derivative of a function with respect to its argument.

Each solution of equation (1) defines a certain submodel of the general model. Our goal is to study all invariant submodels of rank 1.

METHODS

The main method of studying equation (1) is group (symmetry) analysis of differential equations. Group analysis is one of the most effective ways of obtaining maximum information about the properties of solutions of differential equations. The basic concepts and algorithms of modern group analysis of differential equations can be found, for example, in [11–15] and the references provided there.

RESULTS AND DISCUSSION

The following mathematical results were obtained by group (symmetry) analysis methods:

1. The main Lie group of transformations of differential equation (1) was found.
2. Using the obtained group of internal automorphisms, optimal systems of subgroups of the main group were constructed.

3. For each subgroup of these optimal systems, its universal invariant in the space $R^4(t, r, z, u)$ was found. This allows us to write down a representation of each invariant solution of equation (1).

For the convenience of the reader, we omit all the indicated calculations.

Using all the obtained representations of invariant solutions of rank 1, we investigate these solutions. These solutions define 4 submodels of the main model.

The first submodel

This submodel is defined by a solution having the representation

$$p = \left(z^{2-\alpha} f'(t) \right)^{\frac{1}{\lambda-1}} U(\xi), \quad \xi = \frac{r}{z}, \quad (2)$$

where α is arbitrary real number.

After substitution (2) into (1) we obtain the following equation for the function $U = U(\xi)$:

$$\begin{aligned} (\xi^2 + 1)(U^\lambda)'' + \left(\frac{1}{\xi} + 2\beta\xi \right) (U^\lambda)' + \\ + \lambda\beta(\beta + 1)U^\lambda = 0, \quad \beta = \frac{\lambda(\alpha-2)}{\lambda-1}. \end{aligned} \quad (3)$$

1. At $\alpha = 2$ the solution of this equation has a form

$$U = \left(c_1 + c_2 \left(\sqrt{\xi^2 + 1} + \ln \frac{\xi}{1 + \sqrt{\xi^2 + 1}} \right) \right)^\lambda, \quad (4)$$

where c_1, c_2 are arbitrary real numbers.

From (2) and (4) it follows that the pressure is determined by the formula

$$p = \left(f'(t) \right)^{\frac{1}{\lambda-1}} \left(c_1 + c_2 \left(\frac{\sqrt{r^2 + z^2}}{z} + \ln \frac{r}{z + \sqrt{r^2 + z^2}} \right) \right)^\lambda. \quad (5)$$

At each fixed moment in time, the pressure is the same at all points on each conical surface $r = cz$ ($c = \text{const} > 0$).

When $c_1 = 0, c_2 = 1, \lambda = 2, f(t) = 2t^2 + 2t + 1, r = 2, z > 0$ the pressure distribution is shown in Fig. 1.

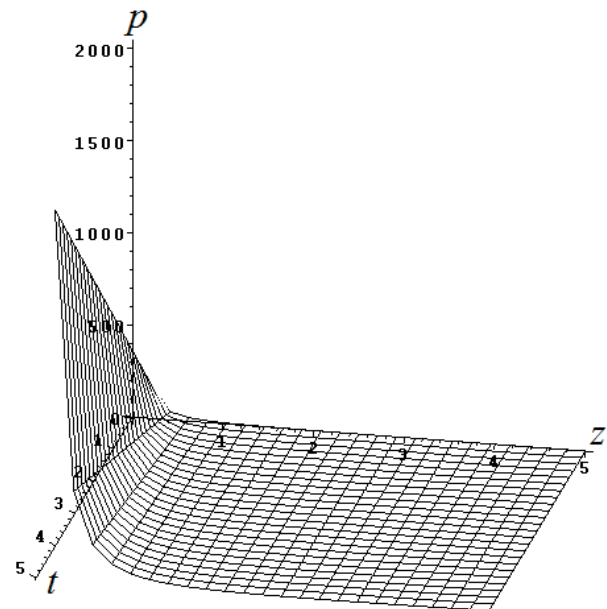


Figure 1. Pressure distributions at $r = 2$

In this case, the pressure increases monotonically over time under the influence of an external source and tends to infinity at $t \rightarrow \infty$.

