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NONLOCAL IN TIME DYNAMIC DEFORMATION MODEL AND ITS CALIBRATION BASED ON THE BEAM VIBRATION EXPERIMENT RESULTS

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Abstract: The article is devoted to the development of a nonlocal in time model of a material dynamic deformation and its calibration on the basis of the beam vibration experiment results. The model is based on the defining relations of nonlocal mechanics. Elastic forces in a system are considered dependent on the entire history of its deformation, and not only on the instantaneous deformed state under consideration. The nonlocal in time model of a dynamic deformation is proposed as an alternative to detailed three-dimensional models when modeling the dynamic behavior of elements and systems made of materials characterized by an inhomogeneous structure or anisotropic properties. The model is incorporated into the FEA algorithm to make it usable for applied engineering problems. As a numerical example, the oscillation of a bending beam is considered. A method for the identification of the governing parameter of a nonlocal in time model based on the least squares method is being implemented based on the results of laboratory tests of bent beams for free vibrations.

Keywords: Nonlocal mechanics, nonlocal damping, numerical simulation, finite element method, beam vibrations

НЕЛОКАЛЬНАЯ ВО ВРЕМЕНИ МОДЕЛЬ ДИНАМИЧЕСКОГО ДЕФОРМИРОВАНИЯ МАТЕРИАЛА И ЕЁ КАЛИБРОВКА ПО РЕЗУЛЬТАТАМ ДИНАМИЧЕСКИХ ИСПЫТАНИЙ ИЗГИБАЕМЫХ БАЛОК

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Аннотация: статья посвящена разработке нелокальной во времени модели динамического деформирования материала и её калибровке по результатам лабораторных испытаний изгибаемых балок. Модель построена на основании определяющих соотношений нелокальной механики. Упругие силы в системе считаются зависящими от всей истории её деформирования, а не только от рассматриваемого мгновенного деформированного состояния. Нелокальная во времени модель динамического деформирования предлагается в качестве альтернативы подробным трёхмерным моделям при моделировании динамического поведения элементов и систем, выполненных из материалов, характеризующихся неоднородной структурой или обладающих анизотропными свойствами. Модель интегрирована в алгоритм метода конечных элементов, чтобы сделать её применимой для решения прикладных инженерных задач. В качестве численного примера рассматривается задача о колебаниях изгибаемой балки. Методика определения управляющего параметра нелокальной во времени модели, основанная на методе наименьших квадратов, реализуется с использованием лабораторных испытаний изгибаемых балок на свободные колебания.

Ключевые слова: Нелокальная механика, нелокальное демпфирование, численное моделирование, метод конечных элементов, колебания балок

INTRODUCTION

Detailed three-dimensional numerical models are often used for the analysis of systems and structures made of composite materials. Such an approach makes it possible to accurately describe material structure and properties. Such models provide high-quality results and are widely used in the calculations of critical structures. However, they have a significant disadvantage — high computational complexity, which limits their use in multivariate calculations, optimization, and in solving problems that require modeling of large systems or their behavior over the long time intervals.

In this regard, development of the alternative less resource-intensive models that are equivalent to detailed three-dimensional models in terms of the adequacy of describing the dynamic behavior of composite elements becomes urgent.

The reduce of three-dimensional models of composite materials that take into account their internal structure or their anisotropy to lower-dimensional ones is the problem which does not have the unique solution [1]. The criterion for the chosen approach effectiveness can be the correspondence of the obtained results to experimental data in a certain class of problems.

One possible approach to this problem involves constructing lower-dimensional models based on specific hypotheses, particularly nonlocal ones [2].

Among nonlocal models, it is worth mentioning the spatially nonlocal model of the elastic properties of materials proposed by Eringen [3]. These studies were further developed and applied e.g. in [4] to the analysis of Euler–Bernoulli beams made of porous viscoelastic materials. Another nonlocal in space model was described in the book [5] (Kunin chains) and it was noted that this model can be used as an approximate description of a homogeneous three-dimensional medium. Nonlocal in space model of internal damping was proposed by Russel [6]. The model was applied to the problems of dynamics and stability of rods and shells by V.D. Potapov [7,8]. His approach was further studied and expanded in research implemented by the authors [9, 10]. The nonlocal in time model of dynamic deformation of the material presented in [10] shares common features with Boltzmann-Volterra hereditary model [11].