2. At $\alpha \neq 2$ the solution of equation (3) is equivalent to the following system

$$\begin{aligned} W' &= \frac{1}{\xi^2 + 1} \left(V - \left(\frac{1}{\xi} + (2\beta - 1)\xi \right) W \right), \\ V' &= - \left(\beta^2 + 3\beta - 2 + \frac{1}{\xi} \right) W. \end{aligned} \quad (6)$$

Namely: 1) for any solution $(W(\xi), V(\xi))$ of system (6) the function $U = W^{\frac{1}{\lambda}}$ satisfies to equation (3), 2) conversely, for any solution $U(\xi)$ of equation (3) pair of the functions

$$\begin{aligned} W(\xi) &= U^\lambda(\xi), \\ V(\xi) &= \lambda(\xi^2 + 1)U^{\lambda-1}(\xi)U'(\xi) + \\ &+ \left(\frac{1}{\xi} + 2(\beta-1)\xi\right)U^\lambda(\xi) \end{aligned}$$

satisfies to system (6).

Let us find the pressure distribution if at the initial moment of time $t = t_0 \geq 0$ at a fixed point (r_0, z_0) the pressure and its radial derivative are given

$$\begin{aligned} p(t_0, r_0, z_0) &= p_0 > 0, \quad p_r(t_0, r_0, z_0) = p_1, \\ r_0 > 0, \quad z_0 > 0, \end{aligned} \tag{7}$$

where $p_0 > 0$, p_1 are arbitrary real numbers.

In this case the initial data for system (6) have the form

$$\begin{aligned} W(\xi_0) &= \left(z_0^{2-\alpha} f'(t_0)\right)^\mu p_0^\lambda, \quad \mu = \frac{\lambda}{1-\lambda}, \\ V(\xi_0) &= z_0^{\beta-1} \left(f'(t_0)\right)^\mu p_0^{\lambda-1} \cdot \\ &\left(\lambda(r_0^2 + z_0^2)p_1 + \frac{1}{r_0}(2(\beta-1)r_0^2 + z_0^2)p_0\right), \\ \xi_0 &= \frac{r_0}{z_0}. \end{aligned} \tag{8}$$

Due to the smoothness of the right-hand sides of system (6), the solution to the Cauchy problem (6), (8) exists and is unique in the neighborhood of the point ξ_0 .

According to the formula

$$p = \left(z^{2-\alpha} f'(t)\right)^{\frac{1}{1-\lambda}} W^{\frac{1}{\lambda}}(\xi), \quad \xi = \frac{r}{z}. \tag{9}$$

we obtain a unique solution of equation (1) satisfying to conditions (7), for which the value

$\left(z^{\alpha-2} h(t)\right)^{\frac{1}{1-\lambda}} p(t, r, z)$ is constant along each conical surface $r = cz$ ($c = const > 0$).

For example, when $t_0 = 0$, $\lambda = \frac{3}{2}$, $\alpha = 1$,

$r_0 = 1$, $z_0 = 1$, $p_0 = 16$, $p_1 = -1$, $z > 0$,

$f(t) = t^2 + t + 1$ for the Cauchy problem (6), (8) the numerical solution by the Runge-Kutta-Felberg method [16] (with an order of accuracy of 4) is shown in Fig. 2.

From this graph, by virtue of formula (9), it follows that the pressure increases monotonically over time under the influence of an external source and tends to infinity at $t \rightarrow \infty$.

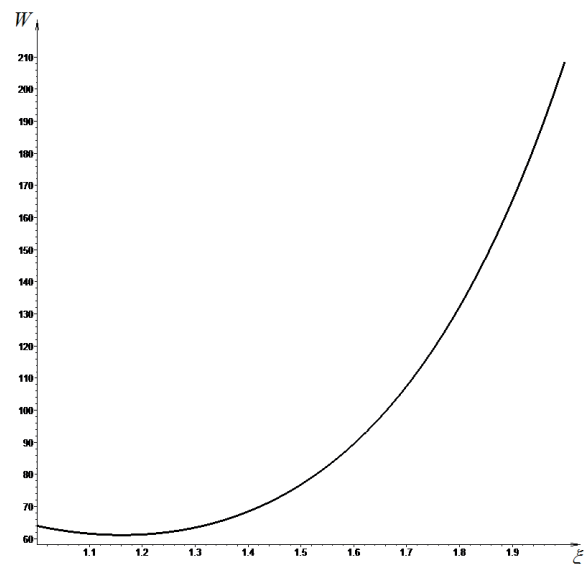


Figure 2. Graph of the function $W(\xi)$

From this graph, by virtue of formula (9), it follows that the pressure increases

monotonically over time under the influence of an external source and tends to infinity at $t \rightarrow \infty$.