NONLOCAL IN TIME MODEL OF THE MATERIAL DEFORMATION

In the most general case, the nonlocal formulation of the dynamics of solids problem assumes that elastic and damping forces at a certain point of the body depend not only on the local values of strains and strain rates in time and space, respectively, but also on the values of strains and strain rates in a certain area adjacent to the point under consideration throughout the deformation history. The influence of points in space on each other diminishes with increasing distance, and the impact of previous deformation history on the current state of the system decreases over time.

In the classical local formulation, the defining relations for viscoelastic material, as described by the Kelvin-Voigt model, take the following form:

$$\sigma^{ij} = C^{ijkl}\varepsilon_{kl} + X^{ijkl}\frac{\partial \varepsilon_{kl}}{\partial t}, \tag{1}$$

where C^{ijkl} is elastic moduli tensor, $X^{ijkl} = C^{ijkl}t_e$ is viscosity moduli tensor, t_e is the retardation time, σ^{ij} is stress tensor, ε_{kl} is strain tensor.

The Cauchy relation:

$$\varepsilon_{ij} = \frac{1}{2}\left(\nabla_i u_j + \nabla_j u_i\right), \tag{2}$$

Where ∇_i is the differential operator and u_i is the components of displacements.

When transitioning to a nonlocal formulation, relation (1) takes the following form. [2]:

$$\begin{aligned}\sigma^{ij} &= C^{ijkl}\int\limits_{\Omega}\int\limits_{-\infty}^{t} K(\bar{r},\bar{\theta})\,R(t,\tau)\varepsilon_{kl}(\theta,\tau)\,d\tau\, d\theta \\ &+ X^{ijkl}\int\limits_{\Omega}\int\limits_{-\infty}^{t} D(\bar{r},\bar{\theta})\,G(t,\tau)\dot{\varepsilon}_{kl}(\theta,\tau)\,d\tau\, d\theta\,.\end{aligned} \tag{3}$$

Here, K and D are spatial kernel functions of elasticity and damping operators respectively; R

and G are temporal kernel functions of elasticity and damping operators respectively; $\bar{r}$ and $\bar{\theta}$ are spatial coordinate vectors; t and τ represent time; and Ω denotes the neighborhood of the point under consideration. The kernel functions are considered invariant with respect to shifts in time and space, thus:

$$\sigma^{ij} = C^{ijkl}\int_{\Omega}\int_{-\infty}^{t} K\left(\left|\bar{r}-\bar{\theta}\right|\right) R(t-\tau)\varepsilon_{kl}(\theta,\tau)d\tau d\theta + X^{ijkl}\int_{\Omega}\int_{-\infty}^{t} D\left(\left|\bar{r}-\bar{\theta}\right|\right) G(t-\tau)\dot{\varepsilon}_{kl}(\theta,\tau)d\tau d\theta. \tag{4}$$

In (4), the kernel functions are considered isotropic, meaning that the influence of different points in time and space on each other decreases at the same rate in all directions. In the most general case, the kernels can be considered anisotropic, i.e. they have different rates of influence decay in different directions. Taking into account all four types of nonlocality simultaneously when solving dynamic problems makes sense only if it reflects the real features of the material's dynamic behavior. To construct a phenomenological description of the dynamics of elements made of composite materials, various special cases of expression (4) are used, which allows to achieve sufficient computational accuracy, but do not overload the model.

By using the Dirac δ-functions as kernels, specific cases of nonlocal models can be obtained from the most general nonlocal stress-strain relationship (4).

In case of representing all four kernels as the δ-functions, we obtain:

$$\sigma^{ij} = C^{ijkl}\int_{\Omega}\int_{-\infty}^{t} \delta\left(\left|\bar{r}-\bar{\theta}\right|\right) \delta(t-\tau)\varepsilon_{kl}(\theta,\tau)d\tau d\theta + X^{ijkl}\int\int^{t} \delta\left(\left|\bar{r}-\bar{\theta}\right|\right) \delta(t-\tau)\dot{\varepsilon}_{kl}(\theta,\tau)d\tau d\theta \tag{5}$$

It is obvious that expression (5) is equivalent to the local classical viscoelastic model (1), which is thus a special case of a nonlocal model.

By various combinations of δ-functions with nonlocal kernels it is possible to derive from (4) various types of non-local models discussed in the literature, such as nonlocal in space model of elasticity (similar to Eringen model [3]), a nonlocal in space damping model [7,8], a nonlocal in time damping model [10].

The model presented in this paper is based on the assumption that elastic forces in a system depend not only on its current deformed state, but also on the previous deformation history. Such a model refers to the fourth type of nonlocality in which the elastic properties of the material are considered to be nonlocal in time:

$$\sigma^{ij} = C^{ijkl}\int_{\Omega}\int_{-\infty}^{t} \delta\left(\left|\bar{r}-\bar{\theta}\right|\right) R(t-\tau)\varepsilon_{kl}(\theta,\tau)d\tau d\theta + X^{ijkl}\int_{\Omega}\int_{-\infty}^{t} \delta\left(\left|\bar{r}-\bar{\theta}\right|\right) \delta(t-\tau)\dot{\varepsilon}_{kl}(\theta,\tau)d\tau d\theta \tag{6}$$

By integrating the δ-functions in (6) and taking into account only the time interval from the beginning of the system`s oscillations [12], we can obtain:

$$\sigma^{ij} = C^{ijkl}\int_{0}^{t} R(t-\tau)\varepsilon_{kl}(\tau)d\tau + X^{ijkl}\dot{\varepsilon}_{kl} \tag{7}$$

For now, the type of kernel function is not correlated with the specific features of the material structure and is mainly selected for computational efficiency reasons. Thus the kernel function in the nonlocal model can take different forms such as exponential function, error function, etc [2]. However, all types of kernels have to meet the normalization requirement:

$$\int^{t} R(t-\tau)\, d\tau = 1 \tag{8}$$

A common property of these functions, taking into account the normalization condition (8), is that, in some cases, the kernel function reduces to the Dirac δ-function, and, consequently, the model transforms to a local classical one.

THE FINITE ELEMENT FORMULATION

In order to make the nonlocal in time dynamic deformation model suitable for practical calculations, it has been integrated into the finite element method algorithm.

The physical interpretation of the finite element dynamic problems can be based on the principle of virtual work [13]:

$$\int\limits_V \rho \ddot{u}_i \delta u_i dV + \int\limits_V \sigma_{ij} \delta \varepsilon_{ij} dV = \int\limits_A q_i \delta u_i dA + \int\limits_V F_i \delta u_i dV. \tag{9}$$

Here ρ is a material density; q_i and F_i are surface and body forces; δu_i is the field of virtual displacements; $\delta \varepsilon_{ij}$ is the field of virtual strains. The two dots denote the second time derivative. Using the defining relation of a nonlocal in time dynamic deformation model (7) and considering the components of elastic moduli tensor constant in time, we can rearrange (9) as:

$$\begin{aligned} &\int\limits_V \rho \ddot{u}_i \delta u_i dV \\ &\quad + \int\limits_V C^{ijkl} \int\limits_0^t R_t(t-\tau) \varepsilon_{kl}(\tau) d\tau \, \delta \varepsilon_{ij} dV \\ &\quad + \int\limits_V X^{ijkl} \dot{\varepsilon}_{kl} \delta \varepsilon_{ij} dV \\ &\quad = \int\limits_A q_i \delta u_i dA + \int\limits_V F_i \delta u_i dV. \end{aligned} \tag{10}$$

When employing the finite element approach, the actual strain and displacement fields for an element are replaced by approximate ones [14]:

$$\tilde{u} \approx \boldsymbol{Nd}\,, \tag{11}$$

where $\boldsymbol{N}$ is the shape functions matrix, $\boldsymbol{d}$ is the vector of displacements along the nodal degrees of freedom.