The second submodel

This submodel is defined by a solution having the representation

$$p = (f'(t))^{\frac{1}{\lambda-1}} \exp\left(\frac{z}{1-\lambda}\right) U(r). \quad (10)$$

Substituting (10) into (1) yields the equation

$$(U^\lambda)'' + \frac{1}{r}(U^\lambda)' + \left(\frac{\lambda}{\lambda-1}\right)^2 U^\lambda = 0$$

whose general solution is

$$U = \left(c_3 J_0\left(\left|\frac{\lambda}{\lambda-1}\right|r\right) + c_4 Y_0\left(\left|\frac{\lambda}{\lambda-1}\right|r\right) \right)^{\frac{1}{\lambda}}$$

where $J_0(x)$ is the Bessel function of the first kind of the zero order, and $Y_0(x)$ is the Bessel function of the second kind of the zero order, c_3, c_4 are arbitrary real numbers.

The pressure is determined by the formula

$$p = (f'(t))^{\frac{1}{\lambda-1}} \exp\left(\frac{z}{1-\lambda}\right) \cdot \left(c_3 J_0\left(\left|\frac{\lambda}{\lambda-1}\right|r\right) + c_4 Y_0\left(\left|\frac{\lambda}{\lambda-1}\right|r\right) \right)^{\frac{1}{\lambda}}. \quad (11)$$

On each cylindrical surface $r = const > 0$ the pressure is distributed according to the formula

$$p = c (f'(t))^{\frac{1}{\lambda-1}} \exp\left(\frac{z}{1-\lambda}\right).$$

For example, at $c_3 = 1, c_4 = 0, \lambda = \frac{1}{4}, r = 2,$

$z > 0, f(t) = (t+1)^2$ the pressure distribution is shown in Fig. 3.

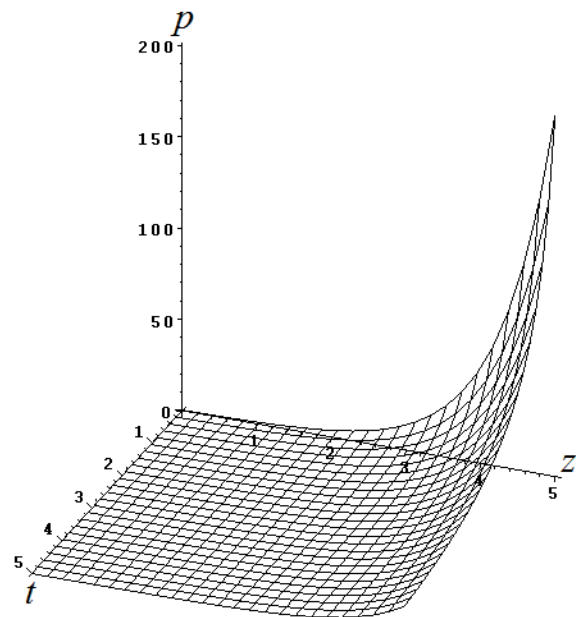


Figure 3. Pressure distribution on a cylindrical surface $r = 2$

In this case, the pressure on the cylindrical surface $r = 2$ decreases monotonically over time under the influence of external absorption and tends to zero at $t \rightarrow \infty$.