With the matrix operator of Cauchy relations $\boldsymbol{A}$ we can write $\boldsymbol{AN} = \boldsymbol{B}$, then assumed element strain field is:

$$\tilde{\varepsilon} = \boldsymbol{Bd}. \tag{12}$$

Considering this and omitting the virtual nodal displacements according to the principle of virtual work [13] we can represent (10) in matrix form:

$$\begin{aligned} &\int\limits_V \rho \boldsymbol{N^T N} dV \ddot{\boldsymbol{d}} + \int\limits_V \boldsymbol{B^T X B} dV \dot{\boldsymbol{d}} \\ &\quad + \int\limits_V \boldsymbol{B^T C} \int\limits_0^t R(t-\tau) \boldsymbol{Bd}(\tau) d\tau \, dV \\ &\quad = \int\limits_A \boldsymbol{Nq} dA + \int\limits_V \boldsymbol{NF} dV \end{aligned} \tag{13}$$

For beam vibrations, we have only one component of $\boldsymbol{C}$ matrix: Young modulus *E;* and one component of strain:

$$\varepsilon_{11} = -y \frac{d^2 v}{dx^2}. \tag{14}$$

Here x is the longitudinal coordinate of the beam, y is the distance from the neutral layer of the bending beam to the point at which the deformation ε_{11} is calculated, v is the beam deflection function. The assumed solution $\tilde{v} \approx \boldsymbol{Nd}$ for each beam element can be written using Hermit interpolation functions:

$$\boldsymbol{N} = \begin{bmatrix} \frac{2s^3}{l^3} - \frac{3s^2}{l^2} + 1 \\ \frac{s^3}{l^2} - \frac{2s^2}{l} + s \\ \frac{3s^2}{l^2} - \frac{2s^3}{l^3} \\ \frac{s^3}{l^2} - \frac{s^2}{l} \end{bmatrix}, \tag{15}$$

where l is an element length, $s \in [0, l]$ is an element longitudinal coordinate.

Using $\frac{d^2}{ds^2}$ as $\boldsymbol{A}$ we have:

$$\boldsymbol{B} = \frac{d^2\boldsymbol{N}}{ds^2} = \begin{bmatrix} \frac{12s}{l^3} - \frac{6}{l^2} \\ \frac{6s}{l^2} - \frac{4}{l} \\ \frac{6}{l^2} - \frac{12s}{l^3} \\ \frac{6s}{l^2} - \frac{2s}{l} \end{bmatrix}. \tag{16}$$

Since all elements of $\boldsymbol{B}$ matrix are time independent we can take it out of the integral over time in (13). Then taking into account that cross-sectional moment of inertia is $I_z = \int_A y^2 dA$ we obtain:

$$\int_l \rho A \boldsymbol{N}^T \boldsymbol{N} ds \ddot{\boldsymbol{d}} + \int_l \boldsymbol{B}^T \chi I_z \boldsymbol{B} ds \dot{\boldsymbol{d}} + \int_l E I_z \boldsymbol{B}^T \boldsymbol{B} \int_0^t R(t-\tau) \boldsymbol{d}(\tau) d\tau \, ds = \boldsymbol{F}. \tag{17}$$

Here $\boldsymbol{M_i} = \int_l \rho A \boldsymbol{N}^T \boldsymbol{N} ds$ is the beam element mass matrix, $\boldsymbol{D_i} = \int_l \boldsymbol{B}^T \chi I_z \boldsymbol{B} ds$ is the element damping matrix, $\boldsymbol{K_i} = \int_l E I_z \boldsymbol{B}^T \boldsymbol{B} ds$ is the element stiffness matrix, $\boldsymbol{F}$ – is the load vector.

In (17) the damping matrix is proportional to the stiffness matrix. The same result was obtained in [10], where the damping matrix was derived using the stationary condition of the system's total energy during vibration.

After assembling $\boldsymbol{M_i}$, $\boldsymbol{D_i}$, $\boldsymbol{K_i}$ to a global mass matrix $\boldsymbol{M}$, damping matrix $\boldsymbol{D}$ and stiffness matrix $\boldsymbol{K}$ of the beam, we can write the finite element formulation of equation of motion:

$$\boldsymbol{M}\ddot{\boldsymbol{d}} + \boldsymbol{D}\dot{\boldsymbol{d}} + \boldsymbol{K} \int_0^t R(t-\tau) \boldsymbol{d}(\tau) d\tau = \boldsymbol{F}. \tag{18}$$

For the material with constant in time Young modulus in equation (18) the kernel function convolute only with the nodal displacement vector, and stiffness matrix of the beam $\boldsymbol{K}$ remains outside of the integral.

In this research error function was used as the kernel function:

$$R(t-\tau) = \frac{2\eta}{\sqrt{\pi}} \cdot e^{-\eta^2 (t-\tau)^2}. \tag{19}$$

Here η is the parameter that characterizes the rate at which the kernel function decreases. It determines how much of the deformation history significantly impacts the current state of the system. The smaller the value of the scale parameter η, the more nonlocal the model becomes. Conversely, the larger the η, the closer is the model to the local classical one. Figure 1 illustrates how this parameter influences the shape of the kernel.

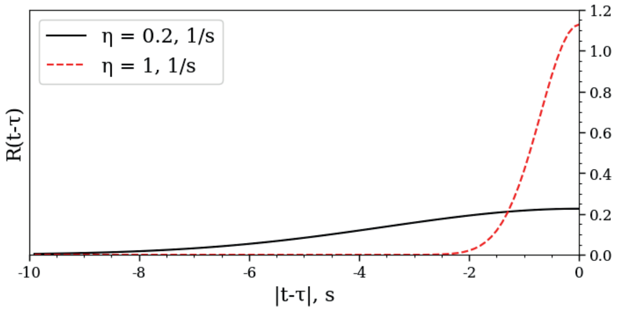

Figure 1 Error kernel functions for different values of η

Since the solution of the dynamic problem using the finite element method is discretized in time, the kernel function (19) was replaced by its discrete equivalent:

$$R = \sum_{j=1}^{i} \frac{2\eta}{\sqrt{\pi}} e^{-\eta^2 \left(t_i - \left(\tau_j - \frac{\Delta t}{2}\right)\right)^2} \Delta t, \tag{20}$$

where i and j are numbers of time steps, Δt is the time increment.

BEAM VIBRATIONS MODELLING CONSIDERING THE ELASTIC PROPERTIES OF THE MATERIAL NONLOCAL IN TIME

To solve the equation of motion in matrix form, an implicit scheme [15] was employed. In this case, the nodal velocities and accelerations are represented as:

$$\begin{aligned} \dot{\boldsymbol{d}}_{i+1} &= \frac{\boldsymbol{d}_{i+1} - \boldsymbol{d}_i}{\Delta t}; \\ \ddot{\boldsymbol{d}}_{i+1} &= \frac{2}{\Delta t^2}\left(\boldsymbol{d}_{i+1} - \boldsymbol{d}_i - \dot{\boldsymbol{d}}_i \cdot \Delta t\right) - \ddot{\boldsymbol{d}}_i \end{aligned} \tag{21}$$

Since, in the case of employing an implicit scheme, the memory function is applied over the entire deformation process, ending at time t_{i+1}, the discrete integral kernel (20) has been divided into two parts [16]: α - the weighting coefficient associated with $\boldsymbol{d}_{i+1}$, β - the sum of all other weights (fig. 2):

$$\begin{aligned} \alpha &= \frac{2\eta}{\sqrt{\pi}} \cdot e^{-\eta^2 (t_{i+1} - t_i)^2} \cdot \Delta t; \\ \beta &= \Delta t \sum_{j=1}^{i} \frac{2\eta}{\sqrt{\pi}} \cdot e^{-\eta^2 (t_i - t_{j-1})^2} \end{aligned} \tag{22}$$

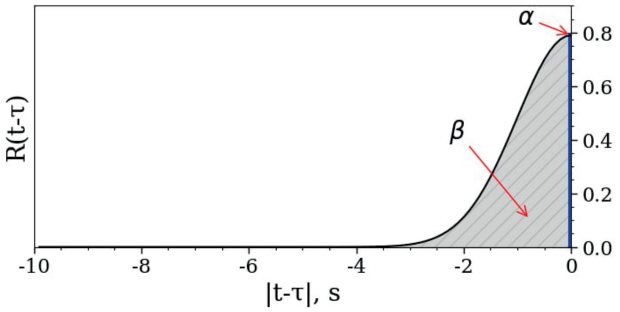

Figure 2. Kernel function divided into two parts

Then the computational scheme for solving the equation of motion by the Newmark method is as follows:

$$\boldsymbol{Q}\boldsymbol{d}_{i+1} = \boldsymbol{F}_{i+1} + \mathbf{M}\ddot{\boldsymbol{d}}_i + \boldsymbol{Q_1}\dot{\boldsymbol{d}}_i + \boldsymbol{Q_2}\boldsymbol{d}_i - \boldsymbol{K}\beta, \tag{23}$$

where:

$$\begin{aligned} \boldsymbol{Q} &= \frac{2}{\Delta t^2}\boldsymbol{M} + \frac{1}{\Delta t}\boldsymbol{D} + \boldsymbol{K}\alpha, \\ \boldsymbol{Q_1} &= \frac{2}{\Delta t}\boldsymbol{M}, \\ \boldsymbol{Q_2} &= \frac{2}{\Delta t^2}\boldsymbol{M} + \frac{1}{\Delta t}\boldsymbol{D}, \end{aligned} \tag{24}$$

To assess the effect of changes in the scale parameter η on the results of mathematical modeling, the vibrations of a 12-meter fixed-end beam (Fig. 3) made of thermoactive vinyl ester glassfiber-reinforced plastic were considered as a numerical example. The beam was loaded with an instantaneously applied and evenly distributed load with an intensity of -10 kN/m.

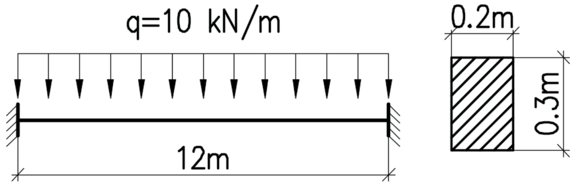

Figure 3. The calculation scheme of the considered beam

The characteristics of the material are given in Table 1.

Table 1. Characteristics of thermoactive vinyl ester glass-reinforced plastic

Young's Modulus (longitudinal), E_{lw}	17.2 GPa
Poisson's Ratio (longitudinal), μ_{lw}	0.32
Density, ρ	1900 kg/m^3
Damping coefficient, ξ	0.042

A software module written in Python was developed to solve the equation of motion and compute the nodal displacements, velocities, and accelerations of the beam accounting for nonlocal in time dynamic deformation. The results of the computer simulation are presented on fig. 4.

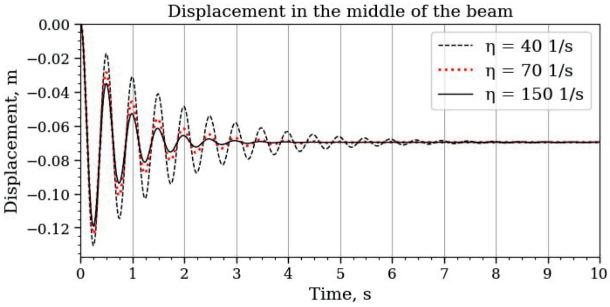

Figure 4. Vertical displacements history of the central node of the beam, obtained for three different scale parameters

Analyzing the presented graphs, it can be observed that as the parameter η decreases (indicating a higher degree of non-locality), the amplitude of oscillations increases.

IDENTIFICATION OF THE SCALE PARAMETER OF THE NONLOCAL MODEL BASED ON THE BEAM VIBRATION EXPERIMENT

One of the key challenges when using nonlocal models in applied problems is the necessity of determining the scale parameter η. In this study, a method for identifying the scale parameter through model calibration based on experimental data is proposed. The approach is based on the least squares method and involves minimizing the standard deviation between the results of numerical simulations and experimental measurements. To find the optimal value of the scale parameter that provides the best match between the model and the experimental data, the golden ratio method [17] was employed.

The proposed methodology was implemented based on the results of laboratory tests conducted on beams made of lightweight concrete filled with ceramic microspheres. The experiment was carried out with the support of the Scientific Research Center "Nanomaterials and Nanotechnologies" and the Research Institute of Experimental Mechanics of the Moscow State University of Civil Engineering. The characteristics of the material are presented in Table 2. The properties are isotropic, but the material has a heterogeneous structure.