The third submodel

This submodel is defined by a solution having the representation

$$p = (rf'(t))^{\frac{1}{\lambda-1}} U(\xi), \quad \xi = \frac{f(t)-z}{r}. \quad (12)$$

Substituting (12) into (1) yields the equation

$$\begin{aligned} & (\xi^2 + 1)(U^\lambda)'' - \frac{\lambda + 1}{\lambda - 1} \xi (U^\lambda)' - U' + \\ & + \left(\frac{\lambda}{\lambda - 1} \right)^2 U^\lambda = 0. \end{aligned} \tag{13}$$

Equation (13) is equivalent to the system

$$\begin{aligned} U' &= U^{1-\lambda} W, \\ W' &= \frac{1}{\xi^2 + 1} \left(\frac{1}{\lambda} U^{1-\lambda} W - \frac{\lambda}{(\lambda - 1)^2} U^\lambda + \right. \\ & \left. + \frac{\lambda + 1}{\lambda - 1} \xi W \right). \end{aligned} \tag{14}$$

Let the pressure and its rate of change along the axis Oz are given at the initial moment of time $t = t_0 \geq 0$ at a fixed point with coordinates $r = r_0 > 0, z = z_0$:

$$p(t_0, r_0, z_0) = p_0 > 0, \quad p_z(t_0, r_0, z_0) = p_1, \tag{15}$$

where $p_0 > 0, p_2$ are arbitrary real numbers.

The initial data for system (14) have the form

$$\begin{aligned} U(\xi_0) &= (r_0 f'(t_0))^{1-\lambda} p_0, \\ W(\xi_0) &= r_0^{1-\lambda} (f'(t_0))^{1-\lambda} p_0^{1-\lambda} p_2, \\ \xi_0 &= \frac{f(t_0) - z_0}{r_0}. \end{aligned} \tag{16}$$

Due to the smoothness of the right-hand sides of system (14), the solution of the Cauchy problem (14), (16) exists and is unique in the neighborhood of the point ξ_0 .

Applying formula (12), we obtain a unique solution to equation (1) that satisfies to the conditions (15), for which the value $(rf'(t))^{1-\lambda} p(t, r, z)$ is constant along each Trajectory $z = f(t) - cr$ ($c = const$).

For example, at $t_0 = 0, \beta = 2, r_0 = 1, z_0 = 1, p_0 = 4, p_2 = 1, z > 0$ the Cauchy problem (14), (16) is solved numerically by the Runge-Kutta-Felberg method [16] (with an order of accuracy of 4). The graph of the function $U(\xi)$ is shown in Fig. 4.

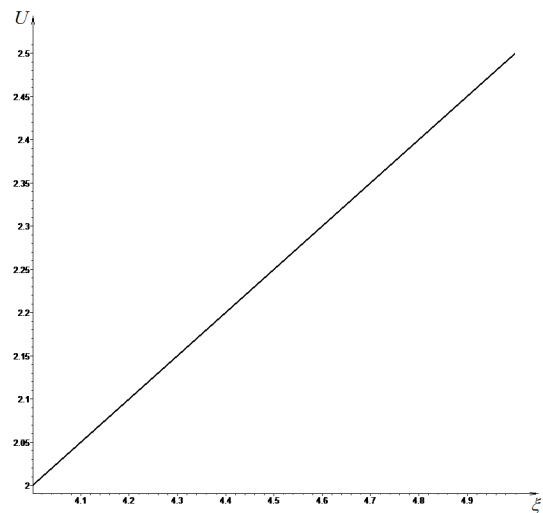


Figure 4. Graph of the function $U(\xi)$

From this graph, by virtue of formula (12), it follows that the pressure increases monotonically over time under the influence of an external source and tends to infinity at $t \rightarrow \infty$.

The fourth submodel

This submodel is defined by a solution having the representation

$$p = \left(\frac{f'(t)}{f(t)} z^2 \right)^{\frac{1}{\lambda-1}} U(\xi), \quad \xi = \frac{r}{z}. \tag{17}$$

After substitution (17) into (1) we obtain the following equation for the function $U = U(\xi)$:

$$(\xi^2 + 1)(U^\lambda)'' + \left(\frac{1}{\xi} - \frac{2(2\lambda - 1)}{\lambda - 1}\xi\right)(U^\lambda)' - \frac{\lambda^2(3\lambda - 13)}{\lambda - 1}U^\lambda - \frac{1}{(\lambda - 1)}U = 0.$$

This equation is equivalent to the system

$$U' = U^{1-\lambda}W, \\ W' = \frac{1}{\xi^2 + 1} \left(\frac{1}{\lambda}U^{1-\lambda}W - \frac{\lambda}{(\lambda - 1)^2}U^\lambda + \frac{\lambda + 1}{\lambda - 1}\xi W \right). \quad (18)$$