Table 2. Properties of the lightweight concrete filled with the ceramic microspheres

Young's Modulus (longitudinal), E_{lw}	78.5 GPa
Density, ρ	1700 kg/m^3
Damping coefficient, ξ	0.0765

Identification of the model parameters was carried out based on the accelerations history of the free vibrations of a fixed-end beam initiated by a test hammer strike. To register the accelerations of the structure in the middle of the beam, a single-axis accelerometer was used (fig 5).

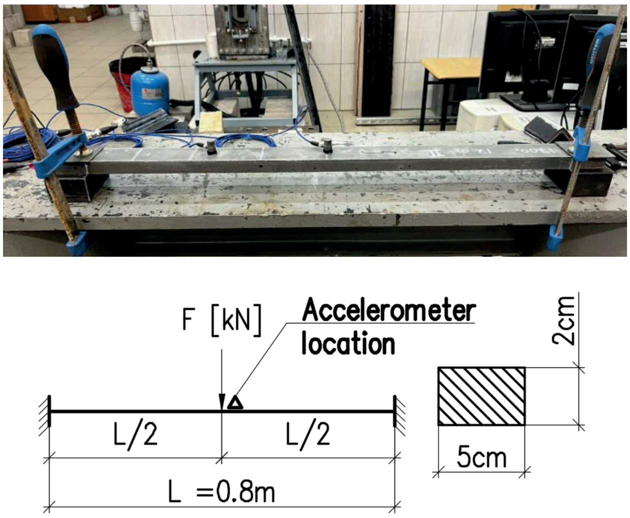

Figure 5. lightweight concrete beam used in the experiment

The received signal was synchronized and recorded using the National Instruments PXIe-1082 multichannel measuring system, which provides a sampling frequency of up to 12 kHz. This accuracy of registration made it possible to record in detail the accelerogram of the oscillatory process.

The acceleration time history at the mid-cross-section of the beam was used as a reference against which the results of one-dimensional modeling, based on local and non-local dynamic deformation models, were compared.

On figure 6 the comparison of the results obtained by local mathematical model with the experimental acceleration history is shown.

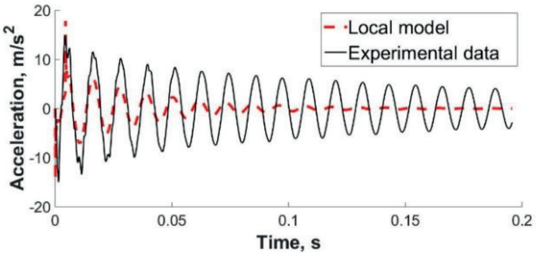

Figure 6. comparison of the reference accelerations with the results obtained by local classical model

Nonlocal in time dynamic deformation model of the beam vibration was calibrated by the experimental data. The comparison of those results with the experimental accelerations is presented on figure 7.

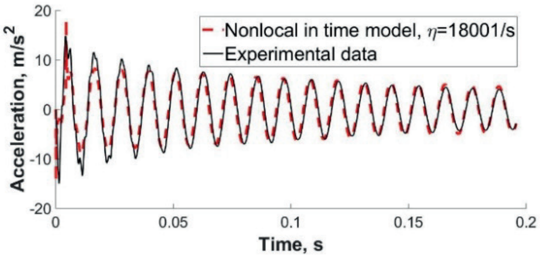

Figure 7. comparison of the results obtained by the nonlocal in time dynamic deformation model with the experimental accelerations

It is evident that the nonlocal in time model provides a better correspondence with the experimental data compared to the classical local model of beam vibration, which does not account for the inhomogeneous structure of the material. Although the nonlocal in time model also does not explicitly consider this inhomogeneity, it remains sufficiently flexible to accurately describe the dynamic behavior of beams made of such materials.

CONCLUSION

Nonlocal in time models of dynamic deformation of the material can serve as an effective alternative to detailed three-dimensional models when describing the dynamic behavior of materials with complex internal structure. The developed model has been integrated into the finite element method algorithm, enabling its application to practical problems. The method for determining the scale parameter of a nonlocal in time model was implemented based on the results of a laboratory experiment. It is shown that in comparison with the local classical model, the developed model makes it possible to achieve a better match with the experimental results.

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