If at the initial moment of time $t = t_0 \geq 0$ at a fixed point with coordinates $r = r_0 > 0$, $z = z_0 > 0$ the pressure and its radial derivative are given by formulas (7), then the initial data for system (18) have the form

$$U(\xi_0) = \left(z_0^2 \frac{f'(t_0)}{f(t_0)} \right)^{\frac{1}{1-\lambda}} p_0, \quad \xi_0 = \frac{r_0}{z_0}, \quad (19)$$

$$W(\xi_0) = z_0^{\frac{\lambda+1}{1-\lambda}} \left(\frac{f'(t_0)}{f(t_0)} \right)^{\frac{\lambda}{1-\lambda}} p_0^{\lambda-1} p_1.$$

Due to the smoothness of the right-hand sides of system (18), the solution of the Cauchy problem (18), (19) exists and is unique in the neighborhood of the point ξ_0 .

Applying formula (17), we obtain a unique solution of equation (1) satisfying to conditions (7), for which the value

$$\left(\frac{f'(t)}{f(t)} z^2 \right)^{\frac{1}{1-\lambda}} p(t, r, z)$$

is constant along each conical surface $r = cz$ ($c = \text{const} > 0$). For example, when $t_0 = 0$, $\alpha = 1.8$, $r_0 = 1$, $z > 0$ for the Cauchy problem (18), (19) the numerical solution by the Runge-Kutta-Felberg method [16] (with an order of accuracy of 4) is shown in Fig. 5.

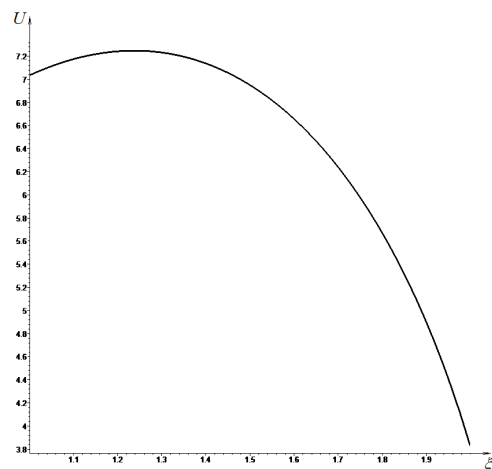


Figure 5. Graph of the function $U(\xi)$

From this graph, by virtue of formula (17), it follows that the pressure decreases monotonically over time under the influence of external absorption and tends to zero at $t \rightarrow \infty$.

CONCLUSION

By the methods of group (symmetry) analysis of differential equations we used to study the generalized axisymmetric model of liquid or gas motion in a porous medium in the presence of a non-stationary external source or absorption, defined by equation (1).

All invariant submodels of rank 1 were obtained. They are defined by five invariant solutions of rank 1 of equation (1). These are solutions (5) and (11), found in explicit form, and solutions (9), (12) and (17), the search for which is reduced to systems of ordinary

differential equations of the first order (6), (14) and (18), respectively.

For solutions (5) and (11) for some specific values of the parameters included in them, pressure distribution graphs are constructed. These graphs are shown in Figures 1 and 2. The physical meaning of these solutions is indicated. Solutions (9), (12) and (17) were used to study physically meaningful problems for which at the initial moment of time at a fixed point of the medium the pressure and either the radial rate of its change or the rate of its change along the axis of symmetry are specified. These problems are solved numerically for some specific values of the parameters included in them. Graphs of the functions determining these solutions are obtained. These graphs are shown in Figures 3, 4 and 5.

The significance of the solutions found is as follows:

1. They describe specific physical processes.
2. These solutions can be used as tests in numerical calculations when studying the motion of liquid or gas in a porous medium in the presence of a non-stationary external source or absorption.
3. The solutions found in the work depend on 5 arbitrary numerical parameters and one arbitrary function, which are determined empirically depending on the physical process under study. This allows using the submodels determined by these solutions to study other physically meaningful problems different from those considered in this work.

ACKNOWLEDGEMENTS

The research was carried out with the financial support of the Industry Consortium “Construction and Architecture”.

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